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# BOUNDS OF THE STATIONARY DISTRIBUTION IN M/G/1 RETRIAL QUEUE WITH TWO-WAY COMMUNICATION AND $n$ TYPES OF OUTGOING CALLS

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**Abstract:** In this article we analyze the $M/G/1$ retrial queue with two-way communication and $n$ types of outgoing calls from a stochastic comparison viewpoint. The main idea is that given a complex Markov chain that cannot be analyzed numerically, we propose to bound it by a new Markov chain, which is easier to solve by using a stochastic comparison approach. Particularly, we study the monotonicity of the transition operator of the embedded Markov chain relative to the stochastic and convex orderings. Bounds are also obtained for the stationary distribution of the embedded Markov chain at departure epochs. Additionally, the performance measures of the considered system can be estimated by those of an $M/M/1$ retrial queue with two-way communication and $n$ types of outgoing calls when the service time distribution is NBUE (respectively, NWUE). Finally, we test numerically the accuracy of the proposed bounds.

**Keywords:** Retrial Queues, Outgoing Calls, Markov Chain, Stochastic Comparison, Simulation.

**MSC:** 60K25, 60E15, 60K10.

## 1. INTRODUCTION

Retrial queues play an important role insolving many practical problems of computer and other communication networks. A detailed analysis of this type of queues can be found in Artalejo and Gómez-Corral [1]. Retrial queues have been mainly studied from a theoretical viewpoint [15, 26, 28].

Approximation methods are essential to deal with the complexity of communication protocols especially in a call center context, where the server not only serves incoming calls, but in idle time it makes outgoing calls to the outside. This type of systems with both incoming and outgoing calls is referred to as a two-way communication retrial queueing model [21, 22]. Many researchers undertook to examine such a model in the literature of queues. Falin [16] analyzed an $M/G/1$ retrial queue with two-way communication by assuming that service times of incoming and outgoing calls follow the same arbitrary distribution and obtained an integral formula for the partial generating functions and explicit expressions for some expected performance measures. Afterward, the same model with constant retrial rate has been investigated by Artalejo and Martin [2], whereas Artalejo and Phung-Duc [3] considered a single server retrial queue with two-way communication where incoming calls and outgoing calls follow distinct exponential distributions. They derived explicit expressions for the generating functions as well as the joint stationary distribution of the number of calls in the orbit and the state of the server. Artalejo and Phung-Duc [4] extended their analysis to an $M/G/1$ retrial queue with two-way communication, which turns out to have a close relation with retrial queues and priority (see, Falin et al. [17]) where an infinite buffer is available for outgoing calls. In all the previous works, there is at most one flow of outgoing calls. However, in practice there are various types of outgoing calls whose durations may be extremely different. Recently, Sakurai and Phung-Duc [23] considered models where multiple types of outgoing calls follow distinct distributions. Using the generating function approach, they obtained the mathematical expressions for the joint stationary distribution of the number of calls in the orbit and the state of the server of a two-way communication retrial queues with multiple types of outgoing calls whose durations follow distinct exponential distributions. Boutarfa and Djellab [14], considered an $M_1, M_2/G_1, G_2/1$ priority retrial queue with preemptive resume policy by using the method of supplementary variables. A wide variety of techniques were used for providing the solution of queueing problems with two-way communication in different frameworks (namely, supplementary variable technique, probability generating function, matrix method, etc.). Therefore, there are different approaches to study retrial queues. We place emphasis on the stochastic comparison method because it leads to simplifications when solving complex models. Stochastic bounds take place in various retrial models. More precisely, in the literature of queues, many researchers have undertaken to examine such a question in the same spirit, among others, we cite: Boualem [5], Boualem et al. [6, 7, 8, 9, 10, 11, 12, 13], Khalil and Falin [18], Liang [19], Liang and Kulkarni [20], Shin [25] and references therein.

In this paper, we propose stochastic bounds for the main performance characteristics given in Sakurai and Phung-Duc [23] for the single server retrial queue with two-way communication and $n$ types of outgoing calls. The analysis resorts mainly to the embedded Markov chain approach and stochastic ordering comparisons to study the monotonicity properties for this model relative to the stochastic and convex orderings.

In the next section, we consider the mathematical model formulation. In Section 3, we give some preliminary results. In Section 4, we analyze the notions of monotonicity for Markov chain in continuous time. In Section 5, we show the comparability conditions of stationary distributions of the number of costumers in the system. Finally, in order to further validate stochastic comparison bounds, we present some numerical results illustrating the interest of our approach, based on a simulation study.

## 2. MATHEMATICAL MODEL FORMULATION

The model considered in this work comes from Sakurai and Phung-Duc [23], where a single server retrial queues with two-way communication and $n$ types of outgoing calls is considered.

Primary incoming calls arrive according to a Poisson process with rate $\lambda$. if the server is idle at arriving of incoming call, it will be served immediately. Else, incoming call joins the orbit and repeats its request after an exponentially distributed time with rate $\nu$.

If the server is free, it makes an outgoing call of type $\ell$ in an exponentially distributed time with rate $\sigma_\ell$ $(\ell = 2, ..., n+1)$.

We assume that the two types of calls (incoming calls and outgoing calls) receive different service times. Indeed, the service times of incoming calls and an outgoing call of type $\ell$ $(\ell = 2, ..., n+1)$ are characterized by the distribution functions $S_1(x)$ and $S_\ell(x)$ $(\ell = 2, ..., n+1)$, respectively. The Laplace$-$Stieltjes transform and the $k^{th}$ moment of $S_l(x)$ by $\delta_l(s)$ and $\delta_l^k$ for $l = 1, ..., n+1$.

Let $C(t)$ denote the state of the server at time $t \geq 0$,

$$C(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy with an incoming call,} \\ 2, & \text{if an outgoing call of type } \ell \text{ is in service,} \end{cases}$$

and $N(t)$ denote the number of incoming calls in the orbit at time $t$.

The embedded Markov chain in this context represents the number of customers in the orbit at the service completion epoch of either an incoming call or an outgoing call. It is easy to see that the one-step transition probabilities of this

Markov chain are given as follows (see [23])

$$p_{\imath,j}=\begin{cases}\dfrac{\imath\nu}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+\imath\nu}k_1^0, & \imath\geq 1;\\ \dfrac{\lambda}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+n\nu}k_1^{j-\imath}+\dfrac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+n\nu}k_\ell^{j-\imath}+\dfrac{n\nu}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+n\nu}k_1^{j-\imath+1}, & 0\leq\imath\leq j,\quad \ell=2,...,n+1;\\ 0, & \imath-1\geq 2,\end{cases}$$

where

$$k_l^j=\int\limits_0^\infty e^{-\lambda x}\frac{(\lambda x)^j}{j!}dS_l(x),\ l=1,2,...,n+1,\ j\in\mathbf{Z}_+.$$

## 3. PRELIMINARY RESULTS

In this part, we define some notions of the stochastic orders, which will be used in the context of the paper. For more details, we refer to [24, 27].

Let $X$ and $Y$ be two random variables with distribution functions $F$ and $G$, respectively.

**Definition 1.** 1. *If $F(t)\geq G(t)$, for all real $t$, then $X$ is smaller than $Y$ in the usual stochastic order, which is denoted by $X\leq_{st}Y$ or $F\leq_{st}G$.*

2. *$X$ is less than $Y$ in convex order (written $X\leq_v Y$ or $F\leq_v G$)) if and only if $\int\limits_x^{+\infty}(1-F(t))dt\leq\int\limits_x^{+\infty}(1-G(t))dt$, for all real $t$.*
3. *If $X$ and $Y$ are discret random variables taking values in $\mathbb{N}$, with distribution $p_i=P(X=i)$ and $q_i=P(Y=i)$, $i\in\mathbb{N}$, respectively, then $X\leq_{st}Y$ ($X\leq_v Y$) if and only if, $\overline{p}_i\leq\overline{q}_i$ ($\overline{\overline{p}}_i\leq\overline{\overline{q}}_i$), for all $i\in\mathbb{N}$, respectively, where $\overline{p}_i=\sum\limits_{j\geq i}p_j$, $\overline{\overline{p}}_i=\sum\limits_{j\geq i}\overline{p}_j$.*
4. *Let $\{X_i,i=1,2,...\}$ and $\{Y_i,i=1,2,...\}$ be two sequences of random variables such that $X_i\rightarrow_{st}X$ and $Y_i\rightarrow_{st}Y$ as $i\rightarrow\infty$, where " $\rightarrow_{st}$ " denotes convergence in distribution.*
   *If $X_i\leq_{st}Y_i, i=1,2,...,$ then $X\leq_{st}Y$.*
5. *$F$ is NBUE (NWUE) if and only if, $F_e\leq_{st}(\geq_{st})F$,*
   *where $F_e(x)=\frac{1}{m}\int\limits_0^x\overline{F}(t)dt, x\geq 0$.*

Now, let $\Sigma_1$ and $\Sigma_2$ be two $M/G/1$ retrial queues with two-way communication and $n$ types of arbitrarily distributed outgoing calls with parameters $\lambda^{(i)}$, $\nu^{(i)}$, $\sigma_\ell^{(i)}$, $S_1^{(i)}(x)$, $S_\ell^{(i)}(x)$, $k_l^{(i)}$, and $\pi_n^{(i)}$ (the stationary distribution in $\Sigma_i$), $i=1,2$; $l=1,...,n$; $\ell=2,...,n+1$.

The following lemma gives the conditions under which the probabilities of the number of incoming arrivals during the service of an incoming or outgoing call of type $\ell$ of two systems $M/G/1$ retrial queue with two-way communication and $n$ types of outgoing calls $\{k_n^{(i)},\ i=1,2;\ n\in\mathbb{N}\}$ are comparable relatively to stochastic and convex ordering.

**Lemma 2.** *If* $\lambda^{(1)}\le\lambda^{(2)},\ S_1^{(1)}\le_{so}S_1^{(2)}$ *and* $(S_2^{(1)},\ldots,\ S_{n+1}^{(1)})\le_{so}(S_2^{(2)},\ldots,\ S_{n+1}^{(2)})$, *then* $\{k_n^{(1)}\}\le_{so}\{k_n^{(2)}\}$, $so=(st,v)$.

*Proof.* Suppose that $\lambda^{(1)}\le\lambda^{(2)}$, $S_1^{(1)}\le_{st}S_1^{(2)}$ and $(S_2^{(1)},\ldots,S_{n+1}^{(1)})\le_{so}(S_2^{(2)},\ldots,S_{n+1}^{(2)})$.

By definition we have

$$\overline{k}_n^{(i)}=\sum_{j=n}^{+\infty}k_j^{(i)}=\int_0^{+\infty}\sum_{j=n}^{+\infty}\frac{(\lambda^{(i)}x)^j}{j!}\exp\{-\lambda^{(i)}x\}dS_l^{(i)}(x),\quad l=1,\ldots,n+1,\quad i=1,2.$$

Since $f_n(x,\ \lambda)=\sum\limits_{j=n}^{+\infty}\dfrac{(\lambda^{(i)}x)^j}{j!}\exp\{-\lambda^{(i)}x\}$ is increasing in $\lambda$ and $x$, then

$$\begin{aligned}\frac{\partial}{\partial x}f_n(x,\ \lambda)&=\lambda\frac{(\lambda x)^{n-1}}{(n-1)!}\exp\{-\lambda x\}>0,\quad\forall\ x\ge\ 0,\\ \frac{\partial}{\partial\lambda}f_n(x,\ \lambda)&=x\exp\{-\lambda x\}\frac{(\lambda x)^{n-1}}{(n-1)!}>0.\end{aligned}$$

Also, since $\lambda^{(1)}\le\lambda^{(2)}$ and $S_l^{(1)}\le_{st}S_l^{(2)},\quad l=1,\ldots,n+1$, then

$$\int_0^{+\infty}f_n(x,\ \lambda^{(1)})dS_l^{(1)}(x)\le\int_0^{+\infty}f_n(x,\ \lambda^{(1)})dS_l^{(2)}(x)\le\int_0^{+\infty}f_n(x,\ \lambda^{(2)})dS_l^{(2)}(x),l=1,\ldots,n+1.$$

Therefore, $\{k_n^{(1)}\}\le_{st}\{k_n^{(2)}\}$.

On the other hand, we have

$$\begin{aligned}\{k_n^{(1)}\}\le_v\{k_n^{(2)}\}\quad&\Leftrightarrow\quad\overline{\overline{k}}_n^{(1)}=\sum_{m=n}^{+\infty}\overline{k}_m^{(1)}\le\sum_{m=n}^{+\infty}\overline{k}_m^{(2)}=\overline{\overline{k}}_n^{(2)}\\ &\Leftrightarrow\quad\int_0^{+\infty}\sum_{m=n}^{+\infty}\sum_{l=m}^{+\infty}\frac{(\lambda^{(1)}x)^l}{l!}\exp\{-\lambda^{(1)}x\}dS_l^{(1)}(x)\\ &\qquad\le\int_0^{+\infty}\sum_{m=n}^{+\infty}\sum_{l=m}^{+\infty}\frac{(\lambda^{(2)}x)^l}{l!}\exp\{-\lambda^{(2)}x\}dS_l^{(2)}(x)\\ &\Leftrightarrow\quad\int_0^{+\infty}\sum_{m=n}^{+\infty}f_m(x,\ \lambda^{(1)})dS_l^{(1)}(x)\le\int_0^{+\infty}\sum_{m=n}^{+\infty}f_m(x,\ \lambda^{(2)})dS_l^{(2)}(x),\end{aligned}$$

with $f_m(x,\ \lambda^{(i)}) = \sum\limits_{l=m}^{+\infty} \frac{(\lambda^{(i)}x)^l}{l!} \exp\{-\lambda^{(i)}x\},\ l = 1, 2, ..., n+1$. The function $\overline{f}_n(x,\ \lambda) = \sum\limits_{m=n}^{+\infty} f_m(x,\ \lambda)$ is increasing in $\lambda$ and is increasing and convex in $x$. Indeed,

$$\frac{\partial^2}{\partial x^2}\overline{f}_n(x,\ \lambda) = \lambda \frac{\partial}{\partial x} f_{n-1}(x,\ \lambda) = \lambda^2 \left( \frac{(\lambda x)^{n-2}}{(n-2)!} \right) \exp\{-\lambda x\} > 0.$$

So, applying Theorem 1.3.1 given in [27], and by monotonicity of $\overline{f}_n(x,\ \lambda)$ with respect to $\lambda$, we obtain the result. □

## 4. MONOTONICITY PROPERTIES OF THE EMBEDDED MARKOV CHAIN

To every distribution $p = (p_n)_{n \geq 0}$, the transition operator $r$ of the embedded Markov chain associates a distribution $r_p = q = (q_m)_{m \geq 0}$ such that

$$q_m = \sum_{n \geq 0} p_n p_{n,m}.$$

**Theorem 3.** *Under the condition* $\sup\limits_{2 \leq \ell \leq n+1} \{S_\ell\} \leq_{st} S_1$*, the transition operator* $r$ *is monotone with respect to the stochastic order* $\leq_{st}$*, i.e., for any two distribution* $p^{(1)}$ *and* $p^{(2)}$*, the inequality*

$$p^{(1)} \leq_{st} p^{(2)} \Rightarrow rp^{(1)} \leq_{st} rp^{(2)}.$$

*Proof.* The operator $r$ is monotone with respect to $\leq_{st}$ if and only if [27]

$$\overline{p}_{n-1,m} \leq \overline{p}_{n,m}, \quad \forall\ n,\ m, \tag{1}$$

with

$$\begin{aligned}
\overline{p}_{n,m} &= \sum_{k=m}^{+\infty} p_{n,k} \\
&= \sum_{k=m}^{+\infty} \left[ \frac{\lambda}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} k_1^{k-n} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} k_\ell^{k-n} + \frac{n\nu}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} k_1^{k-n+1} \right] \\
&= \frac{\lambda + n\nu}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} \overline{k}_1^{m-n+1} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} \overline{k}_\ell^{m-n} + \frac{\lambda}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} k_1^{m-n},
\end{aligned}$$

$$\overline{p}_{n-1,m} = \frac{\lambda+(n-1)\nu}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu}\overline{k}_1^{m-n+1} + \frac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu}\overline{k}_\ell^{m-n+1} - \frac{(n-1)\nu}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu}k_1^{m-n+1}.$$

Finally, we get

$$\begin{aligned}\overline{p}_{n,m}-\overline{p}_{n-1,m} &= \frac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell\nu}{(\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+n\nu)(\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu)}\left[\overline{k}_1^{m-n+1}-\overline{k}_\ell^{m-n+1}\right] \\ &+\frac{\lambda}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+n\nu}k_1^{m-n}+\frac{(n-1)\nu}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu}k_1^{m-n+1} \\ &+\frac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+n\nu}k_\ell^{m-n}\geq 0.\end{aligned}$$

Consequently, since $\sup\limits_{2\leq\ell\leq n+1}\{S_\ell\}\leq_{st} S_1$, inequality (1) is verified, which proves the monotonicity of the transition operator $(r)$ with respect to stochastic order. □

**Theorem 4.** *Under the condition* $\inf\limits_{2\leq\ell\leq n+1}\{S_\ell\}\geq_v S_1$*, the transition operator* $r$ *is monotone with respect to the convex order* ($\leq_v$)*, i.e., for any two distribution* $p^{(1)}$ *and* $p^{(2)}$*, we have*

$$p^{(1)}\leq_v p^{(2)}\Rightarrow rp^{(1)}\leq_v rp^{(2)}.$$

*Proof.* The operator $r$ is monotone with respect to the convex order if and only if (see [27])

$$2\overline{\overline{p}}_{n,m}\leq\overline{\overline{p}}_{n-1,m}+\overline{\overline{p}}_{n+1,m},\ \forall\ n,\ m, \tag{2}$$

where,

$$
\begin{aligned}
\overline{\overline{p}}_{n,m} &= \sum_{k=m}^{+\infty} \overline{p}_{n,k} \\
&= \sum_{k=m}^{+\infty} \left[ \frac{\lambda}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} \overline{k}_1^{k-n} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} \overline{k}_\ell^{k-n} + \frac{n\nu}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} k_1^{m-n+1} \right] \\
&= \frac{\lambda + n\nu}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} \overline{\overline{k}}_1^{m-n+1} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} \overline{\overline{k}}_\ell^{m-n} + \frac{\lambda}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} \overline{k}_1^{m-n},
\end{aligned}
$$

$$
\begin{aligned}
\overline{\overline{p}}_{n-1,m} &= \frac{\lambda + (n-1)\nu}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n-1)\nu} \overline{\overline{k}}_1^{m-n} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n-1)\nu} \overline{\overline{k}}_\ell^{m-n} \\
&\quad - \left( \frac{\lambda + 2(n-1)\nu}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n-1)\nu} \right) \overline{k}_1^{m-n} - \frac{(n-1)\nu}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n-1)\nu} k_1^{m-n} \\
&\quad - \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n+1)\nu} \overline{k}_\ell^{m-n},
\end{aligned}
$$

$$
\begin{aligned}
\overline{\overline{p}}_{n+1,m} &= \frac{\lambda + (n+1)\nu}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n+1)\nu} \overline{\overline{k}}_1^{m-n} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n+1)\nu} \overline{\overline{k}}_\ell^{m-n} \\
&\quad + \frac{\lambda}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n+1)\nu} \overline{k}_1^{m-n} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n+1)\nu} \overline{k}_\ell^{m-n} \\
&\quad + \frac{\lambda}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + n\nu} k_1^{m-n} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell}{\lambda + \sum\limits_{\ell=2}^{n+1} \sigma_\ell + (n+1)\nu} k_\ell^{m-n-1}.
\end{aligned}
$$

Finally, after some algebraic manipulations, we get

$$\begin{aligned}
\overline{\overline{p}}_{n-1,m} &+ \overline{\overline{p}}_{n+1,m} - 2\overline{\overline{p}}_{n,m} \\
&= \frac{2\sum_{\ell=2}^{n+1}\sigma_\ell\nu^2}{(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n+1)\nu)(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+n\nu)(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu)}\left[\overline{\overline{k}}_\ell^{m-n+1}-\overline{\overline{k}}_1^{m-n+1}\right] \\
&+\frac{2\sum_{\ell=2}^{n+1}\sigma_\ell\nu^2}{(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n+1)\nu)(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+n\nu)(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu)}\left[\overline{k}_\ell^{m-n}-\overline{k}_1^{m-n}\right] \\
&+\frac{2\sum_{\ell=2}^{n+1}\sigma_\ell\nu(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+n\nu)}{(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n+1)\nu)(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu)(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+n\nu)}\left[\overline{k}_1^{m-n}-\overline{k}_\ell^{m-n}\right] \\
&+\frac{2\nu^2(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell)}{(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n+1)\nu)(\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+n\nu)}\overline{k}_1^{m-n}+\frac{(n-1)\nu}{\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n-1)\nu}k_1^{m-n} \\
&+\frac{\lambda}{\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n+1)\nu}k_1^{m-n-1}+\frac{\sum_{\ell=2}^{n+1}\sigma_\ell}{\lambda+\sum_{\ell=2}^{n+1}\sigma_\ell+(n+1)\nu}k_\ell^{m-n-1}\geq 0.
\end{aligned}$$

Hence, $r$ is monotone with respect to convex order ($\leq_v$) if the condition $\inf_{2\leq\ell\leq n+1}\{S_\ell\}\geq_v S_1$ is fulfilled. □

Now, let $r^{(1)}$ and $r^{(2)}$ be the transition operators of the embedded Markov chains added to each model $\Sigma_1$ and $\Sigma_2$. Comparability conditions of $r^{(1)}$, $r^{(2)}$ relatively to stochastic and convex orders are given in the following theorems.

**Theorem 5.** *If $\lambda^{(1)}\leq\lambda^{(2)}$, $\nu^{(1)}\geq\nu^{(2)}$, $\sum_{\ell=2}^{n+1}\sigma_\ell^{(1)}\leq\sum_{\ell=2}^{n+1}\sigma_\ell^{(2)}$, $S_1^{(1)}\leq_{st}S_1^{(2)}$ and $S_\ell^{(1)}\leq_{st}S_\ell^{(2)}$ ($\ell=2,...,n+1$), then $r^{(1)}\leq_{st}r^{(2)}$ (i.e., for any distribution $p$, we have $r^{(1)}p\leq_{st}r^{(2)}p$).*

*Proof.* We must prove that (see [27])

$$\overline{p}_{nm}^{(1)} \leq \overline{p}_{nm}^{(2)},\ \forall\ 0\leq n\leq m,$$

which is to show

$$\begin{aligned}
&\frac{\lambda^{(1)} + n\nu^{(1)}}{\lambda^{(1)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)} + n\nu^{(1)}} (\overline{k}_1^{m-n})^{(1)} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)}}{\lambda^{(1)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)} + n\nu^{(1)}} (\overline{k}_\ell^{m-n})^{(1)} \\
&- \frac{n\nu^{(1)}}{\lambda^{(1)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)} + n\nu^{(1)}} (k_1^{m-n})^{(1)} \\
\leq\ &\frac{\lambda^{(2)} + n\nu^{(2)}}{\lambda^{(2)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)} + n\nu^{(2)}} (\overline{k}_1^{m-n})^{(2)} + \frac{\sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)}}{\lambda^{(2)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)} + n\nu^{(2)}} (\overline{k}_\ell^{m-n})^{(2)} \\
&- \frac{n\nu^{(2)}}{\lambda^{(2)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)} + n\nu^{(2)}} ({k_1}^{m-n})^{(2)}.
\end{aligned}$$

By Lemma 2 (with respect to stochastic order), we have $\{k_l^{n(1)}\} \leq_{st} \{k_l^{n(2)}\}$, $l = 1, ..., n+1$.

As $\lambda^{(1)} \leq \lambda^{(2)}$, $\sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)} \leq \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)}$ and $\nu^{(1)} \geq \nu^{(2)}$, implies that

$$\frac{\lambda^{(1)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)}}{\nu^{(1)}} \leq \frac{\lambda^{(2)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)}}{\nu^{(2)}},$$

and since the function $\frac{m}{x+m}$ is decreasing, so

$$\frac{n\nu^{(1)}}{\lambda^{(1)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)} + n\nu^{(1)}} \geq \frac{n\nu^{(2)}}{\lambda^{(2)} + \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)} + n\nu^{(2)}}, \tag{3}$$

we have also the increasing function $\frac{x}{x+m}$ and $\frac{\lambda^{(1)}}{\nu^{(1)}} \leq \frac{\lambda^{(2)}}{\nu^{(2)}}$, so

$$\frac{\lambda^{(1)}}{\lambda^{(1)} + n\nu^{(1)}} \leq \frac{\lambda^{(2)}}{\lambda^{(2)} + n\nu^{(2)}},$$

whence,

$$\lambda^{(1)} + n\nu^{(1)} \geq \lambda^{(2)} + n\nu^{(2)},$$

Consequently, $\sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)} \leq \sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)}$ and $\lambda^{(1)} + n\nu^{(1)} \geq \lambda^{(2)} + n\nu^{(2)}$, implies that

$$\frac{\lambda^{(1)} + n\nu^{(1)}}{\sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(1)}} \geq \frac{\lambda^{(2)} + n\nu^{(2)}}{\sum\limits_{\ell=2}^{n+1} \sigma_\ell^{(2)}},$$

indeed,

$$\frac{\lambda^{(1)}+n\nu^{(1)}}{\lambda^{(1)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}+n\nu^{(1)}} \geq \frac{\lambda^{(2)}+n\nu^{(2)}}{\lambda^{(2)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}+n\nu^{(2)}}. \tag{4}$$

From Lemma 2 and inequalities (3)–(4), we get the following result

$$\begin{aligned}
&\frac{\lambda^{(1)}+n\nu^{(1)}}{\lambda^{(1)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}+n\nu^{(1)}}(\overline{k}_1^{m-n})^{(1)} - \frac{n\nu^{(1)}}{\lambda^{(1)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}+n\nu^{(1)}}(k_1^{m-n})^{(1)} \\
&-\frac{\lambda^{(2)}+n\nu^{(2)}}{\lambda^{(2)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}+n\nu^{(2)}}(\overline{k}_1^{m-n})^{(2)} + \frac{n\nu^{(2)}}{\lambda^{(2)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}+n\nu^{(2)}}(k_1^{m-n})^{(2)} \\
\leq\; &\frac{\lambda^{(1)}+n\nu^{(1)}}{\lambda^{(1)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}+n\nu^{(1)}}(\overline{k}_1^{m-n})^{(1)} - \frac{n\nu^{(2)}}{\lambda^{(2)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}+n\nu^{(2)}}(k_1^{m-n})^{(2)} \\
&-\frac{\lambda^{(1)}+n\nu^{(1)}}{\lambda^{(1)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}+n\nu^{(1)}}(\overline{k}_1^{m-n})^{(1)} + \frac{n\nu^{(2)}}{\lambda^{(2)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}+n\nu^{(2)}}(k_1^{m-n})^{(2)}.
\end{aligned}$$

Furthermore,

$$\frac{\lambda+n\nu}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+n\nu} = 1-\frac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell}{\lambda+\sum\limits_{\ell=2}^{n+1}\sigma_\ell+n\nu},$$

so,

$$\frac{\lambda^{(1)}+n\nu^{(1)}}{\lambda^{(1)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}+n\nu^{(1)}} \geq \frac{\lambda^{(2)}+n\nu^{(2)}}{\lambda^{(2)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}+n\nu^{(2)}},$$

becomes

$$1-\frac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}}{\lambda^{(1)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}+n\nu^{(1)}} \geq 1-\frac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}}{\lambda^{(2)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}+n\nu^{(2)}},$$

which implies

$$\frac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}}{\lambda^{(1)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(1)}+n\nu^{(1)}} \leq \frac{\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}}{\lambda^{(2)}+\sum\limits_{\ell=2}^{n+1}\sigma_\ell^{(2)}+n\nu^{(2)}}. \tag{5}$$

Consequently, if $S_1^{(1)} \leq_{st} S_1^{(2)}$, and $S_\ell^{(1)} \leq_{st} S_\ell^{(2)}$, $\ell = 2, ..., n+1$, we obtain the result. $\square$

**Theorem 6.** *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $\nu^{(1)} \geq \nu^{(2)}$, $\sum_{\ell=2}^{n+1} \sigma_\ell^{(1)} \leq \sum_{\ell=2}^{n+1} \sigma_\ell^{(2)}$, $S_1^{(1)} \leq_v S_1^{(2)}$ *and* $S_\ell^{(1)} \leq_v S_\ell^{(2)}$, $\ell = 2, ..., n+1$, *then* $r^{(1)} \leq_v r^{(2)}$ *(i.e., for any distribution* $p$*, we have* $r^{(1)}p \leq_v r^{(2)}p$*).*

*Proof.* The proof is similar to that of Theorem 5. It is sufficient to substitute $\overline{p}_{nm}^{(1)} \leq \overline{p}_{nm}^{(2)}$ by $\overline{\overline{p}}_{nm}^{(1)} \leq \overline{\overline{p}}_{nm}^{(2)}$ and using Lemma 2 (with respect to convex order), which means substitute $\overline{k}_n^{(i)}$ by $\overline{\overline{k}}_n^{(i)}$, which implies, using inequalities (3)–(5)

$$
\begin{aligned}
& \frac{\lambda^{(1)} + n\nu^{(1)}}{\lambda^{(1)} + \sum_{\ell=2}^{n+1} \sigma_\ell^{(1)} + n\nu^{(1)}} (\overline{\overline{k}}_1^{m-n+1})^{(1)} + \frac{\sum_{\ell=2}^{n+1} \sigma_\ell^{(1)}}{\lambda^{(1)} + \sum_{\ell=2}^{n+1} \sigma_\ell^{(1)} + n\nu^{(1)}} (\overline{\overline{k}}_\ell^{m-n})^{(1)} \\
& - \frac{n\nu^{(1)}}{\lambda^{(1)} + \sum_{\ell=2}^{n+1} \sigma_\ell^{(1)} + n\nu^{(1)}} (\overline{\overline{k}}_1^{m-n})^{(1)} \\
\leq \; & \frac{\lambda^{(2)} + n\nu^{(2)}}{\lambda^{(2)} + \sum_{\ell=2}^{n+1} \sigma_\ell^{(2)} + n\nu^{(2)}} (\overline{\overline{k}}_1^{m-n+1})^{(2)} + \frac{\sum_{\ell=2}^{n+1} \sigma_\ell^{(2)}}{\lambda^{(2)} + \sum_{\ell=2}^{n+1} \sigma_\ell^{(2)} + n\nu^{(2)}} (\overline{\overline{k}}_\ell^{m-n})^{(2)} \\
& - \frac{n\nu^{(2)}}{\lambda^{(2)} + \sum_{\ell=2}^{n+1} \sigma_\ell^{(2)} + n\nu^{(2)}} (\overline{\overline{k}}_1^{m-n})^{(2)}.
\end{aligned}
$$

$\square$

## 5. STOCHASTIC COMPARABILITY CONDITIONS FOR THE STATIONARY NUMBER OF CUSTOMERS IN THE SYSTEM

The following theorems give comparability conditions of stationary distributions of the number of customers in the orbit for two system $M/G/1$ retrial queue with two-way communication and $n$ types of outgoing calls, with respect to stochastic and convex orders.

**Theorem 7.** *Let* $\pi_n^{(1)}$ *and* $\pi_n^{(2)}$ *be the stationary distributions of the number of customers in* $\Sigma_1$ *and* $\Sigma_2$*.If* $\lambda^{(1)} \leq \lambda^{(2)}$, $\nu^{(1)} \geq \nu^{(2)}$, $\sum_{\ell=2}^{n+1} \sigma_\ell^{(1)} \leq \sum_{\ell=2}^{n+1} \sigma_\ell^{(2)}$, $S_1^{(1)} \leq_{so} S_1^{(2)}$, $S_\ell^{(1)} \leq_{so} S_\ell^{(2)}$, $\sup_{2 \leq \ell \leq n+1} \{S_\ell\}^{(1)} \leq_{st} S_1^{(2)}$ *(resp.* $\inf_{2 \leq \ell \leq n+1} \{S_\ell\}^{(2)} \geq_v S_1^{(2)}$*), where* $so = st(or\ v)$ *then,*

$$\{\pi_n^{(1)}\} \leq_{st} \{\pi_n^{(2)}\}, \ (resp. \ \{\pi_n^{(1)}\} \geq_v \{\pi_n^{(2)}\}).$$

*Proof.* It is well known that the distribution of the number of costumers in the system at a steady state coincides with that of the system at the departure epoch. The stationary distribution coincides with the limite distribution, since the corresponding embedded Markov chain $\{Z_n, n \geq 1\}$ is ergodic. So, using Theorems 5 and 6 saying that $r^i$ are monotone, we have by induction

$$r^{(1)}p^{(1)} \leq_{so} r^{(2)}p^{(2)}, \tag{6}$$

for any two distributions $p^{(1)}$, $p^{(2)}$.

$$r^{(1)}p_n^{(1)} \quad = \quad P(Z_k^{(1)} = (C,n)) \leq_{so} P(Z_k^{(2)} = (C,n)) = r^{(2)}p_n^{(2)}.$$

When $k \longrightarrow \infty$, we have $\{\pi_n^{(1)}\} \leq_{so} \{\pi_n^{(2)}\}$, $so = st(or\ v)$. □

**Theorem 8.** *If the service time distributions of ingoing and outgoing call are* $NBUE$ *(New Better than Used in Expectation) (respectively* $NWUE$*- New Worse than Used in Expectation) in the* $M/G/1$ *retrial queue with two-way communication and n types of outgoing calls, and if* $S_1^{(1)} \leq_v S_1^{(2)} \equiv S_1^*$, $S_\ell^{(1)} \leq_v S_\ell^{(2)} \equiv S_\ell^*$ *and* $\inf_{2\leq\ell\leq n+1}\{S_\ell\} \geq_v S_1$*,then* $(\pi_n) \leq_v (\pi_n^*)$ *(respectively, greater relative to the convex ordering) where* $(\pi_n^*)$ *is the stationary distribution of the number of customers in the orbit for the* $M/M/1$ *retrial queue with two-way communication and n types of outgoing calls.*

*Proof.* Consider an $M/M/1$ retrial queue with two-way communication and $n$ types of outgoing calls with the same parameters as in the system $M/G/1$ retrial queue with two-way communication and $n$ types of outgoing calls: arrival rate (ingoing call) $\lambda$, retrial rate $\nu$, outgoing call rate $\sigma_\ell$, means service time $\delta_1^1$ and $\delta_\ell^1$ $\ell = 2, ..., n+1$, but with exponentially distributed service times, $\theta_1 = \frac{1}{\delta_1^1}$ and $\theta_\ell = \frac{1}{\delta_\ell^1}$ $\ell = 2, ..., n+1$.

$$S_1^*(x) = \begin{cases} 1 - e^{-\frac{x}{\delta_1^1}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \quad \text{and} \quad S_\ell^*(x) = \begin{cases} 1 - e^{-\frac{x}{\delta_\ell^1}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

From Stoyan [27], if $S_l(x)$, $l = 1, ..., n+1$ are $NBUE$ (respectively $NWUE$), then

$$S_l(x) \leq_v S_l^*(x), \quad \text{(respectively } \ S_l(x) \geq_v S_l^*(x)) \ \ l = 1, ..., n+1.$$

Since $S_\ell^{(1)} \leq_v S_\ell^{(2)}$ and $S_1^* \leq_v \inf_{2\leq\ell\leq n+1}\{S_\ell\}$, then using Theorem 7, we deduce that the stationary distribution of the number of costumers in the orbit of $M/G/1$ retrial queue with two-way communication and $n$ types of outgoing calls is smaller (respectively greater) than the stationary distribution of the number of costumers in the orbit of an $M/M/1$ retrial queue with two-way communication and $n$ types of outgoing calls. □

## 6. NUMERICAL EXAMPLE

In this section, we give a numerical illustration concerning Theorem 8. After developing a simulator, with Matlab environment, describing the behavior of the model $M/G/1$ retrial queue with two-way communication and $n$ types of outgoing calls, we assume that $n = 3$, we estimated the stationary probabilities of this system when the service time distribution is $NBUE$ (resp. $NWUE$). The results are being compared to those of the system $M/M/1$ with two-way communication and $n$ types of outgoing calls relative to the convex ordering. Thus, we set the incoming call rate $\lambda = 0.3$, the outgoing call rate $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $\sigma_3 = 0.15$, retrial rate $\nu = 1$, the simulation time $T_max = 1000$ time units and $k = 100$ (the number of replications).

| | $S_1(x)$ | $S_2(x)$ | $S_3(x)$ | $S_4(x)$ |
|---|---|---|---|---|
| $NBUE$ | $Wbl(2.5, 2)$ | $Wbl(2, 4)$ | $Wbl(3.11, 2)$ | $Wbl(1.78, 2)$ |
| $EXP$ | $\exp(0.61)$ | $\exp(1.00)$ | $\exp(0.65)$ | $\exp(0.55)$ |
| $NWUE$ | $Wbl(0.5, 0.3333)$ | $Wbl(0.66, 0.46)$ | $Wbl(0.5, 0.46)$ | $Wbl(0.42, 0.3333)$ |
| | $\Gamma(0.53, 2)$ | $\Gamma(0.56, 4)$ | $\Gamma(0.6, 4)$ | $\Gamma(0.48, 2)$ |

Table 1: Parameter setting for the numerical examples considered in all figures

By choosing one type of probability distributions $NBUE$ (a Weibull distribution ($Wbl(a, b)$ with $a > 1$), two other laws type $NWUE$ (a Weibull distribution ($Wbl(a, b)$ with $a \leq 1$) and a Gamma distribution ($\Gamma(a, b)$ with $0 \leq a < 1$) for service times of incoming and outgoing call ($S_1(x), S_\ell(x)$, $\ell = 2, 3, 4$) with different parameters (see Table 1), we observe that the following Figures (see Figure 1 which shows comparison of stationary distribution with respect to stochastic order, and Figure 2 representing comparison with respect to convex order) obtained by simulation confirm theoretical results. Consequently, performance measures of the considered system can be estimated by those of the system $M/M/1$ retrial queue with two-way communication and $n$ types of outgoing calls.

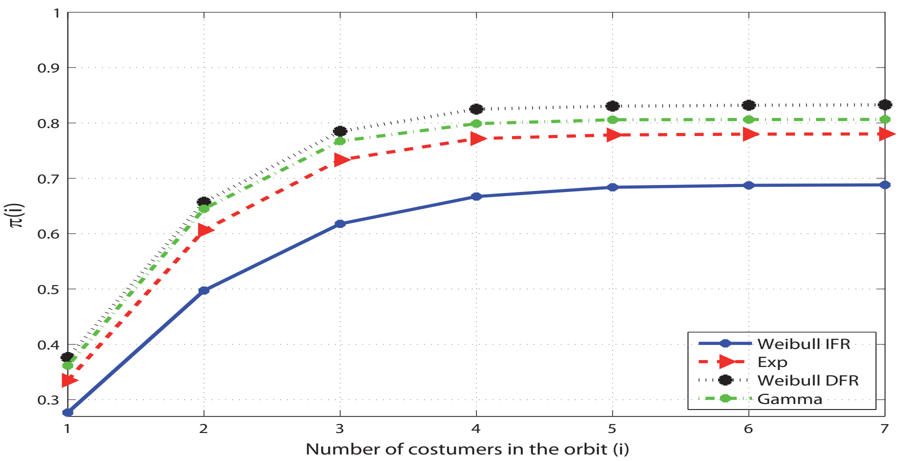


Figure 1: Comparison of the stationary probabilities for different laws with respect to stochastic ordering.

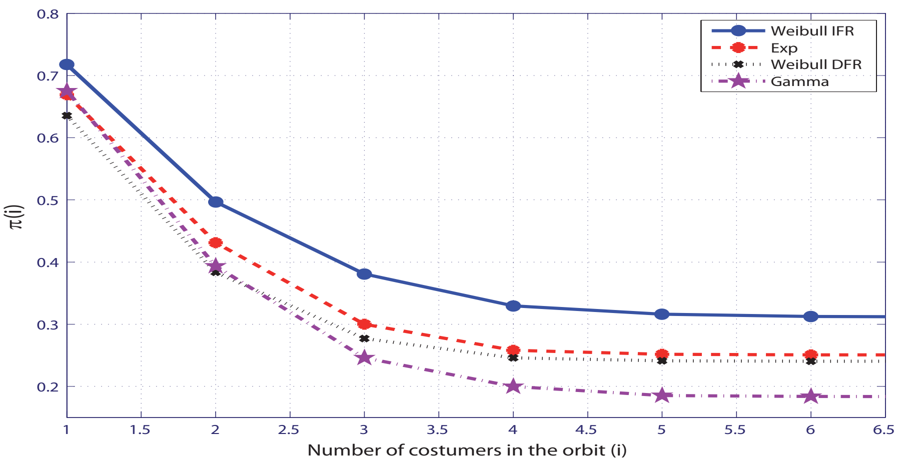


Figure 2: Comparison of the stationary probabilities for different laws with respect to the convex ordering.

## 7. CONCLUSION

In this work, we used the general theory of stochastic orderings to investigate the monotonicity properties of the considered model. We showed the monotonicity of the transition operator of the embedded Markov chain relative to the strong stochastic ordering, convex ordering, and we obtained comparability conditions of the number of customers in the system. Inequalities are also obtained for the stationary distribution of the number of costumers in the orbit. We discussed the conditions under which the comparison of this model with an $M/M/1$ retrial queue with two-way communication and $n$ types of outgoing calls, where all distributions are exponential, is valid. Hence, bounds of performance measures are derived. An illustrative numerical example is presented to confirm the theoretical results.

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