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JOINT OPTIMAL DECISIONS ON PRICING AND WARRANTY POLICY OF DUOPOLY SUPPLY CHAIN WITH ONE COMMON RETAILER

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Abstract: Early researches related to the interaction between manufactures for complementary products, mainly considered price as only the dimension of competition. With the increasing competition in capturing the market share, manufactures cannot compete by only lowering prices. In this paper, we assume that besides the price, the manufactures choose warranty as the competitive strategy of two different but substitutable products in a duopoly supply chain with one common retailer. Furthermore, two cases are considered (i) only one manufacturer adopts warranty policy as a competitive strategy against the other, (ii) both manufacturers offer warranty on their product, to study under which situation offering a warranty becomes more profitable for a manufacturer while the other competitive manufacturer has already adopted warranty policy. The profit functions of the manufacturers and the retailer are then maximized under manufacturers' cooperative and non-cooperative strategies. We then compare the scenarios under different decision strategies numerically, which gives some insights on changes of key parameters to help the decision makers to capture the market.

Keywords: Warranty, Pricing, Supply Chain Management, Quality, Stackleberg Game.

MSC: 91B24.

1. INTRODUCTION

With rapid trends in business globalization and current competitive environment, the marketing strategies of the business have to be renovated to face the challenges of the global competitive marketplace. Today, various brands of a single kind of product (e.g., smart-phone manufactured by Samsung, Vivo, Apple etc) are often sold by the same retailer. Thus, the business models have experienced significant changes to improve customer service reputation in highly competitive market.

The competition among the companies was mainly concerned with prices, but in this modern age of social networking, the trust and support of the customers play a vital role in the business world. Thus, a good reputation of a business in terms of quality of the product and consumer service becomes crucial to its survival. Customer can forecast the durability of the product based on its length of warranty (Boulding and Kirmani [4]). To avoid the risk whether the product will serve as expected or not, the majority of the customers favor to buy a product from a manufacturing company who offers a warranty period guaranteeing replacement, refunding or repairing of the product during this period. As a result, the manufacturer can explore the market strategy that offer warranty on their product, e.g., Hyundai, Acura, Audi, Mercedes-Benz in the automobile market, Hewlett-Packard, Panasonic, Samsung, Cannon in the electronics market.

Therefore, it becomes important for manufacturers to decide on how to set the optimal wholesale price for their product because the demand of the product not only depends on its own price but also on the price of its complementary product. It is also observed that the manufacturer adopts some marketing strategies such as warranty for competition. As a result, it turns out to be more challenging for the manufacturers to decide how to set warranty period and wholesale price to increase their profit individually and for retailer, it becomes a crucial task to set their retail prices to satisfy the customer demand. We have addressed this issue by considering the demand of each product decreasing with its own price and the competitor's warranty period and increasing with its own warranty period and the competitor's price, which corresponds to reality in many practical situations. To examine the situation under which offering a warranty becomes more economical for a manufacturer while the other competitive manufacturer has already adopted warranty policy, we consider two scenarios (i)one manufacturer offers warranty on his product and the other does not (case 1) (ii) both manufacturers offer warranty as the competitive strategy (case 2).

2. LITERATURE REVIEW

Dealing with warranty policy for products has gained much interest from researchers. Regarding the agreement of warranty policies, manufacturers adopt different types of warranties such as (i) free replacement warranty policy, (ii) money back warranty (full refund) policy, (iii) outsourcing maintenance service policy, (iv) Pro-rata warranty (replacement at a cost or refunding a fraction of its purchasing price) policy. Boom [3] discussed a situation where a monopolist supplier reimburses the risk-averse consumers by three types of warranty rules (a) no warranty, (b) money back guarantee, (c) renewing free replacement. Rinsaka and Sandoh [6] considered the case in which the manufacturer replaces the product or system with a new one for its first failure but minimal repairs are conducted with the succeeding failures during the warranty period. Asgharizadeh and Murthy [2] developed a game theoretic model where repairs are carried out by an external agent under a service contract when the equipments fail. Alqahtani and Gupta [1] studied a renewable two-dimensional Pro-Rata warrantee policy for end-of-life products.

In the present market, it is observed that multiple brands of a single type of product (e.g., sunglass made by Ray Ban, Gucci, Oakley etc) is often sold by the same retailer. In this situation, price, discount, warranty, or other service contracts are significant sale factors in capturing market share. There are numerous studies involving pricing problem (e.g., Choi [10]; Raju *et al.* [7]; Zhao *et al.* [8]; Tsay and Agrawal [20]). Choi [10] developed three types of pricing games of different power structures between two manufactures and a retailer in a two-echelon supply chain to examine how channel profits split among the channel members. Choi [11] extended this monopoly common retailer channel model by introducing price competition between duopoly common retailers where each manufacturer sells the same product to both retailers. Luo *et al.* [12] investigated the price competition between two manufacturers and a retailer in which the retailer sells differentiated brands, a good brand and an average brand, supplied by two manufacturers.

To solve the problem of gaining the market share, many researchers have focused on both price and warranty/ service /capacity/location as the dimensions of competition (e.g., Wei *et al.* [13]; Tsay and Agrawal [14]; Hall and Porteus [15]; Iyer [16]; Tsao and Su [17]). In this study, we consider a pricing and warranty period decision problem in a supply chain consisting of two competing manufacturer and a common retailer. Lu *et al.* [18] examined a pricing and warranty decisions problem in a two-echelon dual supply chain model. Taleizadeh *et al.* [19] analyzed two markets with different level of willingess to pay for product with a common manufacturer at both markets who offers warranty as a competing factor when a third party distributer acts as a gray market. However, most of the studies which consider warranty as the effective strategy to boost the sales tend to ignore warranty cost as the function of product quality, and consider warranty cost as the function of length of warranty period and failure rate. But warranty cost mainly emerges due to poor quality level of a product. In the recent years, industries are continuously trying to reduce warranty costs by increasing product quality.

3. ASSUMPTIONS and NOTATIONS

To develop the model, we make the following assumptions and notations

3.1. Assumptions

- The model structure is developed for two different but substitutable products consisting of two manufacturers and a common retailer.
- Warranty cost of manufacturer depends on quality level of product and warranty period.
- Manufacturer bears a quality improvement cost to lessen the warranty cost.
- The manufacturer is more powerful in making decision than retailer.

3.2. Notations

α_{p_i}	The market potential of the product produced by manufacturer $i(\alpha_{p_i} > 0)$
w_{ij}	The wholesale price per unit by the manufacturer i in case j
p_{ij}	The retail price per unit product produced by manufacturer i in case j
$T_{L_{ij}}$	The warranty period offered by the manufacturer i in case j
β_c	The price sensitivity factor ($\beta_c > 0$)
β_t	The warranty period sensitivity factor ($\beta_t > 0$)
η_c	The degree of price competition between the manufacturers ($\eta_c > 0$)
η_t	The degree of warranty period competition between the manufacturers ($\eta_t > 0$)
q_i	The quality level of the product produce by the manufacturer i ($q_i \in [0, 1]$)
c_i	The production cost per unit of the manufacturer i

4. MODEL FORMULATION

In this paper, we develop a two-echelon supply chain model, where a common retailer sells two complementary products produced by two manufacturers indexed by $i \in \{1, 2\}$. Thus, it leads to a competition between manufacturers. Besides the price, to attract the customers, the manufacturers provide a free repair warranty policy as a competitive strategy against each other. Manufacturer i faces the warranty cost $C_{Li} = \lambda_i T_{Li}^{\gamma_i} q_i^{-\delta_i}$, which is convex and decreasing with respect to the quality level q_i for any $\delta_i > 0$ (i.e., $\frac{\partial C_{Li}}{\partial q_i} < 0$, $\frac{\partial^2 C_{Li}}{\partial q_i^2} > 0$) (Noll [21]). We also see that this cost function C_{Li} is increasing and convex with respect to warranty period T_{Li} for any $\gamma_i > 1$(i.e., $\frac{\partial C_{Li}}{\partial T_{Li}} > 0$, $\frac{\partial^2 C_{Li}}{\partial T_{Li}^2} > 0$). To reduce the warranty cost, manufacturer i expends cost $C_{m_i}(q_i) = c_{m_i} \frac{q_i}{1-q_i}$ in improving his product quality level, which is increasing and convex with respect to q_i, (i.e., $\frac{\partial C_{m_i}}{\partial q_i} > 0$, $\frac{\partial^2 C_{m_i}}{\partial q_i^2} > 0$), $lim_{q_i \to 0} C_{m_i} = 0$ and $lim_{q_i \to 1} C_{m_i} = \infty$ in the range $q_i \in [0, 1]$.

4.1. Only one manufacturer offers warranty (Case 1)

In this situation, only the manufacturer 1 offers warranty on his product. We consider that the demand function for a product is decreasing with respect to its own retail price and increasing with respect to the complementary product's retail price. On the other hand, increasing warranty period offered by manufacturer 1, increases manufacturer 1's demand and decreases manufacturer 2's demand. Thus, we design the demand functions of manufacturers respectively as follows

$$D_{11}(p_{11}, p_{21}, T_{L_{11}}) = \alpha_{p_1} - (\beta_c + \eta_c)p_{11} + \eta_c p_{21} + (\beta_t + \eta_t)T_{L_{11}} \tag{1}$$

and

$$D_{21}(p_{11}, p_{21}, T_{L_{11}}) = \alpha_{p_2} - (\beta_c + \eta_c)p_{21} + \eta_c p_{11} - \eta_t T_{L_{11}}. \tag{2}$$

The profit functions of two manufacturers and the retailer can be written respectively as follows

$$TP_{m_{11}} = \left(w_{11} - c_1 - c_{m_1}\frac{q_1}{1-q_1} - \lambda_1 q_1^{-\delta_1} T_{L_{11}}^{\gamma_1}\right) D_{11}, \tag{3}$$

$$TP_{m_{21}} = (w_{21} - c_2)D_{21} \tag{4}$$

and

$$\begin{aligned} TP_{r_1} = & \quad (p_{11} - w_{11})\{\alpha_{p_1} - (\beta_c + \eta_c)p_{11} + \eta_c p_{21} + (\beta_t + \eta_t)T_{L_{11}}\} \\ & + \quad (p_{21} - w_{21})\{\alpha_{p_2} - (\beta_c + \eta_c)p_{21} + \eta_c p_{11} - \eta_t T_{L_{11}}\}. \end{aligned} \tag{5}$$

4.1.1. Decentralized decision

In decentralized decision making, considering the reality, we assume that the manufacturers are more powerful in decision making than the retailer, i.e., the manufacturers act as leaders and the common retailer is their follower. Based on the reaction of the retailer on retail prices, the manufacturers make decisions on their wholesale prices and warranty periods. To determine the retailer best response on retail price, we first optimize retailer profit function for the given manufacturers' decision variables. That is

$$max \;\; TP_{r_1}(p_{11}, p_{21} | w_{11}, w_{21}, T_{L_{11}}). \tag{6}$$

The optimal values of p_{11} and p_{21} are obtained by solving $\frac{\partial TP_{r_1}}{\partial p_{11}} = 0$ and $\frac{\partial TP_{r_1}}{\partial p_{21}} = 0$ as follows

$$p_{11}^* = \frac{w_{11}}{2} + \frac{(\beta_c + \eta_c)(\beta_t + \eta_t) - \eta_c\eta_t}{2\beta_c(\beta_c + 2\eta_c)} T_{L_{11}} + \frac{(\beta_c + \eta_c)\alpha_{p_1} + \eta_c\alpha_{p_2}}{2\beta_c(\beta_c + 2\eta_c)} \tag{7}$$

and

$$p_{21}^* = \frac{w_{21}}{2} + \frac{(\beta_t + \eta_t)\eta_c - (\beta_c + \eta_c)\eta_t}{2\beta_c(\beta_c + 2\eta_c)} T_{L_{11}} + \frac{\eta_c\alpha_{p_1} + (\beta_c + \eta_c)\alpha_{p_2}}{2\beta_c(\beta_c + 2\eta_c)} \tag{8}$$

Note that

$$\frac{\partial^2 TP_{r_1}}{\partial p_{11}^2} = -2(\beta_c + \eta_c) < 0, \frac{\partial^2 TP_{r_1}}{\partial p_{21}^2} = -2(\beta_c + \eta_c) < 0$$

and

$$\frac{\partial^2 TP_{r_1}}{\partial p_{11}^2}\frac{\partial^2 TP_{r_1}}{\partial p_{21}^2} - \frac{\partial^2 TP_{r_1}}{\partial p_{11}\partial p_{21}}\frac{\partial^2 TP_{r_1}}{\partial p_{21}\partial p_{11}} = 4(\beta_c + \eta_c)^2 - 4\eta_c^2 > 0.$$

That is TP_{r1} is a concave function of p_{11} and p_{21}. Now the manufacturers make decisions, taking into account the retailer's best response on retail prices, with the objective of maximizing their own profit. We develop two decision models by considering the manufacturers' cooperative and noncooperative decision strategies.

Manufacturers' noncooperative decision (MNC) strategy

In this situation, two manufacturers maximize their profits non-cooperatively and make their decisions on wholesale prices and warranty periods independently subject to the constraints imposed by equations in (7) and (8). Hence, the manufacturers' decision problem is formulated as follows

$$\begin{cases} \begin{cases} \max\limits_{(w_{11}, T_{L_{11}})} & TP_{m_{11}}(w_{11}, w_{21}, T_{L_{11}}, p_{11}^*(w_{11}, w_{21}, T_{L_{11}}), p_{21}^*(w_{11}, w_{21}, T_{L_{11}})) \\ \max\limits_{w_{21}} & TP_{m_{21}}(w_{11}, w_{21}, T_{L_{11}}, p_{11}^*(w_{11}, w_{21}, T_{L_{11}}), p_{21}^*(w_{11}, w_{21}, T_{L_{11}})) \end{cases} \\ \textit{subject to } (7) \textit{ and } (8). \end{cases} \tag{9}$$

The partial derivatives of $TP_{m_{11}}(w_{11}, w_{21}, T_{L_{11}}, p_{11}^*, p_{21}^*)$ with respect to $w_{11}, T_{L_{11}}$ and $TP_{m_{21}}(w_{11}, w_{21}, T_{L_{11}}, p_{11}^*, p_{21}^*)$ with respect to w_{21} are respectively as follows

$$\begin{aligned} \frac{\partial TP_{m_{11}}}{\partial w_{11}} &= -(\beta_c + \eta_c)w_{11} + \frac{1}{2}\eta_c w_{21} + \frac{1}{2}(\beta_t + \eta_t)T_{L_{11}} \\ &+ \frac{1}{2}(\beta_c + \eta_c)\lambda_1 T_{L_{11}}^{\gamma} q_1^{-\delta_1} + \frac{1}{2}\left\{\alpha_{p_1} + (\beta_c + \eta_c)\left(c_1 + c_{m_1}\frac{q_1}{1-q_1}\right)\right\}, \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\partial TP_{m_{11}}}{\partial T_{L_{11}}} &= \frac{1}{2}(\beta_t + \eta_t)\left(w_{11} - c_1 - c_{m_1}\frac{q_1}{1-q_1} - \lambda_1 T_{L_{11}}^{\gamma_1} q_1^{-\delta_1}\right) \\ &- \frac{1}{2}\lambda_1\gamma_1 T_{L_{11}}^{\gamma_1 - 1} q_1^{-\delta_1}\left\{\alpha_{p_1} - (\beta_c + \eta_c)w_{11} + \eta_c w_{21} + (\beta_t + \eta_t)T_{L_{11}}\right\} \end{aligned} \tag{11}$$

and

$$\frac{\partial TP_{m_{21}}}{\partial w_{21}} = \frac{1}{2}\eta_c w_{11} - (\beta_c + \eta_c)w_{21} - \frac{1}{2}\eta_t T_{L_{11}} + \frac{1}{2}\left\{\alpha_{p_2} + (\beta_c + \eta_c)c_2\right\}. \tag{12}$$

Solving equations $\frac{\partial TP_{m_{11}}}{\partial w_{11}}=0$, $\frac{\partial TP_{m_{11}}}{\partial T_{L_{11}}}=0$ and $\frac{\partial TP_{m_{21}}}{\partial w_{21}}=0$, we have

$$\begin{aligned}
w_{11}^{mnc*} &= \frac{2(\beta_c+\eta_c)\{\alpha_{p_1}+(\beta_c+\eta_c)c_1\}+\eta_c\{\alpha_{p_2}+(\beta_c+\eta_c)c_2\}}{4(\beta_c+\eta_c)^2-\eta_c^2} \\
&+ \frac{2c_{m_1}(\beta_c+\eta_c)^2 q_1}{(1-q_1)\{4(\beta_c+\eta_c)^2-\eta_c^2\}} \\
&+ \frac{\left\{2(\beta_c+\eta_c)(\beta_t+\eta_t)\left(1+\frac{1}{\gamma_1}\right)-\eta_c\eta_t\right\}\left\{\frac{q_1^{\delta_1}(\beta_t+\eta_t)}{\lambda_1\gamma_1(\beta_c+\eta_c)}\right\}^{\frac{1}{\gamma_1-1}}}{4(\beta_c+\eta_c)^2-\eta_c^2}, \quad (13)\\
w_{21}^{mnc*} &= \frac{\eta_c\{\alpha_{p_1}+(\beta_c+\eta_c)c_1\}+2(\beta_c+\eta_c)\{\alpha_{p_2}+(\beta_c+\eta_c)c_2\}}{4(\beta_c+\eta_c)^2-\eta_c^2} \\
&+ \frac{c_{m_1}\eta_c(\beta_c+\eta_c)q_1}{(1-q_1)\{4(\beta_c+\eta_c)^2-\eta_c^2\}} \\
&+ \frac{\left\{\eta_c(\beta_t+\eta_t)\left(1+\frac{1}{\gamma_1}\right)-2(\beta_c+\eta_c)\eta_t\right\}\left\{\frac{q_1^{\delta_1}(\beta_t+\eta_t)}{\lambda_1\gamma_1(\beta_c+\eta_c)}\right\}^{\frac{1}{\gamma_1-1}}}{4(\beta_c+\eta_c)^2-\eta_c^2}, \quad (14)
\end{aligned}$$

and

$$T_{L_{11}}^{mnc*} = \left\{\frac{q_1^{\delta_1}(\beta_t+\eta_t)}{\lambda_1\gamma_1(\beta_c+\eta_c)}\right\}^{\frac{1}{\gamma_1-1}}. \quad (15)$$

The corresponding retail prices under MNC strategy respectively are as follows:

$$p_{11}^{mnc*} = \frac{w_{11}^{*mnc}}{2}+\frac{(\beta_c+\eta_c)(\beta_t+\eta_t)-\eta_c\eta_t}{2\beta_c(\beta_c+2\eta_c)}T_{L_{11}}^{*mnc}+\frac{(\beta_c+\eta_c)\alpha_{p_1}+\eta_c\alpha_{p_2}}{2\beta_c(\beta_c+2\eta_c)} \quad (16)$$

and

$$p_{21}^{mnc*} = \frac{w_{21}^{*mnc}}{2}+\frac{(\beta_t+\eta_t)\eta_c-(\beta_c+\eta_c)\eta_t}{2\beta_c(\beta_c+2\eta_c)}T_{L_{11}}^{*mnc}+\frac{\eta_c\alpha_{p_1}+(\beta_c+\eta_c)\alpha_{p_2}}{2\beta_c(\beta_c+2\eta_c)}, \quad (17)$$

where $w_{11}^{*mnc}, w_{21}^{*mnc}, T_{L_{11}}^{*mnc}$ are given in Equations (13), (14), and (15).

Proposition 1. *The profit function $TP_{m_{11}}$ under decentralized MNC strategy is a concave function in w_{11} and $T_{L_{11}}$ if $(\gamma_1-1)\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}^{mnc*}+\eta_c w_{21}^{mnc*}\}+(\gamma_1+1)(\beta_t+\eta_t)T_{L_{11}}^{mnc*}>0$ and $(\gamma_1-1)\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}^{mnc*}+\eta_c w_{21}^{mnc*}+(\beta_t+\eta_t)T_{L_{11}}^{mnc*}\}>0$.*

Proof. The profit function $TP_{m_{11}}$ under decentralized MNC strategy would be concave in w_{11} and $T_{L_{11}}$ if at the stationary point $(w_{11}^{mnc*}, T_{L_{11}}^{mnc*})$, the Hessian matrix of $TP_{m_{11}}$ is negative definite. Here, at $(w_{11}^{mnc*}, T_{L_{11}}^{mnc*})$

$$
\begin{aligned}
\frac{\partial^2 TP_{m_{11}}}{\partial w_{11}^2} &= -(\beta_c+\eta_c) < 0,\\
\frac{\partial^2 TP_{m_{11}}}{\partial T_{L_{11}}^2} &= -\frac{(\gamma_1-1)(\beta_t+\eta_t)\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}^{mnc*}+\eta_c w_{21}^{mnc*}\}}{2(\beta_c+\eta_c)T_{L_{11}}^{mnc*}}\\
&- \frac{(\gamma_1+1)(\beta_t+\eta_t)^2}{2(\beta_c+\eta_c)} < 0
\end{aligned}
$$

if $(\gamma_1-1)\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}^{mnc*}+\eta_c w_{21}^{mnc*}\}+(\gamma_1+1)(\beta_t+\eta_t)T_{L_{11}}^{mnc*} > 0$ holds.

$$
\frac{\partial^2 TP_{m_{11}}}{\partial w_{11}^2}\frac{\partial^2 TP_{m_{11}}}{\partial T_{L_{11}}^2} - \frac{\partial^2 TP_{m_{11}}}{\partial w_{11}T_{L_{11}}}\frac{\partial^2 TP_{m_{11}}}{\partial T_{L_{11}}w_{11}} = -(\beta_t+\eta_t)^2+(\beta_c+\eta_c)
$$

$$
\times\left[\frac{(\gamma_1-1)(\beta_t+\eta_t)\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}^{mnc*}+\eta_c w_{21}^{mnc*}\}}{2(\beta_c+\eta_c)T_{L_{11}}^{mnc*}}+\frac{(\gamma_1+1)(\beta_t+\eta_t)^2}{2(\beta_c+\eta_c)}\right] > 0
$$

if $(\gamma_1-1)\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}^{mnc*}+\eta_c w_{21}^{mnc*}+(\beta_t+\eta_t)T_{L_{11}}^{mnc*}\} > 0$ holds. This completes the proof. □

Proposition 2. *The profit function $TP_{m_{21}}$ under decentralized MNC strategy is a concave function in w_{21}.*

Proof. Here at $w_{21} = w_{21}^{mnc*}$,

$$
\frac{\partial^2 TP_{m_{21}}}{\partial w_{21}^2} = -(\beta_c+\eta_c) < 0.
$$

Hence, the profit function $TP_{m_{21}}$ under decentralized MNC strategy is a concave function in w_{21} . This completes the proof. □

Manufacturers' cooperative (MC) decision strategy

In this situation, two manufacturers cooperate and make decisions jointly to find their maximum total profit after seeing the retailer's reaction on retail prices. After optimization, their joint profit would be divided between the two manufacturers. Hence, the manufacturers' decision problem is formulated as follows.

$$
\begin{aligned}
\max_{(w_{11},w_{21},T_{L_{11}})} \quad & [TP_{m_{11}}+TP_{m_{21}}](w_{11},w_{21},T_{L_{11}},p^*_{11(w_{11},w_{21},T_{L_{11}})},p^*_{21(w_{11},w_{21},T_{L_{11}})})\\
& \textit{subject to } (7) \textit{ and } (8). \qquad (18)
\end{aligned}
$$

The partial derivatives of $TP_{m_{11}}+TP_{m_{21}}$ with respect to $w_{11}, T_{L_{11}}$ and w_{21} are respectively as follows:

$$\begin{aligned}\frac{\partial(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{11}} &= -(\beta_c+\eta_c)w_{11}+\eta_c w_{21}\\ &+ \frac{1}{2}\left\{(\beta_t+\eta_t)T_{L_{11}}+(\beta_c+\eta_c)\lambda_1 q_1^{-\delta_1}T_{L_{11}}^{\gamma_1}\right\}\\ &+ \frac{1}{2}\left\{\alpha_{p_1}+(\beta_c+\eta_c)c_1-\eta_c c_2+c_{m_1}(\beta_c+\eta_c)\frac{q_1}{1-q_1}\right\}, \qquad (19)\end{aligned}$$

$$\begin{aligned}\frac{\partial(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{21}} &= \eta_c w_{11}-(\beta_c+\eta_c)w_{21}-\frac{1}{2}\{\eta_t T_{L_{11}}+\eta_c\lambda_1 q_1^{-\delta_1}T_{L_{11}}^{\gamma_1}\}\\ &+ \frac{1}{2}\left\{\alpha_{p_2}+(\beta_c+\eta_c)c_2-\eta_c c_1-c_{m_1}\eta_c\frac{q_1}{1-q_1}\right\} \qquad (20)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial(TP_{m_{11}}+TP_{m_{21}})}{\partial T_{L_{11}}} &= -\frac{1}{2}\eta_t(w_{21}-c_2)\\ &+ \frac{1}{2}(\beta_t+\eta_t)\left\{w_{11}-c_1-c_{m_1}\frac{q_1}{1-q_1}-\lambda_1 q_1^{-\delta_1}T_{L_{11}}^{\gamma_1}\right\}\\ &- \frac{1}{2}\lambda_1\gamma_1 q_1^{-\delta_1}T_{L_{11}}^{\gamma_1-1}\left\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}+\eta_c w_{21}\right.\\ &+ \left.(\beta_t+\eta_t)T_{L_{11}}\right\}. \qquad (21)\end{aligned}$$

Solving equations $\frac{\partial(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{11}}=0, \frac{\partial(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{21}}=0$ and $\frac{\partial(TP_{m_{11}}+TP_{m_{21}})}{\partial T_{L_{11}}}=0$, we obtain the optimal values of w_{11}, w_{21}, $T_{L_{11}}$. Analytically it is difficult to solve these equation. We solve the equation numerically by using Matlab2013 software. Let the solution be $w_{11}=w_{11}^{mc*}$, $w_{21}=w_{21}^{mc*}$, and $T_{L_{11}}=T_{L_{11}}^{mc*}$.

Proposition 3. *The profit function $(TP_{m_{11}}+TP_{m_{21}})(w_{11},w_{21},T_{L_{11}})$ is a concave function if $(\beta_c+\eta_c)^2u_1+2\eta_c u_2u_3+(\beta_c+\eta_c)u_3^2+(\beta_c+\eta_c)u_2^2-u_1\eta_c^2<0$ where $u_1=-\frac{1}{2}\lambda_1\gamma_1(\gamma_1-1)q_1^{-\delta_1}\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}^{mc*}+\eta_c w_{21}^{mc*}\}(T_{L_{11}}^{mc*})^{\gamma_1-2}-\frac{1}{2}\lambda_1\gamma_1(\gamma_1+1)(\beta_t+\eta_t)q_1^{-\delta}(T_{L_{11}}^{mc*})^{\gamma_1-1}$, $u_2=\frac{1}{2}(\beta_t+\eta_t)+\frac{1}{2}(\beta_c+\eta_c)\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{11}}^{mc*})^{\gamma_1-1}$ and $u_3=-\frac{1}{2}\eta_t-\frac{1}{2}\eta_c\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{11}}^{mc*})^{\gamma_1-1}$.*

Proof. The second order partial derivatives of $(TP_{m_{11}}+TP_{m_{21}})$ at stationary point $S_1=(w_{11}^{mc*},w_{21}^{mc*},T_{L_{11}}^{mc*})$ are

$$\left.\frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{11}^2}\right|_{atS_1} = -(\beta_c+\eta_c), \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{21}^2}|_{atS_1}=-(\beta_c+\eta_c),$$

$$
\begin{aligned}
\frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial T_{L_{11}}^2}|_{atS_1} &= -\frac{1}{2}\lambda_1\gamma_1(\gamma_1-1)q_1^{-\delta_1}\Big\{\alpha_{p_1}-(\beta_c+\eta_c)w_{11}^{mc*}+\eta_c w_{21}^{mc*}\Big\} \\
&\times \ (T_{L_{11}}^{mc*})^{\gamma_1-2}-\frac{1}{2}\lambda_1\gamma_1(\gamma_1+1)(\beta_t+\eta_t)q_1^{-\delta}(T_{L_{11}}^{mc*})^{\gamma_1-1} \\
&= u_1(say), \\
\frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{11}\partial w_{21}}|_{atS_1} &= \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{21}\partial w_{11}}|_{atS_1}=\eta_c, \\
\frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{11}\partial T_{L_{11}}}|_{atS_1} &= \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial T_{L_{11}}\partial w_{11}}|_{atS_1} \\
&= \frac{1}{2}(\beta_t+\eta_t)+\frac{1}{2}(\beta_c+\eta_c)\lambda_1\gamma_1 q_1^{-\delta_1}(T_{L_{11}}^{mc*})^{\gamma_1-1}=u_2(say), \\
\frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{21}\partial T_{L_{11}}}|_{atS_1} &= \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial T_{L_{11}}\partial w_{21}}|_{atS_1} \\
&= -\frac{1}{2}\eta_t-\frac{1}{2}\eta_c\lambda_1\gamma_1 q_1^{-\delta_1}(T_{L_{11}}^{mc*})^{\gamma_1-1}=u_3(say).
\end{aligned}
$$

The Hessian matrix H_1 of $(TP_{m_{11}}+TP_{m_{21}})$ at the stationary point S_1 $(w_{11}^{mc*}, w_{21}^{mc*}, T_{L_{11}}^{mc*})$

$$
H_1 = \begin{pmatrix}
\frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{11}^2} & \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{11}\partial w_{21}} & \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{11}\partial T_{L_{11}}} \\
\frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{21}\partial w_{11}} & \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{21}^2} & \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial w_{21}\partial T_{L_{11}}} \\
\frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial T_{L_{11}}\partial w_{11}} & \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial T_{L_{11}}\partial w_{21}} & \frac{\partial^2(TP_{m_{11}}+TP_{m_{21}})}{\partial T_{L_{11}}^2}
\end{pmatrix} atS_1
$$

The profit function $(TP_{m_{11}}+TP_{m_{21}})$ will be concave function if the principal minors of H_1 are alternatively negative and positive, i.e., if the i^{th} order principal minor D_i of H_1 takes the sign $(-1)^i$. Here,

$$
\begin{aligned}
D_1 &= -(\beta_c+\eta_c)<0, \\
D_2 &= \begin{vmatrix} -(\beta_c+\eta_c) & \eta_c \\ \eta_c & -(\beta_c+\eta_c) \end{vmatrix} \\
&= (\beta_c+\eta_c)^2-\eta_c^2>0
\end{aligned}
$$

and

$$
D_3 = |H_1| = (\beta_c+\eta_c)^2u_1+2\eta_c u_2u_3+(\beta_c+\eta_c)u_3^2+(\beta_c+\eta_c)u_2^2-u_1\eta_c^2<0
$$

if $(\beta_c+\eta_c)^2u_1+2\eta_c u_2u_3+(\beta_c+\eta_c)u_3^2+(\beta_c+\eta_c)u_2^2-u_1\eta_c^2<0$ holds. This completes the proof. □

4.1.2. Centralized decisions

In this decision case, both the manufacturers and their common retailer cooperate to maximize the total profit of the supply chain. The total profit function under this scenario is

$$\begin{aligned} TP_{c_1} &= TP_{m_{11}} + TP_{m_{21}} + TP_{r1} \\ &= \left(p_{11} - c_1 - c_{m_1}\frac{q_1}{1-q_1} - \lambda_1 T_{L_{11}}^{\gamma_1} q_1^{-\delta_1}\right) D_{11} + (p_{21} - c_2) D_{12}. \quad (22) \end{aligned}$$

Hence, the channel members'decision problem is formulated as follows

$$\max_{(p_{11}, p_{21}, T_{L_{11}})} \quad TP_{c_1}(p_{11}, p_{21}, T_{L_{11}}). \quad (23)$$

The partial derivatives of $TP_{c_1}(p_{11}, p_{21}, T_{L_{11}})$ with respect to $p_{11}, T_{L_{11}}$, and p_{21} are respectively as follows:

$$\begin{aligned} \frac{\partial TP_{c_1}}{\partial p_{11}} &= -2(\beta_c + \eta_c) p_{11} + 2\eta_c p_{21} + \{(\beta_t + \eta_t) T_{L_{11}} + (\beta_c + \eta_c)\lambda_1 q_1^{-\delta_1} T_{L_{11}}^{\gamma_1}\} \\ &+ \left\{\alpha_{p_1} + (\beta_c + \eta_c) c_1 - \eta_c c_2 + c_{m_1}(\beta_c + \eta_c)\frac{q_1}{1-q_1}\right\}, \quad (24) \\ \frac{\partial TP_{c_1}}{\partial p_{21}} &= 2\eta_c p_{11} - 2(\beta_c + \eta_c) p_{21} - \{\eta_t T_{L_{11}} + \eta_c \lambda_1 q_1^{-\delta_1} T_{L_{11}}^{\gamma_1}\} \\ &+ \left\{\alpha_{p_2} + (\beta_c + \eta_c) c_2 - \eta_c c_1 - c_{m_1}\eta_c \frac{q_1}{1-q_1}\right\} \quad (25) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial TP_{c_1}}{\partial T_{L_{11}}} &= -\eta_t(p_{21} - c_2) + (\beta_t + \eta_t)\left\{p_{11} - c_1 - c_{m_1}\frac{q_1}{1-q_1} - \lambda_1 q_1^{-\delta} T_{L_{11}}^{\gamma_1}\right\} \\ &- \lambda_1 \gamma_1 q_1^{-\delta_1} T_{L_{11}}^{\gamma_1 - 1}\left\{\alpha_{p_1} - (\beta_c + \eta_c) p_{11} + \eta_c p_{21} + (\beta_t + \eta_t) T_{L_{11}}\right\}. \end{aligned} \quad (26)$$

Solving equations $\frac{\partial TP_{c_1}}{\partial p_{11}} = 0, \frac{\partial TP_{c_1}}{\partial p_{21}} = 0$, and $\frac{\partial TP_{c_1}}{\partial T_{L_{11}}} = 0$, we obtain the optimal values of p_{11}, p_{21}, $T_{L_{11}}$. Analytically it is difficult to solve these equation. We solve the equation numerically by using Matlab2013 software. Let the solution be $p_{11} = p_{11}^{c*}$, $p_{21} = p_{21}^{c*}$, $T_{L_{11}} = T_{L_{11}}^{c*}$

Proposition 4. *The profit function $TP_{c_1}(p_{11}, p_{21}, T_{L_{11}})$ is a concave function if* $4(\beta_c + \eta_c)^2 u_4 + 4\eta_c u_5 u_6 + 2(\beta_c + \eta_c) u_6^2 + 2(\beta_c + \eta_c) u_5^2 - 4u_4 \eta_c^2 < 0$ *where*
$u_4 = -\lambda_1\gamma_1(\gamma_1 - 1) q_1^{-\delta_1}\{\alpha_{p_1} - (\beta_c + \eta_c) p_{11}^{c*} + \eta_c p_{21}^{c*}\}(T_{L_{11}}^{c*})^{\gamma_1 - 2} - \lambda_1\gamma_1(\gamma_1 + 1)(\beta_t + \eta_t) q_1^{-\delta}(T_{L_{11}}^{c*})^{\gamma_1 - 1}$
$u_5 = (\beta_t + \eta_t) + (\beta_c + \eta_c)\lambda_1\gamma_1 q_1^{-\delta_1}(T_{L_{11}}^{c*})^{\gamma_1 - 1}$ *and* $u_6 = -\eta_t - \eta_c\lambda_1\gamma_1 q_1^{-\delta_1}(T_{L_{11}}^{c*})^{\gamma_1 - 1}$.

Proof. The second order partial derivatives of TP_{c_1} at stationary point $S_2 = (p_{11}^{c*}, p_{21}^{c*}, T_{L_{11}}^{c*})$ are

$$\left.\frac{\partial^2 TP_{c_1}}{\partial p_{11}^2}\right|_{atS_2} = -2(\beta_c + \eta_c), \left.\frac{\partial^2 TP_{c_1}}{\partial p_{21}^2}\right|_{atS_2} = -2(\beta_c + \eta_c),$$

$$\left.\frac{\partial^2 TP_{c_1}}{\partial T_{L_{11}}^2}\right|_{atS_2} = -\lambda_1\gamma_1(\gamma_1-1)q_1^{-\delta_1}\{\alpha_{p_1}-(\beta_c+\eta_c)p_{11}^{c*}+\eta_c p_{21}^{c*}\}(T_{L_{11}}^{c*})^{\gamma_1-2}$$
$$-\lambda_1\gamma_1(\gamma_1+1)(\beta_t+\eta_t)q_1^{-\delta_1}(T_{L_{11}}^{c*})^{\gamma_1-1} = u_4(say),$$
$$\left.\frac{\partial^2 TP_{c_1}}{\partial p_{11}\partial p_{21}}\right|_{atS_2} = \left.\frac{\partial^2 TP_{c_1}}{\partial p_{21}\partial p_{11}}\right|_{atS_2} = 2\eta_c,$$
$$\left.\frac{\partial^2 TP_{c_1}}{\partial p_{11}\partial T_{L_{11}}}\right|_{atS_2} = \left.\frac{\partial^2 TP_{c_1}}{\partial T_{L_{11}}\partial p_{11}}\right|_{atS_2} = (\beta_t+\eta_t)+(\beta_c+\eta_c)\lambda_1\gamma_1 q_1^{-\delta_1}(T_{L_{11}}^{c*})^{\gamma_1-1}$$
$$= u_5(say),$$
$$\left.\frac{\partial^2 TP_{c_1}}{\partial p_{21}\partial T_{L_{11}}}\right|_{atS_2} = \left.\frac{\partial^2 TP_{c_1}}{\partial T_{L_{11}}\partial p_{21}}\right|_{atS_2} = -\eta_t-\eta_c\lambda_1\gamma_1 q_1^{-\delta_1}(T_{L_{11}}^{c*})^{\gamma_1-1} = u_6(say).$$

The Hessian matrix H_2 of TP_{c_1} at the stationary point S_2 $(p_{11}^{c*}, p_{21}^{c*}, T_{L_{11}}^{c*})$

$$H_2 = \begin{pmatrix} \frac{\partial^2 TP_{c_1}}{\partial p_{11}^2} & \frac{\partial^2 TP_{c_1}}{\partial p_{11}\partial p_{21}} & \frac{\partial^2 TP_{c_1}}{\partial p_{11}\partial T_{L_{11}}} \\ \frac{\partial^2 TP_{c_1}}{\partial p_{21}\partial p_{11}} & \frac{\partial^2 TP_{c_1}}{\partial p_{21}^2} & \frac{\partial^2 TP_{c_1}}{\partial p_{21}\partial T_{L_{11}}} \\ \frac{\partial^2 TP_{c_1}}{\partial T_{L_{11}}\partial p_{11}} & \frac{\partial^2 TP_{c_1}}{\partial T_{L_{11}}\partial p_{21}} & \frac{\partial^2 TP_{c_1}}{\partial T_{L_{11}}^2} \end{pmatrix} atS_2$$

The profit function TP_{c_1} will be concave function if the principal minors of H_2 are alternatively negative and positive, i.e., if the i^{th} order principal minor D_i of H_2 takes the sign $(-1)^i$. Here,

$$D_{11} = -2(\beta_c+\eta_c) < 0$$
$$D_{21} = \begin{vmatrix} -2(\beta_c+\eta_c) & 2\eta_c \\ 2\eta_c & -2(\beta_c+\eta_c) \end{vmatrix}$$
$$= 4(\beta_c+\eta_c)^2 - 4\eta_c^2 > 0$$

$$D_3 = |H_2| = 4(\beta_c+\eta_c)^2 u_4 + 4\eta_c u_5 u_6 + 2(\beta_c+\eta_c)u_6^2 + 2(\beta_c+\eta_c)u_5^2 - 4u_4\eta_c^2 < 0$$

if $4(\beta_c+\eta_c)^2 u_4 + 4\eta_c u_5 u_6 + 2(\beta_c+\eta_c)u_6^2 + 2(\beta_c+\eta_c)u_5^2 - 4u_4\eta_c^2 < 0$ holds. This completes the proof. □

4.2. Both manufacturers offer warranty (Case 2)

In this case, we assume that demand function of each manufacturer i is symmetric between two complementary products and is expressed as

$$D_{i2}(p_{i2}, p_{k2}, T_{L_{i2}}, T_{L_{k2}}) = \alpha_{p_i} - (\beta_c+\eta_c)p_{i2} + \eta_c p_{k2} + (\beta_t+\eta_t)T_{L_{i2}} - \eta_t T_{L_{k2}}, \quad (27)$$

where $i \in \{1,2\}$ and $k = 3-i$. The profit functions of two manufacturers and the retailer can be written respectively as follows

$$TP_{m_{i2}} = \left(w_{i2} - c_i - c_{m_i}\frac{q_i}{1-q_i} - \lambda_i q_i^{-\delta_i} T_{L_{i2}}^{\gamma_i}\right) D_{i2} \quad (28)$$

and

$$TP_{r_2} = \sum_{i=1}^{2} (p_{i2} - w_{i2}) D_{i2}. \tag{29}$$

4.2.1. Decentralized decisions

In this decentralized decision making, the manufacturers and the retailer operate independently and the manufacturers make decisions first as Stackleberg leader and then the retailer reacts as their follower. So, we first determine the optimal values of p_{12} and p_{22} for given $w_{12}, w_{22}, T_{L_{12}}$ and $T_{L_{22}}$ to maximize the retailer's profit function, that is

$$\max_{p_{12},p_{22}} \; TP_{r_2}(p_{12}, p_{22}|w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}}). \tag{30}$$

The optimal values of p_{12} and p_{22} are obtained by solving $\frac{\partial TP_{r_2}}{\partial p_{12}} = 0$ and $\frac{\partial TP_{r_2}}{\partial p_{22}} = 0$ as follows

$$\begin{aligned} p_{12}^* &= \frac{w_{12}}{2} + \frac{(\beta_c+\eta_c)(\beta_t+\eta_t) - \eta_c\eta_t}{2\beta_c(\beta_c+2\eta_c)} T_{L_{12}} + \frac{(\beta_t+\eta_t)\eta_c - (\beta_c+\eta_c)\eta_t}{2\beta_c(\beta_c+2\eta_c)} T_{L_{22}} \\ &+ \frac{(\beta_c+\eta_c)\alpha_{p_1} + \eta_c\alpha_{p_2}}{2\beta_c(\beta_c+2\eta_c)} \end{aligned} \tag{31}$$

and

$$\begin{aligned} p_{22}^* &= \frac{w_{22}}{2} + \frac{(\beta_t+\eta_t)\eta_c - (\beta_c+\eta_c)\eta_t}{2\beta_c(\beta_c+2\eta_c)} T_{L_{12}} + \frac{(\beta_c+\eta_c)(\beta_t+\eta_t) - \eta_c\eta_t}{2\beta_c(\beta_c+2\eta_c)} T_{L_{22}} \\ &+ \frac{\eta_c\alpha_{p_1} + (\beta_c+\eta_c)\alpha_{p_2}}{2\beta_c(\beta_c+2\eta_c)}. \end{aligned} \tag{32}$$

Note that $\frac{\partial^2 TP_{r_2}}{\partial p_{12}^2} = -2(\beta_c+\eta_c) < 0$, $\frac{\partial^2 TP_{r_2}}{\partial p_{22}^2} = -2(\beta_c+\eta_c) < 0$ and $\frac{\partial^2 TP_{r_2}}{\partial p_{12}^2}\frac{\partial^2 TP_{r_2}}{\partial p_{22}^2} - \frac{\partial^2 TP_{r_2}}{\partial p_{12}\partial p_{22}}\frac{\partial^2 TP_{r_1}}{\partial p_{22}\partial p_{12}} = 4(\beta_c+\eta_c)^2 - 4\eta_c^2 > 0$. That is TP_{r_2} is a concave function of p_{12} and p_{22}. Now, observing the retailer's best response on retail prices, the manufacturers decide to offer wholesale prices and warranty periods with the purpose of maximizing their own profit. We establish two decision models by considering the manufacturers' cooperative and noncooperative decision strategies.

Manufacturers' noncooperative decision (MNC) strategy

In this situation, two manufacturers maximize their profits independently and make their decisions on wholesale prices and warranty periods individually, based on the reaction of the retailer. Hence, the manufacturers' decision problem is formulated, as follows.

$$\begin{cases} \begin{cases} \max\limits_{(w_{12},T_{L_{12}})} & TP_{m_{12}}(w_{12}, T_{L_{12}}, w_{22}, T_{L_{22}}, p_{12}^*(w_{12},w_{22},T_{L_{12}},T_{L_{22}}), p_{22}^*(w_{12},w_{22},T_{L_{12}},T_{L_{22}})) \\ \max\limits_{(w_{22},T_{L_{22}})} & TP_{m_{22}}(w_{12}, T_{L_{12}}, w_{22}, T_{L_{22}}, p_{12}^*(w_{12},w_{22},T_{L_{12}},T_{L_{22}}), p_{22}^*(w_{12},w_{22},T_{L_{12}},T_{L_{22}})) \end{cases} \\ subject\ to\ (31)\ and\ (32). \end{cases}$$

(33)

The partial derivatives of $TP_{m_{i2}}$ with respect to w_{i2} and $T_{L_{i2}}$ are respectively as follows

$$\begin{aligned}\frac{\partial TP_{m_{i2}}}{\partial w_{i2}} &= -(\beta_c+\eta_c)w_{i2}+\frac{1}{2}\eta_c w_{k2}+\frac{1}{2}\{(\beta_t+\eta_t)T_{L_{i2}}+(\beta_c+\eta_c)\lambda_i q_i^{-\delta_i}T_{L_{i2}}^{\gamma_i}\}\\ &- \frac{1}{2}\eta_t T_{L_{k2}}+\frac{1}{2}\left\{\alpha_{p_i}+(\beta_c+\eta_c)\left(c_i+c_{m_i}\frac{q_i}{1-q_i}\right)\right\} \end{aligned} \tag{34}$$

and

$$\begin{aligned}\frac{\partial TP_{m_{i2}}}{\partial T_{L_{i2}}} &= \frac{1}{2}(\beta_t+\eta_t)\left(w_{i2}-c_i-c_{m_i}\frac{q_i}{1-q_i}-\lambda_i q_i^{-\delta_i}T_{L_{i2}}^{\gamma_i}\right)\\ &- \frac{1}{2}\lambda_i\gamma_i q_i^{-\delta_i}T_{L_{i2}}^{\gamma_i-1}\{\alpha_{p_1}-(\beta_c+\eta_c)w_{i2}+\eta_c w_{k2}+(\beta_t+\eta_t)T_{L_{i2}}-\eta_t T_{L_{k2}}\}. \end{aligned} \tag{35}$$

where $i\in\{1,2\}$, and $k=3-i$. Equating the above partial derivatives to zero, we have

$$\begin{aligned} w_{i2}^{mnc*} &= \frac{2(\beta_c+\eta_c)\{\alpha_{p_i}+(\beta_c+\eta_c)c_i\}+\eta_c\{\alpha_k+(\beta_c+\eta_c)c_k\}}{4(\beta_c+\eta_c)^2-\eta_c^2}\\ &+ \frac{2(\beta_c+\eta_c)^2 c_{m_i}q_i}{(1-q_i)\{4(\beta_c+\eta_c)^2-\eta_c^2\}}+\frac{(\beta_c+\eta_c)\eta_c c_{mk}q_k}{(1-q_k)\{4(\beta_c+\eta_c)^2-\eta_c^2\}}\\ &+ \frac{\left\{2(\beta_c+\eta_c)(\beta_t+\eta_t)\left(1+\frac{1}{\gamma_i}\right)-\eta_c\eta_t\right\}}{4(\beta_c+\eta_c)^2-\eta_c^2}\left\{\frac{q_i^{\delta_i}(\beta_t+\eta_t)}{\lambda_i\gamma_i(\beta_c+\eta_c)}\right\}^{\frac{1}{\gamma_i-1}}\\ &+ \frac{(\beta_t+\eta_t)\eta_c\left(1+\frac{1}{\gamma_k}\right)-2\eta_t(\beta_c+\eta_c)}{4(\beta_c+\eta_c)^2-\eta_c^2}\left\{\frac{q_k^{\delta_k}(\beta_t+\eta_t)}{\lambda_k\gamma_k(\beta_c+\eta_c)}\right\}^{\frac{1}{\gamma_k-1}} \end{aligned} \tag{36}$$

and

$$T_{L_{i2}}^{mnc*} = \left\{\frac{q_i^{\delta_i}(\beta_t+\eta_t)}{\lambda_i\gamma_i(\beta_c+\eta_c)}\right\}^{\frac{1}{\gamma_i-1}}. \tag{37}$$

The corresponding retail prices under MNC strategy respectively are as follows:

$$\begin{aligned} p_{i2}^{mnc*} &= \frac{w_{i2}^{mnc*}}{2}+\frac{(\beta_c+\eta_c)(\beta_t+\eta_t)-\eta_c\eta_t}{2\beta_c(\beta_c+2\eta_c)}T_{L_{i2}}^{mnc*}+\frac{(\beta_t+\eta_t)\eta_c-(\beta_c+\eta_c)\eta_t}{2\beta_c(\beta_c+2\eta_c)}T_{L_{k2}}^{mnc*}\\ &+ \frac{(\beta_c+\eta_c)\alpha_{p_1}+\eta_c\alpha_{p_2}}{2\beta_c(\beta_c+2\eta_c)}. \end{aligned} \tag{38}$$

where w_{i2}^{mnc*} and T_{Li1}^{mnc*} are given in Equations (36) and(37).

Proposition 5. *The profit function $TP_{m_{i2}}$ under decentralized MNC strategy is a concave function in w_{i2} and $T_{L_{i2}}$ if $(\gamma_i - 1)\{\alpha_{p_i} - (\beta_c + \eta_c)w_{i2}^{mnc*} + \eta_c w_{k2}^{mnc*} - \eta_t T_{L_{k2}}^{mnc*}\} + (\gamma_i+1)(\beta_t+\eta_t)T_{L_{i2}}^{mnc*} > 0$ and $(\gamma_i - 1)\{\alpha_{p_i} - (\beta_c+\eta_c)w_{i2}^{mnc*} + \eta_c w_{k2}^{mnc*} + (\beta_t + \eta_t)T_{L_{i2}}^{mnc*} - \eta_t T_{L_{k2}}^{mnc*}\} > 0$, where $i \in \{1, 2\}$ and $k = 3 - i$.*

Proof. The profit function TP_{mi1} under decentralized MNC strategy would be concave in w_{i2} and $T_{L_{i2}}$ if at the stationary point $(w_{i2}^{mnc*}, T_{L_{i2}}^{mnc*})$, the Hessian matrix of $TP_{m_{i2}}$ is negative definite. Here, at $(w_{i2}^{mnc*}, T_{L_{i2}}^{mnc*})$

$$
\begin{aligned}
\frac{\partial^2 TP_{m_{i2}}}{\partial w_{i2}^2} &= -(\beta_c + \eta_c) < 0, \\
\frac{\partial^2 TP_{m_{i2}}}{\partial T_{L_{i2}}^2} &= -\frac{(\gamma_i - 1)(\beta_t + \eta_t)\{\alpha_{p_i} - (\beta_c + \eta_c)w_{i2}^{mnc*} + \eta_c w_{k2}^{mnc*} - \eta_t T_{L_{k2}}^{mnc*}\}}{2(\beta_c + \eta_c)T_{L_{i2}}^{mnc*}} \\
&\quad - \frac{(\gamma_i + 1)(\beta_t + \eta_t)^2}{2(\beta_c + \eta_c)} < 0
\end{aligned}
$$

if $(\gamma_i - 1)\{\alpha_{p_i} - (\beta_c + \eta_c)w_{i2}^{mnc*} + \eta_c w_{k2}^{mnc*} - \eta_t T_{L_{k2}}^{mnc*}\} + (\gamma_i + 1)(\beta_t + \eta_t)T_{L_{i2}}^{mnc*} > 0$ holds.

$$
\begin{aligned}
&\frac{\partial^2 TP_{m_{11}}}{\partial w_{11}^2}\frac{\partial^2 TP_{m_{11}}}{\partial T_{L_{11}}^2} - \frac{\partial^2 TP_{m_{11}}}{\partial w_{11} T_{L_{11}}}\frac{\partial^2 TP_{m_{11}}}{\partial T_{L_{11}} w_{11}} = \\
&(\beta_c + \eta_c)\Bigg[\frac{(\gamma_i - 1)(\beta_t + \eta_t)\{\alpha_{p_i} - (\beta_c + \eta_c)w_{i2}^{mnc*} + \eta_c w_{k2}^{mnc*} - \eta_t T_{L_{k2}}^{mnc*}\}}{2(\beta_c + \eta_c)T_{L_{i2}}^{mnc*}} \\
&+ \frac{(\gamma_i + 1)(\beta_t + \eta_t)^2}{2(\beta_c + \eta_c)}\Bigg] - (\beta_t + \eta_t)^2 > 0
\end{aligned}
$$

if $(\gamma_i - 1)\{\alpha_{p_i} - (\beta_c+\eta_c)w_{i2}^{mnc*} + \eta_c w_{k2}^{mnc*} + (\beta_t+\eta_t)T_{L_{i2}}^{mnc*} - \eta_t T_{L_{k2}}^{mnc*}\} > 0$ holds, where $i \in \{1, 2\}$ and $k = 3 - i$ This completes the proof. □

Manufacturers' cooperative (MC) decision strategy

In this strategy, two manufacturers operate jointly and agree to make decisions jointly in order to maximize their total profit, subject to the constraints imposed by equations in (31), and (32). Hence, the manufacturers' decision problem is formulated as follows

$$
\begin{aligned}
\max_{(w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}})} \quad & [TP_{m_{12}} + TP_{m_{22}}](w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}}, p_{12}^*, p_{22}^*) \\
& subject\ to\ (31)\ and\ (32). \qquad (39)
\end{aligned}
$$

The partial derivatives of $TP_{m_{12}} + TP_{m_{22}}$ with respect to $w_{12}, w_{22}, T_{L_{12}}$ and $T_{L_{22}}$ are respectively as follows:

$$\begin{aligned}
\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}} &= -(\beta_c+\eta_c)w_{12}+\eta_c w_{22}+\frac{1}{2}\{(\beta_t+\eta_t)T_{L_{12}} \\
&+ (\beta_c+\eta_c)\lambda_1 q_1^{-\delta_1}T_{L_{12}}^{\gamma_1}\} \\
&+ \frac{1}{2}\left\{\alpha_{p_1}+(\beta_c+\eta_c)c_1+c_{m_1}(\beta_c+\eta_c)\frac{q_1}{1-q_1}\right\} \\
&- \frac{1}{2}(\eta_t T_{L_{22}}+\eta_c\lambda_2 q_2^{-\delta_2}T_{L_{22}}^{\gamma_2})-\frac{1}{2}\eta_c\left\{c_2+c_{m_2}\frac{q_2}{1-q_2}\right\}, \qquad (40)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}} &= \eta_c w_{12}-(\beta_c+\eta_c)w_{22}-\frac{1}{2}\{\eta_t T_{L_{12}}+\eta_c\lambda_1 q_1^{-\delta_1}T_{L_{12}}^{\gamma_1}\} \\
&+ \frac{1}{2}\{(\beta_t+\eta_t)T_{L_{22}}+(\beta_c+\eta_c)\lambda_2 q_2^{-\delta_2}T_{L_{22}}^{\gamma_2}\} \\
&- \frac{1}{2}\eta_c\left\{c_1+c_{m_1}\frac{q_1}{1-q_1}\right\} \\
&+ \frac{1}{2}\left\{\alpha_{p_2}+(\beta_c+\eta_c)c_2+c_{m_2}(\beta_c+\eta_c)\frac{q_2}{1-q_2}\right\}, \qquad (41)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}} &= \frac{1}{2}(\beta_t+\eta_t)\left\{w_{12}-c_1-c_{m_1}\frac{q_1}{1-q_1}-\lambda_1 q_1^{-\delta_1}T_{L_{12}}^{\gamma_1}\right\} \\
&- \frac{1}{2}\lambda_1\gamma_1 q_1^{-\delta_1}T_{L_{12}}^{\gamma_1-1} \\
&\times \left\{\alpha_{p_1}-(\beta_c+\eta_c)w_{12}+\eta_c w_{22}+(\beta_t+\eta_t)T_{L_{12}}-\eta_t T_{L_{22}}\right\} \\
&- \frac{1}{2}\eta_t\left\{w_{22}-c_2-c_{m_2}\frac{q_2}{1-q_2}-\lambda_2 q_2^{-\delta_2}T_{L_{22}}^{\gamma_2}\right\} \qquad (42)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}} &= \frac{1}{2}(\beta_t+\eta_t)\left\{w_{22}-c_2-c_{m_2}\frac{q_2}{1-q_2}-\lambda_2 q_2^{-\delta_2}T_{L_{22}}^{\gamma_2}\right\} \\
&- \frac{1}{2}\lambda_2\gamma_2 q_2^{-\delta_2}T_{L_{22}}^{\gamma_2-1} \\
&\times \left\{\alpha_{p_2}-(\beta_c+\eta_c)w_{22}+\eta_c w_{12}+(\beta_t+\eta_t)T_{L_{22}}-\eta_t T_{L_{12}}\right\} \\
&- \frac{1}{2}\eta_t\left\{w_{12}-c_1-c_{m_1}\frac{q_1}{1-q_1}-\lambda_1 q_1^{-\delta_1}T_{L_{12}}^{\gamma_1}\right\}. \qquad (43)
\end{aligned}$$

Solving equations $\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}}=0$, $\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}}=0$, $\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}}=0$ and $\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}}=0$, we obtain the optimal values of $w_{12}, w_{22}, T_{L_{12}}$ and $T_{L_{22}}$. Analytically it is difficult to solve these equation. We solve the equation numerically by using Matlab2013 software. Let the solution be $w_{12} = w_{12}^{mc*}$, $w_{22} = w_{22}^{mc*}$, $T_{L_{12}} = T_{L_{12}}^{mc*}$ and $T_{L_{22}} = T_{L_{22}}^{mc*}$.

Proposition 6. *The profit function* $(TP_{m_{12}} + TP_{m_{22}})(w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}})$ *is a concave function if* $(\beta_c+\eta_c)^2u_{11}+2\eta_c u_7u_9+(\beta_c+\eta_c)u_9^2-\eta_c^2u_{11}+(\beta_c+\eta_c)u_7^2<0$ *and* $\{(\beta_c+\eta_c)^2-\eta_c^2\}(u_{11}u_{13}-u_{12}^2)+(\beta_c+\eta_c)u_{13}(u_9^2+u_7^2)+(\beta_c+\eta_c)u_{11}(u_{10}^2+u_8^2)+(u_8^2u_9^2+u_7^2u_{10}^2)-2(\beta_c+\eta_c)u_9u_{10}u_{12}-2\eta_c u_8u_9u_{12}-2\eta_c u_7u_{10}u_{12}+2\eta_c u_8u_{10}u_{11}+2\eta_c u_7u_9u_{13}-2(\beta_c+\eta_c)u_7u_8u_{12}-2u_7u_8u_9u_{10}>0$*, where* $u_7=\frac{1}{2}(\beta_t+\eta_t)+\frac{1}{2}(\beta_c+\eta_c)\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-1}$, $u_8=-\frac{1}{2}\eta_t-\frac{1}{2}\eta_c\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-1}$, $u_9=-\frac{1}{2}\eta_t-\frac{1}{2}\eta_c\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-1}$, $u_{10}=\frac{1}{2}(\beta_t+\eta_t)+\frac{1}{2}(\beta_c+\eta_c)\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-1}$, $u_{11}=-\frac{\lambda_1\gamma_1(\gamma_1-1)q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-2}}{2}\{\alpha_{p_1}-(\beta_c+\eta_c)w_{12}^{mc*}+\eta_c w_{22}^{mc*}-\eta_t T_{22}^{mc*}\}-\frac{1}{2}\lambda_1\gamma_1(\gamma_1+1)(\beta_t+\eta_t)q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-1}$, $u_{12}=\frac{1}{2}\eta_t\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-1}+\frac{1}{2}\eta_t\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-1}$ *and* $u_{13}=-\frac{\lambda_2\gamma_2(\gamma_2-1)q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-2}}{2}\{\alpha_{p_2}-(\beta_c+\eta_c)w_{22}^{mc*}+\eta_c w_{12}^{mc*}-\eta_t T_{12}^{mc*}\}-\frac{1}{2}\lambda_2\gamma_2(\gamma_2+1)(\beta_t+\eta_t)q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-1}$.

Proof. The second order partial derivatives of $(TP_{m_{12}}+TP_{m_{22}})$ at stationary point $S_3=(w_{12}^{mc*}, w_{22}^{mc*}, T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*})$ are

$$\begin{aligned}
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}^2}\right|_{atS_3} &= -(\beta_c+\eta_c), \left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}^2}\right|_{atS_3} = -(\beta_c+\eta_c),\\
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}\partial w_{22}}\right|_{atS_3} &= \left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}\partial w_{12}}\right|_{atS_3} = \eta_c,\\
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}\partial T_{L_{12}}}\right|_{atS_3} &= \left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}\partial w_{12}}\right|_{atS_3}\\
&= \frac{1}{2}(\beta_t+\eta_t)+\frac{1}{2}(\beta_c+\eta_c)\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-1} = u_7(say),
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}\partial T_{L_{22}}}\right|_{atS_3} &= \left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}\partial w_{12}}\right|_{atS_3}\\
&= -\frac{1}{2}\eta_t-\frac{1}{2}\eta_c\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-1} = u_8(say),\\
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}\partial T_{L_{12}}}\right|_{atS_3} &= \left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}\partial w_{22}}\right|_{atS_3}\\
&= -\frac{1}{2}\eta_t-\frac{1}{2}\eta_c\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-1} = u_9(say),\\
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}\partial T_{L_{22}}}\right|_{atS_3} &= \left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}\partial w_{22}}\right|_{atS_3}\\
&= \frac{1}{2}(\beta_t+\eta_t)+\frac{1}{2}(\beta_c+\eta_c)\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-1}\\
&= u_{10}, (say)
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}^2}\right|_{atS_3} &= -\frac{\lambda_1\gamma_1(\gamma_1-1)q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-2}}{2} \\
&\quad \times\{\alpha_{p_1}-(\beta_c+\eta_c)w_{12}^{mc*}+\eta_c w_{22}^{mc*}-\eta_t T_{22}^{mc*}\} \\
&\quad -\frac{1}{2}\lambda_1\gamma_1(\gamma_1+1)(\beta_t+\eta_t)q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-1} = u_{11}(say),
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}\partial T_{L_{22}}}\right|_{atS_3} &= \left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}\partial T_{L_{12}}}\right|_{atS_3} \\
&= \frac{1}{2}\eta_t\lambda_1\gamma_1 q_1^{-\delta_1}(T_{L_{12}}^{mc*})^{\gamma_1-1}+\frac{1}{2}\eta_t\lambda_2\gamma_2 q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-1} \\
&= u_{12}(say),
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}^2}\right|_{atS_3} &= -\frac{\lambda_2\gamma_2(\gamma_2-1)q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-2}}{2} \\
&\quad \times\{\alpha_{p_2}-(\beta_c+\eta_c)w_{22}^{mc*}+\eta_c w_{12}^{mc*}-\eta_t T_{12}^{mc*}\} \\
&\quad -\frac{1}{2}\lambda_2\gamma_2(\gamma_2+1)(\beta_t+\eta_t)q_2^{-\delta_2}(T_{L_{22}}^{mc*})^{\gamma_2-1} = u_{13}(say).
\end{aligned}$$

The Hessian matrix H_3 of $(TP_{m_{12}}+TP_{m_{22}})$ at the stationary point S_3 $(w_{12}^{mc*}, w_{22}^{mc*}, T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*})$

$$H_3 = \begin{pmatrix}
\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}^2} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}\partial w_{22}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}\partial T_{L_{12}}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}\partial T_{L_{22}}} \\
\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}\partial w_{12}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}^2} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}\partial T_{L_{12}}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}\partial T_{L_{22}}} \\
\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}\partial w_{12}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}\partial w_{22}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}^2} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}\partial T_{L_{22}}} \\
\frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}\partial w_{12}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}\partial w_{22}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}\partial T_{L_{12}}} & \frac{\partial^2(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}^2}
\end{pmatrix} atS_3$$

The profit function $TP_{m_{12}}+TP_{m_{22}}$ will be concave function if the principal minors of H_3 are alternatively negative and positive, i.e., if the i^{th} order principal minor D_i of H_3 takes the sign $(-1)^i$. Here,

$$\begin{aligned}
D_1 &= -(\beta_c+\eta_c)<0, \\
D_2 &= \begin{vmatrix} -(\beta_c+\eta_c) & \eta_c \\ \eta_c & -(\beta_c+\eta_c) \end{vmatrix} \\
&= (\beta_c+\eta_c)^2-\eta_c^2>0
\end{aligned}$$

and

$$\begin{aligned}
D_3 &= \begin{vmatrix} -(\beta_c+\eta_c) & \eta_c & u_7 \\ \eta_c & -(\beta_c+\eta_c) & u_9 \\ u_7 & u_9 & u_{11} \end{vmatrix} \\
&= (\beta_c+\eta_c)^2u_{11}+2\eta_c u_7u_9+(\beta_c+\eta_c)u_9^2-\eta_c^2u_{11}+(\beta_c+\eta_c)u_7^2<0
\end{aligned}$$

if $(\beta_c+\eta_c)^2u_{11}+2\eta_c u_7u_9+(\beta_c+\eta_c)u_9^2-\eta_c^2u_{11}+(\beta_c+\eta_c)u_7^2<0$ holds.

$$\begin{aligned}
|H_3| &= \{(\beta_c+\eta_c)^2-\eta_c^2\}(u_{11}u_{13}-u_{12}^2)+(\beta_c+\eta_c)u_{13}(u_9^2+u_7^2)\\
&+ (\beta_c+\eta_c)u_{11}(u_{10}^2+u_8^2)+(u_8^2u_9^2+u_7^2u_{10}^2)-2(\beta_c+\eta_c)u_9u_{10}u_{12}-2\eta_c u_8u_9u_{12}\\
&- 2\eta_c u_7u_{10}u_{12}+2\eta_c u_8u_{10}u_{11}+2\eta_c u_7u_9u_{13}-2(\beta_c+\eta_c)u_7u_8u_{12}-2u_7u_8u_9u_{10}\\
&> 0,
\end{aligned}$$

if $\{(\beta_c+\eta_c)^2-\eta_c^2\}(u_{11}u_{13}-u_{12}^2)+(\beta_c+\eta_c)u_{13}(u_9^2+u_7^2)+(\beta_c+\eta_c)u_{11}(u_{10}^2+u_8^2)+(u_8^2u_9^2+u_7^2u_{10}^2)-2(\beta_c+\eta_c)u_9u_{10}u_{12}-2\eta_c u_8u_9u_{12}-2\eta_c u_7u_{10}u_{12}+2\eta_c u_8u_{10}u_{11}+2\eta_c u_7u_9u_{13}-2(\beta_c+\eta_c)u_7u_8u_{12}-2u_7u_8u_9u_{10}>0$ holds. This completes the proof. □

Proposition 7. *Under decentralized MC strategy, the profit of each channel member is equal, that is* $TP_{m_{12}}^{mc*}=TP_{m_{22}}^{mc*}=TP_{r_2}^{mc*}$ *and independent of* η_c *and* η_t *if two manufacturers are identical (that is,* $\alpha_{p_1}=\alpha_{p_2}, c_1=c_2, c_{m_1}=c_{m_2}, q_1=q_2, \lambda_1=\lambda_2$, $\gamma_1=\gamma_2$, *and* $\delta_1=\delta_2$*).*

Proof. Under symmetrical condition of two complementary products, at stationary point S_3 we have

$$\begin{aligned}
&\left.\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{22}}\right|_{atS_3}-\left.\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial w_{12}}\right|_{atS_3}=2(\beta_c+2\eta_c)(w_{12}^{mc*}-w_{22}^{mc*})\\
&-\lambda(\beta_c+2\eta_c)q^{-\delta}(T_{L_{12}}^{mc*^{\gamma}}-T_{L_{22}}^{mc*^{\gamma}})-(\beta_t+2\eta_t)(T_{L_{12}}^{mc*}-T_{L_{22}}^{mc*})=0
\end{aligned}\tag{44}$$

and

$$\begin{aligned}
&\left.\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{12}}}\right|_{atS_3}-\left.\frac{\partial(TP_{m_{12}}+TP_{m_{22}})}{\partial T_{L_{22}}}\right|_{atS_3}=(\beta_t+2\eta_t)(w_{12}^{mc*}-w_{22}^{mc*})\\
&-\lambda(\beta_t+2\eta_t)q^{-\delta}(T_{L_{12}}^{mc*^{\gamma}}-T_{L_{22}}^{mc*^{\gamma}})-\lambda\gamma q^{-\delta}[\{\eta_c w_{22}^{mc*}-(\beta_c+\eta_c)w_{12}^{mc*}\}T_{L_{12}}^{mc*^{\gamma-1}}\\
&+\{(\beta_c+\eta_c)w_{22}^{mc*}-\eta_c w_{12}^{mc*}\}T_{L_{22}}^{mc*^{\gamma-1}}+(\beta_t+\eta_t)(T_{L_{12}}^{mc*^{\gamma}}-T_{L_{22}}^{mc*^{\gamma}})\\
&-\eta_t T_{L_{12}}^{mc*}T_{L_{22}}^{mc*}(T_{L_{12}}^{mc*\gamma-2}-T_{L_{22}}^{mc*^{\gamma-2}})+\alpha(T_{L_{12}}^{mc*^{\gamma-1}}-T_{L_{22}}^{mc*^{\gamma-1}})]=0
\end{aligned}\tag{45}$$

where $\alpha_{p_1}=\alpha_{p_2}=\alpha_p\,(say), c_1=c_2=c\,(say), c_{m_1}=c_{m_2}=c_m\,(say), q_1=q_2=q\,(say)$, $\lambda_1=\lambda_2=\lambda\,(say)$, $\gamma_1=\gamma_2=\gamma\,(say)$ and $\delta_1=\delta_2=\delta\,(say)$. Now, from (44) we can write

$$\begin{aligned}
(\beta_c+2\eta_c)(w_{12}^{mc*}-w_{22}^{mc*}) &= (T_{L_{12}}^{mc*}-T_{L_{22}}^{mc*})g(T_{L_{12}}^{mc*},T_{L_{22}}^{mc*})\,(say) && (46)\\
\Rightarrow \{(\beta_c+\eta_c)w_{22}^{mc*}-\eta_c w_{12}^{mc*}\} &= -\{\eta_c w_{22}^{mc*}-(\beta_c+\eta_c)w_{12}^{mc*}\}\\
&\quad -(T_{L_{12}}^{mc*}-T_{L_{22}}^{mc*})g(T_{L_{12}}^{mc*},T_{L_{22}}^{mc*}) && (47)
\end{aligned}$$

where $(T_{L_{12}}^{mc*})^{\gamma-i+1} - (T_{L_{22}}^{mc*})^{\gamma-i+1} = (T_{L_{12}}^{mc*} - T_{L_{22}}^{mc*})g_i(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*})$ for $i = 1, 2, 3$ and $2g(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*}) = (\beta_t + 2\eta_t) + \lambda q^{-\delta}(\beta_c + 2\eta_c)g_1(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*})$. Hence from (45), we get

$$\begin{aligned}
&(T_{L_{12}}^{mc*} - T_{L_{22}}^{mc*})\Bigg[\frac{\beta_1 + 2\eta_t}{\beta_c + 2\eta_c} g(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*}) - \lambda q^{-\delta}(\beta_t + 2\eta_t)g_1(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*}) \\
&-\lambda\gamma q^{-\delta}\{\eta_c w_{22}^{mc*} - (\beta_c + \eta_c)w_{12}^{mc*}\}g_2(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*}) + \lambda\gamma q^{-\delta} {T_{L_{22}}^{mc*}}^{\gamma-1} g(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*}) \\
&-\lambda\gamma q^{-\delta}(\beta_t + \eta_t)g_1(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*}) + \lambda\gamma q^{-\delta}\eta_t T_{L_{12}}^{mc*} T_{L_{22}}^{mc*} g_3(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*}) \\
&-\lambda\gamma\alpha q^{-\delta} g_2(T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*})\Bigg] = 0. \qquad (48)
\end{aligned}$$

Thus, from Equations (48), and (46) we can conclude that $T_{L_{12}}^{mc*} = T_{L_{22}}^{mc*}$ and $w_{12}^{mc*} = w_{22}^{mc*}$ is a solution of the Equations (40)-(43). Now if $T_{L_{12}}^{mc*} = T_{L_{22}}^{mc*} = T_{L2}^{mc*}(say)$ and $w_{12}^{mc*} = w_{22}^{mc*} = w_2^{mc*}(say)$ is the optimal solution of the manufacturers, then from Equations (31) and (32), we get

$$p_{12}^{mc*} = p_{22}^{mc*} = \frac{1}{2}\left(\frac{\alpha_p}{\beta_c} + \frac{\beta_t}{\beta_c}T_{L2}^{mc*} + w_2^{mc*}\right) = p_2^{mc*}(\text{say})$$

and equating the partial derivative $\frac{\partial(TP_{m_{12}} + TP_{m_{22}})}{\partial w_{12}}$ (given in expression (40)) to zero we get

$$-\frac{\beta_c}{2}\left\{w_2^{mc*} - c - c_m\frac{q}{1-q} - \lambda q^{-\delta}{T_{L2}^{mc*}}^{\gamma}\right\} + \frac{1}{2}(\alpha_p - \beta_c w_2^{mc*} + \beta_t T_{L2}^{mc*}) = 0. \quad (49)$$

Hence the manufacturer's optimal profit becomes

$$TP_{m_{12}}^{mc*} = TP_{m_{22}}^{mc*} = \frac{1}{2}\left\{w_2^{mc*} - c - c_m\frac{q}{1-q} - \lambda q^{-\delta}{T_{L2}^{mc*}}^{\gamma}\right\}(\alpha_p - \beta_c w_2^{mc*} + \beta_t T_{L2}^{mc*})$$

and the retailer optimal profit becomes

$$\begin{aligned}
TP_{r_2}^{mc*} &= (p_2^{mc*} - w_2^{mc*})(\alpha_p - \beta_c w_2^{mc*} + \beta_t T_{L2}^{mc*}) \\
&= \frac{1}{2}\left(\frac{\alpha_p}{\beta_c} + \frac{\beta_t}{\beta_c}T_{L2}^{mc*} - w_2^{mc*}\right)(\alpha_p - \beta_c w_2^{mc*} + \beta_t T_{L2}^{mc*}) \\
&= \frac{1}{2}\left\{w_2^{mc*} - c - c_m\frac{q}{1-q} - \lambda q^{-\delta}{T_{L2}^{mc*}}^{\gamma}\right\}(\alpha_p - \beta_c w_2^{mc*} + \beta_t T_{L2}^{mc*}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(From Equation (49))} \\
&= TP_{m_{12}}^{mc*} = TP_{m_{22}}^{mc*}(\text{ independent of } \eta_c \text{ and } \eta_t).
\end{aligned}$$

This completes the proof. □

4.2.2. Centralized decisions

In this case, both the manufacturers and their common retailer cooperate and together they make the decision that maximizes the overall supply chain profit. The total profit function under this scenario is

$$\begin{aligned} TP_{c2} &= TP_{m_{12}} + TP_{m_{22}} + TP_{r_2} \\ &= \sum_{i=1}^{2} \left(p_{i2} - c_i - c_{m_i} \frac{q_i}{1-q_i} - \lambda_i q_i^{-\delta_i} T_{L_{i2}} \right) D_{i2}. \end{aligned} \tag{50}$$

Hence, the channel members'decision problem is formulated as follows

$$\underset{(p_{12}, p_{22}, T_{L_{12}}, T_{L_{22}})}{max} \quad TP_{c2}(p_{12}, p_{22}, T_{L_{12}}, T_{L_{22}}). \tag{51}$$

The partial derivatives of $TP_{m_{12}} + TP_{m_{22}}$ with respect to $p_{12}, p_{22}, T_{L_{12}}$ and $T_{L_{22}}$ are respectively as follows:

$$\begin{aligned} \frac{\partial TP_{c2}}{\partial p_{12}} &= -2(\beta_c+\eta_c)p_{12} + 2\eta_c p_{22} + \{(\beta_t+\eta_t)T_{L_{12}} + (\beta_c+\eta_c)\lambda_1 q_1^{-\delta_1} T_{L_{12}}^{\gamma_1}\} \\ &- (\eta_t T_{L_{22}} + \eta_c \lambda_2 q_2^{-\delta_2} T_{L_{22}}^{\gamma_2}) + \left\{ \alpha_{p_1} + (\beta_c+\eta_c)c_1 + c_{m_1}(\beta_c+\eta_c)\frac{q_1}{1-q_1} \right\} \\ &- \eta_c \left\{ c_2 + c_{m_2}\frac{q_2}{1-q_2} \right\}, \end{aligned} \tag{52}$$

$$\begin{aligned} \frac{\partial TP_{c2}}{\partial p_{22}} &= 2\eta_c p_{12} - 2(\beta_c+\eta_c)p_{22} - \{\eta_t T_{L_{12}} + \eta_c \lambda_1 q_1^{-\delta_1} T_{L_{12}}^{\gamma_1}\} \\ &+ \{(\beta_t+\eta_t)T_{L_{22}} + (\beta_c+\eta_c)\lambda_2 q_2^{-\delta_2} T_{L_{22}}^{\gamma_2}\} - \eta_c \left\{ c_1 + c_{m_1}\frac{q_1}{1-q_1} \right\} \\ &+ \left\{ \alpha_{p_2} + (\beta_c+\eta_c)c_2 + c_{m_2}(\beta_c+\eta_c)\frac{q_2}{1-q_2} \right\}, \end{aligned} \tag{53}$$

$$\begin{aligned} \frac{\partial TP_{c2}}{\partial T_{L_{12}}} &= (\beta_t+\eta_t)\left\{ p_{12} - c_1 - c_{m_1}\frac{q_1}{1-q_1} - \lambda_1 q_1^{-\delta_1} T_{L_{12}}^{\gamma_1} \right\} \\ &- \lambda_1\gamma_1 q_1^{-\delta_1} T_{L_{12}}^{\gamma_1-1} \left\{ \alpha_{p_1} - (\beta_c+\eta_c)p_{12} + \eta_c p_{22} + (\beta_t+\eta_t)T_{L_{12}} - \eta_t T_{L_{22}} \right\} \\ &- \eta_t \left\{ p_{22} - c_2 - c_{m_2}\frac{q_2}{1-q_2} - \lambda_2 q_2^{-\delta_2} T_{L_{22}}^{\gamma_2} \right\} \end{aligned} \tag{54}$$

and

$$\begin{aligned} \frac{\partial TP_{c2}}{\partial T_{L_{12}}} &= (\beta_t+\eta_t)\left\{ p_{22} - c_2 - c_{m_2}\frac{q_2}{1-q_2} - \lambda_2 q_2^{-\delta_2} T_{L_{22}}^{\gamma_2} \right\} \\ &- \lambda_2\gamma_2 q_2^{-\delta_2} T_{L_{22}}^{\gamma_2-1} \left\{ \alpha_{p_2} - (\beta_c+\eta_c)p_{22} + \eta_c p_{12} + (\beta_t+\eta_t)T_{L_{22}} - \eta_t T_{L_{12}} \right\} \\ &- \eta_t \left\{ p_{12} - c_1 - c_{m_1}\frac{q_1}{1-q_1} - \lambda_1 q_1^{-\delta_1} T_{L_{12}}^{\gamma_1} \right\}. \end{aligned} \tag{55}$$

Solving equations $\frac{\partial TP_{c2}}{\partial p_{12}}=0, \frac{\partial TP_{c2}}{\partial p_{22}}=0, \frac{\partial TP_{c2}}{\partial T_{L_{12}}}=0$ and $\frac{\partial TP_{c2}}{\partial T_{L_{22}}}=0$, we obtain the optimal values of $p_{12}, p_{22}, T_{L_{12}}$ and $T_{L_{22}}$. Analytically it is difficult to solve these equation. We solve the equation numerically by using Matlab2013 software. Let the solution be $p_{12}=p_{12}^{c*}$, $p_{22}=p_{22}^{c*}$, $T_{L_{12}}=T_{L_{12}}^{c*}$ and $T_{L_{22}}=T_{L_{22}}^{c*}$.

Proposition 8. *The profit function $TP_{c2}(p_{12},p_{22},T_{L_{12}},T_{L_{22}})$ is a concave function if $4(\beta_c+\eta_c)^2u_{18}+4\eta_c u_{14}u_{16}+2(\beta_c+\eta_c)u_{16}^2-4\eta_c^2u_{18}+2(\beta_c+\eta_c)u_{14}^2<0$ and $4\{(\beta_c+\eta_c)^2-\eta_c^2\}(u_{18}u_{20}-u_{19}^2)+2(\beta_c+\eta_c)u_{20}(u_{16}^2+u_{14}^2)+2(\beta_c+\eta_c)u_{18}(u_{17}^2+u_{15}^2)+(u_{15}^2u_{16}^2+u_{14}^2u_{17}^2)-4(\beta_c+\eta_c)u_{16}u_{17}u_{19}-4\eta_c u_{15}u_{16}u_{19}-4\eta_c u_{14}u_{17}u_{19}+4\eta_c u_{15}u_{17}u_{18}+4\eta_c u_{14}u_{16}u_{20}-4(\beta_c+\eta_c)u_{14}u_{15}u_{19}-2u_{14}u_{15}u_{16}u_{17}>0$, where $u_{14}=(\beta_t+\eta_t)+(\beta_c+\eta_c)\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-1}$, $u_{15}=-\eta_t-\eta_c\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-1}$, $u_{16}=-\eta_t-\eta_c\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-1}$, $u_{17}=(\beta_t+\eta_t)+(\beta_c+\eta_c)\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-1}$, $u_{18}=-\lambda_1\gamma_1(\gamma_1-1)q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-2}\{\alpha_{p_1}-(\beta_c+\eta_c)p_{12}^{c*}+\eta_c p_{22}^{c*}-\eta_t T_{22}^{c*}\}-\lambda_1\gamma_1(\gamma_1+1)(\beta_t+\eta_t)q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-1}$, $u_{19}=\eta_t\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-1}+\eta_t\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-1}$ and $u_{20}=-\lambda_2\gamma_2(\gamma_2-1)q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-2}$ $\{\alpha_{p_2}-(\beta_c+\eta_c)p_{22}^{c*}+\eta_c p_{12}^{c*}-\eta_t T_{12}^{c*}\}-\lambda_2\gamma_2(\gamma_2+1)(\beta_t+\eta_t)q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-1}$.*

Proof. The second order partial derivatives of TP_{c2} at stationary point $S_4=(p_{12}^{c*},p_{22}^{c*},T_{L_{12}}^{c*},T_{L_{22}}^{c*})$ are

$$\begin{aligned}
\left.\frac{\partial^2TP_{c2}}{\partial p_{12}^2}\right|_{atS_4} &= -2(\beta_c+\eta_c), \left.\frac{\partial^2TP_{c2}}{\partial p_{22}^2}\right|_{atS_4}=-2(\beta_c+\eta_c),\\
\left.\frac{\partial^2TP_{c2}}{\partial p_{12}\partial p_{22}}\right|_{atS_4} &= \left.\frac{\partial^2TP_{c2}}{\partial p_{22}\partial p_{12}}\right|_{atS_4}=2\eta_c,\\
\left.\frac{\partial^2TP_{c2}}{\partial p_{12}\partial T_{L_{12}}}\right|_{atS_4} &= \left.\frac{\partial^2TP_{c2}}{\partial T_{L_{12}}\partial p_{12}}\right|_{atS_4}\\
&= (\beta_t+\eta_t)+(\beta_c+\eta_c)\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-1}=u_{14}(say),\\
\left.\frac{\partial^2TP_{c2}}{\partial p_{12}\partial T_{L_{22}}}\right|_{atS_4} &= \left.\frac{\partial^2TP_{c2}}{\partial T_{L_{22}}\partial p_{12}}\right|_{atS_4}\\
&= -\eta_t-\eta_c\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-1}=u_{15}(say),\\
\left.\frac{\partial^2TP_{c2}}{\partial p_{22}\partial T_{L_{12}}}\right|_{atS_4} &= \left.\frac{\partial^2TP_{c2}}{\partial T_{L_{12}}\partial p_{22}}\right|_{atS_4}\\
&= -\eta_t-\eta_c\lambda_1\gamma_1q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-1}=u_{16}(say),\\
\left.\frac{\partial^2TP_{c2}}{\partial p_{22}\partial T_{L_{22}}}\right|_{atS_4} &= \left.\frac{\partial^2TP_{c2}}{\partial T_{L_{22}}\partial p_{22}}\right|_{atS_4}\\
&= (\beta_t+\eta_t)+(\beta_c+\eta_c)\lambda_2\gamma_2q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-1}=u_{17}(say),
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial^2 TP_{c2}}{\partial T_{L_{12}}^2}\right|_{atS_4} &= -\lambda_1\gamma_1(\gamma_1-1)q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-2}\{\alpha_{p_1}-(\beta_c+\eta_c)p_{12}^{c*}+\eta_c p_{22}^{c*} \\
&\quad -\eta_t T_{22}^{c*}\} - \lambda_1\gamma_1(\gamma_1+1)(\beta_t+\eta_t)q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-1} = u_{18}(say), \\
\left.\frac{\partial^2 TP_{c2}}{\partial T_{L_{12}}\partial T_{L_{22}}}\right|_{atS_4} &= \left.\frac{\partial^2 TP_{c2}}{\partial T_{L_{22}}\partial T_{L_{12}}}\right|_{atS_4} \\
&= \eta_t\lambda_1\gamma_1 q_1^{-\delta_1}(T_{L_{12}}^{c*})^{\gamma_1-1} + \eta_t\lambda_2\gamma_2 q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-1} = u_{19}(say), \\
\left.\frac{\partial^2 TP_{c2}}{\partial T_{L_{22}}^2}\right|_{atS_4} &= -\lambda_2\gamma_2(\gamma_2-1)q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-2}\{\alpha_{p_2}-(\beta_c+\eta_c)p_{22}^{c*}+\eta_c p_{12}^{c*} \\
&\quad -\eta_t T_{12}^{c*}\} - \lambda_2\gamma_2(\gamma_2+1)(\beta_t+\eta_t)q_2^{-\delta_2}(T_{L_{22}}^{c*})^{\gamma_2-1} = u_{20}(say).
\end{aligned}$$

The Hessian matrix H_4 of TP_{c2} at the stationary point $(p_{12}^{c*}, p_{22}^{c*}, T_{L_{12}}^{c*})$

$$H_4 = \begin{pmatrix} \frac{\partial^2 TP_{c2}}{\partial p_{12}^2} & \frac{\partial^2 TP_{c2}}{\partial p_{12}\partial p_{22}} & \frac{\partial^2 TP_{c2}}{\partial p_{12}\partial T_{L_{12}}} & \frac{\partial^2 TP_{c2}}{\partial p_{12}\partial T_{L_{22}}} \\ \frac{\partial^2 TP_{c2}}{\partial p_{22}\partial p_{12}} & \frac{\partial^2 TP_{c2}}{\partial p_{22}^2} & \frac{\partial^2 TP_{c2}}{\partial p_{22}\partial T_{L_{12}}} & \frac{\partial^2 TP_{c2}}{\partial p_{22}\partial T_{L_{22}}} \\ \frac{\partial^2 TP_{c2}}{\partial T_{L_{12}}\partial p_{12}} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{12}}\partial p_{22}} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{12}}^2} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{12}}\partial T_{L_{22}}} \\ \frac{\partial^2 TP_{c2}}{\partial T_{L_{22}}\partial p_{12}} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{22}}\partial p_{22}} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{22}}\partial T_{L_{12}}} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{22}}^2} \end{pmatrix} atS_4$$

The profit function TP_{c2} will be concave function if the principal minors of H_4 are alternatively negative and positive, i.e., if the i^{th} order principal minor D_i of H_4 takes the sign $(-1)^i$. Here,

$$\begin{aligned}
D_1 &= -2(\beta_c+\eta_c) < 0, \\
D_2 &= \begin{vmatrix} -2(\beta_c+\eta_c) & 2\eta_c \\ 2\eta_c & -2(\beta_c+\eta_c) \end{vmatrix} \\
&= 4(\beta_c+\eta_c)^2 - 4\eta_c^2 > 0
\end{aligned}$$

and

$$\begin{aligned}
D_3 &= \begin{vmatrix} -2(\beta_c+\eta_c) & 2\eta_c & u_{14} \\ 2\eta_c & -2(\beta_c+\eta_c) & u_{16} \\ u_{14} & u_{16} & u_{18} \end{vmatrix} \\
&= 4(\beta_c+\eta_c)^2 u_{18} + 4\eta_c u_{14}u_{16} + 2(\beta_c+\eta_c)u_{16}^2 - 4\eta_c^2 u_{18} + 2(\beta_c+\eta_c)u_{14}^2 < 0
\end{aligned}$$

if $4(\beta_c+\eta_c)^2 u_{18} + 4\eta_c u_{14}u_{16} + 2(\beta_c+\eta_c)u_{16}^2 - 4\eta_c^2 u_{18} + 2(\beta_c+\eta_c)u_{14}^2 < 0$ holds.

$$\begin{aligned}
|H_4| &= 4\{(\beta_c+\eta_c)^2-\eta_c^2\}(u_{18}u_{20}-u_{19}^2) + 2(\beta_c+\eta_c)u_{20}(u_{16}^2+u_{14}^2) \\
&+ 2(\beta_c+\eta_c)u_{18}(u_{17}^2+u_{15}^2) + (u_{15}^2u_{16}^2+u_{14}^2u_{17}^2) - 4(\beta_c+\eta_c)u_{16}u_{17}u_{19} \\
&- 4\eta_c u_{15}u_{16}u_{19} - 4\eta_c u_{14}u_{17}u_{19} + 4\eta_c u_{15}u_{17}u_{18} + 4\eta_c u_{14}u_{16}u_{20} \\
&- 4(\beta_c+\eta_c)u_{14}u_{15}u_{19} - 2u_{14}u_{15}u_{16}u_{17} > 0,
\end{aligned}$$

if $4\{(\beta_c+\eta_c)^2-\eta_c^2\}(u_{18}u_{20}-u_{19}^2) + 2(\beta_c+\eta_c)u_{20}(u_{16}^2+u_{14}^2) + 2(\beta_c+\eta_c)u_{18}(u_{17}^2+u_{15}^2) + (u_{15}^2u_{16}^2+u_{14}^2u_{17}^2) - 4(\beta_c+\eta_c)u_{16}u_{17}u_{19} - 4\eta_c u_{15}u_{16}u_{19} - 4\eta_c u_{14}u_{17}u_{19} +$

$4\eta_c u_{15}u_{17}u_{18} + 4\eta_c u_{14}u_{16}u_{20} - 4(\beta_c + \eta_c)u_{14}u_{15}u_{19} - 2u_{14}u_{15}u_{16}u_{17} > 0$ holds. This completes the proof. □

5. NUMERICAL ANALYSIS

In this section, we compare the optimal solutions for different scenarios with the following numerical data: $\alpha_{p_1} = \alpha_{p_2} = 50; c_1 = c_2 = 10; c_{m_1} = c_{m_2} = 0.5; \beta_c = 7.5; \eta_c = .45; \beta_t = 6.2; \eta_t = 0.3; \delta_1 = \delta_2 = 0.5; \lambda_1 = \lambda_2 = .25; \gamma_1 = \gamma_2 = 2.$

In Table 1, we observe that in both cases 1, and 2, centralized decision policy is the better strategy for overall supply chain than the decentralized decision policies. Table 1 also indicates that, in case 1 when only the manufacturer 1 adopts warranty policy, the retail price of product 1 is the highest in MC model, followed by MNC model, and centralized model. For product 2, the retail price is the highest in MNC, followed by MC model and centralized model. In case 2, when both the manufacturers adopt warranty policy, the optimal decisions on pricing and warranty strategies of two manufacturers are the same under the identical manufacturer assumption, and the retail price of each product is highest in MC model, followed by MNC model and centralized model.

Case (j)	Models	p_{1j}	p_{2j}	TP_{rj}	w_{1j}	T_{L1j}	TP_{m1j}	w_{2j}	T_{L2j}	TP_{m2j}	Total Profit
1	MNC	23.33	22.55	264.98	19.05	1.156	259.94	18.43		260.09	785.01
	MC	23.46	22.51	260.15	19.30	1.169	259.55	18.34		260.76	780.46
	Centralized	19.30	18.34			1.169					1040.62
2	MNC	23.34	23.34	275.25	19.06	1.156	259.67	19.06	1.156	259.67	794.59
	MC	23.47	23.47	259.89	19.31	1.169	259.89	19.31	1.169	259.89	779.68
	Centralized	19.31	19.31			1.169			1.169		1039.57

Table 1: Optimal results for different scenarios

We study the changes of optimal profits of the two manufacturers and their common retailer by changing the model parameters under different decision strategies (Tables 2-3) to help decision makers take proper marketing decision strategy and examine when manufacturer 2 generates more profit by offering a warranty period on his product. Based on the optimal solutions provided in Tables 1-3, it is also observed that the retailer makes more profit in MNC strategy than in MC strategy for case 2. In case 1, MNC decision strategy can yield more profit for manufacturer 1 while manufacturer 2 is better off in MC decision strategy. As compared with case 1, the retailer makes more profit in case 2 under MNC decision strategy. From Tables 2-3, we observe the following features and managerial insights:

Table 2 shows that while β_c increases, the optimal profits of manufacturers and retailer decrease in MNC model and MC model for both cases 1 and 2. The profit of manufacturer 2 in case 2 will be higher than his profit in case 1, as long as $\beta_c \leq 7.20$ in MNC and MC models. We also see that in Case 1, as β_c increases above a certain level (≥ 7.60 for MNC model and ≥ 7.20 for MC model) manufacturer 2's profit is greater than the manufacturer 1' s profit, which indicates that as price sensitivity coefficient increases, it becomes unprofitable to adopt warranty policy.

With increase in β_t, the optimal profits of manufacturers and retailer increases in MNC model (for case-2) and MC model (for both cases 1 and 2) but in case-1 the profits of manufacturer 2 and retailer decrease in MNC model (see Table 2). The manufacturer 2 generates more profit by offering warranty period in MNC model if $\beta_t \geq 6.30$ and in MC model if $\beta_t \geq 6.40$.

When η_c increases, the optimal profits of the retailer and manufacturer 2 increase but optimal profit of manufacturer 1 decreases in all model structures for case 1. But an opposite behavior in the optimal profits of channel members is recorded in all model structures for case 1 when η_t increases. In case 2, with the increasing value of η_c, in MNC model, the optimal profit of retailer increases but the optimal profit of each manufacturer decreases and with increasing value of η_t the optimal profit of each manufacturer and their common retailer increase. The optimal profit of each channel member remains unchanged when the sensitivity of MC model in case 2 is investigated for the changes in η_c and η_t, which supports Proposition 4.2.1.

Table 3 shows that the optimal profit of manufacturer i of all model structures for both cases is concave with respect to his product quality level q_i, i.e., initial increment of q_i reduces his warranty cost and increases profit, but after a certain level increase in increment of q_i increases his quality improvement cost and hence, decreases the profit. The optimal profit of manufacturer k of all model structures for both cases increases with increasing value of product quality level q_i, and the optimal profit of the retailer is also concave with respect to q_i.

With the increase in λ_i, optimal profits of manufacturer i and the retailer decrease but optimal profit of manufacturer k increases in all model structures for both cases except retailer's optimal profit of MNC model in case 1, which increases with increasing value of λ_i.

parameter		case 1						case-2					
		MNC			MC			MNC			MC		
		TP_{r1}	$TP_{m_{11}}$	$TP_{m_{21}}$	TP_{r1}	$TP_{m_{11}}$	$TP_{m_{21}}$	TP_{r_2}	$TP_{m_{11}}$	$TP_{m_{22}}$	TP_{r_2}	$TP_{m_{12}}$	$TP_{m_{22}}$
β_c	7.00	309.3539	303.5614	301.4320	302.6757	303.0820	302.2693	322.7641	303.2682	303.2682	303.5667	303.5667	303.5667
	7.20	290.7057	285.2414	284.1025	284.8338	284.8001	284.8676	302.7685	284.9586	284.9586	285.2235	285.2235	285.2235
	7.40	273.2729	268.0974	267.8461	268.1187	267.6906	268.5467	284.1125	267.8258	267.8258	268.0613	268.0613	268.0613
	7.60	256.9543	252.0340	252.5778	252.4397	251.6584	253.2209	266.6816	251.7739	251.7739	251.9838	251.9838	251.9838
	7.80	241.6596	236.9657	238.2214	237.7158	236.6185	238.8130	250.3742	236.7175	236.7175	236.9047	236.9047	236.9047
	8.00	227.3080	222.8158	224.7085	223.8741	222.4944	225.2538	235.0996	222.5795	222.5795	222.7468	222.7468	222.7468
β_t	6.00	265.1013	258.9786	260.1106	259.6768	258.6247	260.7289	274.2379	258.7150	258.7150	258.9359	258.9359	258.9359
	6.10	265.0408	259.4534	260.1023	259.9138	259.0812	260.7464	274.7398	259.1885	259.1885	259.4101	259.4101	259.4101
	6.20	264.9802	259.9362	260.0934	260.1549	259.5454	260.7645	275.2505	259.6703	259.6703	259.8926	259.8926	259.8926
	6.30	264.9194	260.4269	260.0838	260.4002	260.0171	260.7832	275.7700	260.1604	260.1604	260.3834	260.3834	260.3834
	6.40	264.8586	260.9256	260.0735	260.6495	260.4965	260.8026	276.2982	260.6587	260.6587	260.8825	260.8825	260.8825
	6.50	264.7976	261.4323	260.0625	260.9031	260.9835	260.8226	276.8353	261.1654	261.1654	261.3899	261.3899	261.3899
	6.60	264.7365	261.9471	260.0507	261.1608	261.4782	260.8434	277.3812	261.6804	261.6804	261.9058	261.9058	261.9058
η_c	0.40	263.1361	260.1152	259.9364	260.1547	259.7366	260.5729	273.5685	259.7169	259.7169	259.8926	259.8926	259.8926
	0.42	263.8749	260.0441	260.0004	260.1548	259.6598	260.6497	274.2421	259.6990	259.6990	259.8926	259.8926	259.8926
	0.44	264.6122	259.9723	260.0628	260.1549	259.5834	260.7263	274.9147	259.6801	259.6801	259.8926	259.8926	259.8926
	0.46	265.3479	259.8999	260.1236	260.1550	259.5074	260.8025	275.5861	259.6603	259.6603	259.8926	259.8926	259.8926
	0.48	266.0821	259.8269	260.1828	260.1551	259.4317	260.8784	276.2564	259.6395	259.6395	259.8926	259.8926	259.8926
	0.50	266.8147	259.7533	260.2405	260.1552	259.3563	260.9540	276.9256	259.6177	259.6177	259.8926	259.8926	259.8926
η_t	0.25	265.1325	259.7004	260.3304	260.1552	259.3153	260.9952	275.2469	259.6669	259.6669	259.8926	259.8926	259.8926
	0.27	265.0718	259.7945	260.2360	260.1551	259.4073	260.9029	275.2486	259.6685	259.6685	259.8926	259.8926	259.8926
	0.29	265.0108	259.8889	260.1411	260.1550	259.4994	260.8106	275.2500	259.6698	259.6698	259.8926	259.8926	259.8926
	0.31	264.9495	259.9836	260.0455	260.1549	259.5914	260.7183	275.2510	259.6708	259.6708	259.8926	259.8926	259.8926
	0.33	264.8880	260.0786	259.9494	260.1548	259.6835	260.6261	275.2518	259.6715	259.6715	259.8926	259.8926	259.8926
	0.35	264.8262	260.1739	259.8527	260.1547	259.7756	260.5338	275.2523	259.6719	259.6719	259.8926	259.8926	259.8926

Table 2: Optimal profits of channel members for changing the values of β_c,β_t,η_c and η_t under different scenarios

parameter		case 1						case-2					
		MNC			MC			MNC			MC		
		TP_{r1}	TP_{mi1}	TP_{mk1}	TP_{r1}	TP_{mi1}	TP_{mk1}	TP_{r_2}	TP_{mi1}	TP_{mk2}	TP_{r_2}	$TP_{m_{i2}}$	TP_{mk2}
q_i	0.10	272.0064	265.5068	260.0069	262.9373	265.4600	260.4146	278.1992	265.5779	259.3282	262.6772	265.8112	259.5432
	0.20	272.0913	266.2110	260.0103	263.2538	266.0504	260.4572	278.5342	266.1681	259.3705	262.9939	266.4020	259.5858
	0.30	270.2599	265.5820	260.0365	262.9284	265.3331	260.5237	278.1894	265.4518	259.4357	262.6682	265.6841	259.6522
	0.40	267.9931	263.6344	260.0424	261.9656	263.3102	260.6209	277.1693	263.4312	259.5305	261.7046	263.6600	259.7492
	0.50	264.9802	259.9362	260.0934	260.1549	259.5454	260.7645	275.2505	259.6703	259.6703	259.8926	259.8926	259.8926
	0.60	260.6440	253.4555	260.2215	256.9958	253.0040	260.9875	271.9008	253.1356	259.8871	256.7310	253.3468	260.1153
	0.70	253.6964	241.7484	260.5026	251.3046	241.2413	261.3679	265.8598	241.3846	260.2567	251.0356	241.5759	260.4952
	0.80	240.4954	217.8053	261.1653	239.6941	217.2481	262.1401	253.5071	217.4146	261.0072	239.4160	217.5656	261.2663
	0.90	205.3325	151.4288	263.3572	207.6601	150.8424	264.4777	219.1669	151.0676	263.2844	207.3537	151.1067	263.6006
λ_i	0.10	262.2254	284.1747	257.5863	271.7111	282.8403	260.5820	287.4821	282.9446	259.5016	271.4565	283.2028	259.7102
	0.20	264.4160	263.9010	259.8147	262.0463	263.3584	260.7341	277.2544	263.4800	259.6422	261.7852	263.7081	259.8623
	0.22	264.6672	262.0951	259.9483	261.1848	261.6217	260.7479	276.3418	261.7449	259.6550	260.9232	261.9704	259.8761
	0.24	264.8833	260.5949	260.0508	260.4692	260.1789	260.7594	275.5835	260.3033	259.6656	260.2071	260.5266	259.8875
	0.26	265.0707	259.3289	260.1313	259.8652	258.9612	260.7691	274.9435	259.0867	259.6746	259.6026	259.3080	259.8972
	0.28	265.2349	258.2462	260.1958	259.3486	257.9197	260.7774	274.3960	258.0461	259.6824	259.0857	258.2659	259.9056
	0.30	265.3797	257.3096	260.2482	258.9017	257.0188	260.7847	273.9224	257.1459	259.6891	258.6386	257.3643	259.9128
	0.40	265.9053	254.0452	260.4070	257.3439	253.8780	260.8099	272.2708	254.0078	259.7125	257.0797	254.2214	259.9380

Table 3: Optimal profits of channel members for changing the values of q_i and λ_i under different scenarios, where $i \in \{1, 2\}$ and $k = 3 - i$

	Models	TP_{m12}	TP_{m22}	TP_{r2}	Total Profit
Our Model	MNC	9354.50	9354.50	9305.10	28014.1
	MC	9250.40	9250.40	9250.40	27751.2
Wei's Model	MNC	7553.80	7553.80	4835.70	19943.3
	MC	8350.60	8350.60	10105.0	26806.2

Table 4: Comparison of optimal results with Wei *et al.*'s [13] model when both manufacturers adopt warranty policty

Table 4, we have compared our model with the model in Wei *et al.* [13] using their numerical data as follows: $\alpha_{p_1} = \alpha_{p_2} = 100; c_1 = c_2 = 30; \eta_c = .25; \beta_c + \eta_c = .30; \eta_t = 0.2; \beta_t = .3$ and remaining parameters of our model remain unchanged. We observed that in our model when both manufacturers adopt warranty policy, profits of manufacturers and retailer are higher than that of Wei *et al.*'s [13] model. Because Wei *et al.* [13] expressed the product's demand function as decreasing function of its selling price, as well as its complementary product's selling price and increasing function of its warranty period and its complementary product's warranty period. So, in order to maximize the market demand, the manufacturers decrease product's price and increase the warranty period which amplify warranty cost and result in lower values of profits. Differing from their study, in this model we consider the demand of each product decreasing with its own selling price and the competitor's warranty period and increasing with its own warranty period and the competitor's product selling price, that corresponds with reality in many practical situations.

6. CONCLUSION

In this article, we studied the importance of price and warranty in the interactions between two manufacturers and their common retailer for two complementary products under decentralized and centralized decision strategies. We consider that the demand of products depend not only on price but also on warranty period. The role of warranty as a competitive strategy was explored by examining the model through two different scenarios: (i) only one manufacturer offers warranty on his product, (ii) both manufacturers offer warranty on their product. We observed that as price sensitivity factor increases, the adoption of warranty policy becomes more unprofitable for the manufacture, but with the increase of warranty period sensitivity factor, the manufacturer inclines to adopt the warranty policy. Numerical analysis also reveals that in case 1 if a manufacturer adopts warranty policy, then he will be more profitable under MNC decision strategy. In both cases, the retailer always earns more profit under MNC decision strategy as compared to the MC strategy, since under MC strategy the manufacturers make decision jointly instead of independently and retailer acts as their follower. We also find that the manufacturer profit function is concave with respect to product quality level.

The proposed model could be extended in many aspects such as developing the model under stochastic demand pattern, introducing competitive strategies among multiple retailers and incorporating some contract mechanisms (e.g., price discount contract, revenue sharing contract, wholesale price sharing contract, etc) to coordinate the supply chain.

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