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SECOND-ORDER SYMMETRIC DUALITY IN MULTIOBJECTIVE VARIATIONAL PROBLEMS

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Abstract: In this work, we introduce a pair of multiobjective second-order symmetric dual variational problems. Weak, strong, and converse duality theorems for this pair are established under the assumption of η-bonvexity/η-pseudobonvexity. At the end, the static case of our problems has also been discussed.

Keywords: Multiobjective Programming, Variational Problem, Second-Order Duality, Efficient Solutions, η-bonvexity/η-pseudobonvexity.

MSC: 46N10, 52A41, 90C46, 49N15, 49M29.

1. INTRODUCTION

A pair of dual problems is called symmetric if the dual of the dual is the original problem, i.e., if we remodel the dual program in the form of the primal, its dual is the primal. The concept of symmetric dual programs was introduced and developed by Dorn [9] and Dantzig et al. [8].

Mond and Hanson [18] extended symmetric duality to variational problems. Since then, many authors [2, 3, 4, 11, 16, 19, 21] have worked on variational problems. Bector and Husain [6] formulated Wolfe and Mond-Weir type dual variational problems and established various duality results to relate properly efficient solutions of the primal and dual problems. Kim and Lee [15] constructed a pair of multiobjective symmetric dual variational programs and proved duality results for efficient solutions under invexity.

Mangasarian [17] introduced the concept of second and higher order duality for nonlinear problems. Since then, many authors [1, 5, 10, 13, 20] have worked in this area. Second-order duality for variational problems has been discussed in [7, 12, 14]. Husain et al. [14] formulated the following pair of the variational problem (CP) and its second-order dual (CD):

(CP) Minimize $\int_a^b f(t, x, \dot{x})dt$

Subject to

$$x(a) = 0 = x(b),$$

$$g(t, x, \dot{x}) \leqq 0, \ t \in I,$$

(CD) Maximize $\int_a^b (f(t, u, \dot{u}) - \frac{1}{2}\beta(t)^T F\beta(t))dt$

Subject to

$$u(a) = 0 = u(b),$$

$$f_u + y(t)^T g_u - D(f_{\dot{u}} + y(t)^T g_{\dot{u}}) + (F + H)\beta(t) = 0, \ t \in I,$$

$$\int_a^b \{y(t)^T g(t, u, \dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t)\}dt \geq 0,$$

$$y(t) \geq 0, \ t \in I,$$

where $f : I \times R^n \times R^n \to R,\ g : I \times R^n \times R^n \to R^m,\ x : I \to R^n,\ y : I \to R^m,\ \beta(t) : I \to R^n,\ t \in I,\ F = f_{uu} - Df_{u\dot{u}} + D^2 f_{\dot{u}\dot{u}}$ and $H = (y(t)^T g_u)_u - D(y(t)^T g_u)_{\dot{u}} + D^2(y(t)^T g_u)_{\dot{u}}$.

Gulati and Mehndiratta [12] modified the above dual as below:

($\widehat{\mathbf{CD}}$) Maximize $\int_a^b (f(t, u, \dot{u}) - \frac{1}{2}\beta(t)^T F\beta(t))dt$

Subject to

$$u(a) = 0 = u(b),$$

$$f_x(t,u,\dot{u}) + g_x(t,u,\dot{u})y(t) - D(f_{\dot{x}}(t,u,\dot{u}) + g_{\dot{x}}(t,u,\dot{u})y(t))$$

$$+(F+H)\beta(t) = 0,\ t \in I,$$

$$y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t)dt \geqq 0,\ t \in I,$$

$$y(t) \geqq 0, \quad t \in I,$$

where

$$H(t,u,\dot{u},\ddot{u},\dddot{u},\ddddot{u},y(t),\dot{y}(t),\ddot{y}(t),\dddot{y}(t)) = (g_x(t,u,\dot{u})y(t))_x - 2D(g_x(t,u,\dot{u})y(t))_{\dot{x}}$$

$$+D^2(g_{\dot{x}}(t,u,\dot{u})y(t))_{\dot{x}} - D^3(g_{\dot{x}}(t,u,\dot{u})y(t))_{\ddot{x}},\ t \in I$$

and

$$F(t,x,\dot{x},\ddot{x},\dddot{x},\ddddot{x}) = f_{xx}(t,x,\dot{x}) - 2Df_{x\dot{x}}(t,x,\dot{x})$$

$$+ D^2 f_{\dot{x}\dot{x}}(t,x,\dot{x}) - D^3 f_{\dot{x}\ddot{x}}(t,x,\dot{x}), \quad t \in I$$

The symbols are as defined above.

In this work, we introduce a pair of multiobjective second-order symmetric dual variational problems. Weak, strong and converse duality theorems for this pair are established under the assumption of η-bonvexity/η-pseudobonvexity. At the end, the static case of our problems has also been discussed.

2. PREREQUISITES

Let $K = \{1,2,...,k\}$ and for $r \in K$, the set $K_r = K - \{r\}$. The following convention for vector inequalities will be used:
for $a,\ b \in R^n$,

$a \geqq b \Leftrightarrow a_i \geqq b_i, i = 1,2,...,n$;
$a \geqslant b \Leftrightarrow a \geqq b$ and $a \neq b$;
$a > b \Leftrightarrow a_i > b_i, i = 1,2,...,n$.

We consider the following multiobjective variational problem (P) :

$$\textbf{(P)}\quad \text{Minimize } \left(\int_a^b \phi^1(t,x,\dot{x})dt,\ \int_a^b \phi^2(t,x,\dot{x})dt, ...,\ \int_a^b \phi^k(t,x,\dot{x})dt\right)$$

$$\text{Subject to}\quad x(a)=\alpha,\quad x(b)=\beta,$$

$$g(t,x,\dot{x}) \leqq 0,\ t\in I,$$

where $I=[a,b]$ is a real interval and $x(t),\ t\in I$ is an n-dimensional piecewise smooth continuous function with derivative $\dot{x}$. $\phi^i : I\times R^n\times R^n \to R\ (i\in K)$ and $g=(g^1,g^2,...,g^m)^T : I\times R^n\times R^n\to R^m$ are continuously differentiable functions. The symbols ϕ^i_x and $\phi^i_{\dot{x}}$ denote the column vectors of partial derivatives with respect to x and $\dot{x}$, respectively, i.e., $\phi^i_x=\left(\frac{\partial\phi^i}{\partial x^1},\frac{\partial\phi^i}{\partial x^2},...,\frac{\partial\phi^i}{\partial x^n}\right)^T$ and $\phi^i_{\dot{x}}=\left(\frac{\partial\phi^i}{\partial\dot{x}^1},\frac{\partial\phi^i}{\partial\dot{x}^2},...,\frac{\partial\phi^i}{\partial\dot{x}^n}\right)^T$. Similarly ϕ^i_{xx} denotes the $n\times n$ matrix with respect to x, i.e.,

$$\begin{pmatrix} \frac{\partial^2\phi^i}{\partial x^1\partial x^1} & \frac{\partial^2\phi^i}{\partial x^1\partial x^2} & \cdots & \frac{\partial^2\phi^i}{\partial x^1\partial x^n}\\ \frac{\partial^2\phi^i}{\partial x^2\partial x^1} & \frac{\partial^2\phi^i}{\partial x^2\partial x^2} & \cdots & \frac{\partial^2\phi^i}{\partial x^2\partial x^n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^2\phi^i}{\partial x^n\partial x^1} & \frac{\partial^2\phi^i}{\partial x^n\partial x^2} & \cdots & \frac{\partial^2\phi^i}{\partial x^n\partial x^n} \end{pmatrix}$$

and g_x denotes the $m\times n$ Jacobian matrix with respect to x, i.e.,

$$\begin{pmatrix} \frac{\partial g_1}{\partial x^1} & \frac{\partial g_1}{\partial x^2} & \cdots & \frac{\partial g_1}{\partial x^n}\\ \frac{\partial g_2}{\partial x^1} & \frac{\partial g_2}{\partial x^2} & \cdots & \frac{\partial g_2}{\partial x^n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial g_m}{\partial x^1} & \frac{\partial g_m}{\partial x^2} & \cdots & \frac{\partial g_m}{\partial x^n} \end{pmatrix}$$

The partial derivatives $\phi^i_{x\dot{x}},\ \phi^i_{\dot{x}x}$ and $g_{\dot{x}}$ are defined similarly.

Let

$$M^i(t,x,\dot{x})=\phi^i_{xx}-2D\phi^i_{x\dot{x}}+D^2\phi^i_{\dot{x}\dot{x}}-D^3\phi^i_{\dot{x}\ddot{x}},\quad t\in I.$$

Definition 1. *[14] The functional* $\int_a^b \phi^i(t,x,\dot{x})dt$ *is said to be* η*-bonvex at* $u(t) \in R^n$ *if there exists a function* $\eta : I \times R^n \times R^n \to R^n$ *such that for all* $x(t) \in R^n,\ q^i(t) \in R^n,\ t \in I,$

$$\int_a^b \phi^i(t,x,\dot{x})dt - \int_a^b \phi^i(t,u,\dot{u})dt + \frac{1}{2}\int_a^b q^i(t)^T M^i(t,u,\dot{u})q^i(t)dt$$
$$\geqq \int_a^b \eta(t,x,u)^T(\phi_x^i(t,u,\dot{u}) - D\phi_{\dot{x}}^i(t,u,\dot{u}) + M^i(t,u,\dot{u})q^i(t))dt.$$

Let X denote the set of all feasible solutions of (P).

Definition 2. *[12] A point* $x^0(t) \in X$ *is said to be an efficient solution of (P) if there exists no* $x(t) \in X$ *such that*

$$\int_a^b \phi^r(t,x,\dot{x})dt < \int_a^b \phi^r(t,x^0,\dot{x}^0)dt, \quad for\ some \quad r \in K$$

and

$$\int_a^b \phi^i(t,x,\dot{x})dt \leqq \int_a^b \phi^i(t,x^0,\dot{x}^0)dt, \quad for\ all \quad i \in K_r.$$

3. SECOND-ORDER MOND-WEIR TYPE SYMMETRIC DUALITY

We present the following second-order symmetric dual multiobjective variational problems and prove duality theorems under η-bonvexity assumptions :

Primal (VP)**:**

Minimize $(\int_a^b (f^1(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p^1(t)^T A^1 p^1(t))dt, ..., \int_a^b (f^k(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p^k(t)^T A^k p^k(t))dt)$

Subject to

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b), \tag{1}$$

$$y(a) = 0 = y(b), \quad \dot{y}(a) = 0 = \dot{y}(b), \tag{2}$$

$$\sum_{i=1}^k \lambda^i(f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) + A^i p^i(t)) \leqq 0, \quad t \in I, \tag{3}$$

$$y(t)^T \sum_{i=1}^{k} \lambda^i (f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) + A^i p^i(t)) \geqq 0, \quad t \in I, \tag{4}$$

$$\lambda > 0, \tag{5}$$

Dual (VD)**:**

Maximize $(\int_a^b (f^1(t,u,\dot{u},v,\dot{v}) - \frac{1}{2} q^1(t)^T B^1 q^1(t))dt, ..., \int_a^b (f^k(t,u,\dot{u},v,\dot{v}) - \frac{1}{2} q^k(t)^T B^k q^k(t))dt)$

Subject to

$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b), \tag{6}$$

$$v(a) = 0 = v(b), \quad \dot{v}(a) = 0 = \dot{v}(b), \tag{7}$$

$$\sum_{i=1}^{k} \lambda^i (f_x^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{x}}^i(t,u,\dot{u},v,\dot{v}) + B^i q^i(t)) \geqq 0, \quad t \in I, \tag{8}$$

$$u(t)^T \sum_{i=1}^{k} \lambda^i (f_x^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{x}}^i(t,u,\dot{u},v,\dot{v}) + B^i q^i(t)) \leqq 0, \quad t \in I, \tag{9}$$

$$\lambda > 0, \tag{10}$$

where, for all $i \in K$,

(i) $\lambda^i \in R, \quad \lambda = (\lambda^1, \lambda^2, ..., \lambda^k),$
(ii) $f^i : I \times R^n \times R^n \times R^m \times R^m \to R,$
(iii) $A^i(t,x,\dot{x},y,\dot{y}) = f_{yy}^i - 2Df_{y\dot{y}}^i + D^2 f_{\dot{y}\dot{y}}^i - D^3 f_{\dot{y}\ddot{y}}^i, \quad t \in I,$
(iv) $B^i(t,x,\dot{x},y,\dot{y}) = f_{xx}^i - 2Df_{x\dot{x}}^i + D^2 f_{\dot{x}\dot{x}}^i - D^3 f_{\dot{x}\ddot{x}}^i \quad t \in I,$
(v) $p^i : I \to R^m, \quad q^i : I \to R^n.$

All the derivatives of x, and all the partial and total derivatives of f used in this paper are assumed to be continuous.

4. DUALITY THEOREMS

Let F and G be sets of all feasible solutions of the primal problem (VP) and its Mond-Weir type dual problem (VD) respectively. Let $\eta_1 : I \times R^n \times R^n \to R^n$ and $\eta_2 : I \times R^m \times R^m \to R^m$.

Theorem 3. *(Weak duality). Let*

(i) $(x(t), y(t), \lambda, p(t)) \in F$ *and* $(u(t), v(t), \lambda, q(t)) \in G,$
(ii) $\eta_1(t,x,u) + u \geqq 0$ *and* $\eta_2(t,v,y) + y \geqq 0,$
(iii) $\int_a^b f^i(t,.,.,v(t),\dot{v}(t))dt$ *be* η_1*-bonvex at* $u(t)$ *for fixed* $v(t)$*, and*

(iv) $-\int_a^b f^i(t, x(t), \dot{x}(t), ., .)dt$ *be* η_2*-bonvex at* $y(t)$ *for fixed* $x(t)$.

Then

$$\int_a^b (f^r(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p^r(t)^T A^r p^r(t))dt < \int_a^b (f^r(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q^r(t)^T B^r q^r(t))dt, \tag{11}$$

for some $r \in K$ *and*

$$\int_a^b (f^i(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p^i(t)^T A^i p^i(t))dt \leqq \int_a^b (f^i(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q^i(t)^T B^i q^i(t))dt, \tag{12}$$

for all $i \in K_r$, *can not hold.*

Proof: Suppose, to the contrary, that the inequalities (11) and (12) hold. Since $\lambda > 0$, we get

$$\int_a^b \sum_{i=1}^k \lambda^i [f^i(t,x,\dot{x},y,\dot{y}) - f^i(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}p^i(t)^T A^i p^i(t) + \frac{1}{2}q^i(t)^T B^i q^i(t)]dt < 0. \tag{13}$$

Inequality (8) and hypothesis (ii) yield,

$$(\eta_1(t,x,u) + u)^T \sum_{i=1}^k \lambda^i [f_x^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{x}}^i(t,u,\dot{u},v,\dot{v}) + B^i q^i(t)] \geqq 0, \quad t \in I.$$

Using the constraint (9), it reduces to

$$\eta_1^T(t,x,u) \sum_{i=1}^k \lambda^i [f_x^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{x}}^i(t,u,\dot{u},v,\dot{v}) + B^i q^i(t)] \geqq 0, \quad t \in I,$$

which implies

$$\int_a^b \eta_1^T(t,x,u) \sum_{i=1}^k \lambda^i [f_x^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{x}}^i(t,u,\dot{u},v,\dot{v}) + B^i q^i(t)]dt \geqq 0. \tag{14}$$

Since $\int_a^b f^i(t,.,.,v(t),\dot{v}(t))dt$ is η_1-bonvex at $u(t)$ for fixed $v(t)$,

$$\int_a^b [f^i(t,x,\dot{x},v,\dot{v}) - f^i(t,u,\dot{u},v,\dot{v}) + \tfrac{1}{2}q^i(t)^T B^i q^i(t)]dt \geqq$$

$$\int_a^b \eta_1^T(t,x,u)[f_x^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{x}}^i(t,u,\dot{u},v,\dot{v}) + B^i q^i(t)]dt. \quad (15)$$

Multiplying (15) by $\lambda^i > 0$, summing over all $i \in K$ and then using the inequality (14), we obtain

$$\int_a^b \sum_{i=1}^k [\lambda^i (f^i(t,x,\dot{x},v,\dot{v}) - f^i(t,u,\dot{u},v,\dot{v}) + \frac{1}{2}q^i(t)^T B^i q^i(t))]dt \geqq 0. \quad (16)$$

Similarly, inequality (3) and hypothesis (ii) give

$$(\eta_2(t,v,y) + y)^T \sum_{i=1}^k \lambda^i (f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) + A^i p^i(t)) \leqq 0, \ t \in I.$$

This along with inequality (4) yields

$$\eta_2^T(t,v,y) \sum_{i=1}^k \lambda^i (f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{y},y,\dot{y}) + A^i p^i(t)) \leqq 0, \ t \in I,$$

or

$$\int_a^b \eta_2^T(t,v,y) \sum_{i=1}^k \lambda^i (f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) + A^i p^i(t))dt \leqq 0. \quad (17)$$

Now, η_2-bonvexity of $-\int_a^b f^i(t,x(t),\dot{x}(t),.,.)dt$ at $y(t)$ for fixed $x(t)$, implies

$$\int_a^b f^i(t,x,\dot{x},y,\dot{y}) - f^i(t,x,\dot{x},v,\dot{v}) - \frac{1}{2}p^i(t)^T A^i p^i(t))dt$$

$$\geqq -\int_a^b \eta_2^T(t,v,y)(f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) + A^i p^i(t))dt.$$

Using $\lambda^i > 0$, $i = 1, 2, \ldots, k$, and (17), we get

$$\int_a^b \sum_{i=1}^k \lambda^i [f^i(t,x,\dot{x},y,\dot{y}) - f^i(t,x,\dot{x},v,\dot{v}) - \frac{1}{2}p^i(t)^T A^i p^i(t))dt \geqq 0. \quad (18)$$

The above inequality, along with (16), yields

$$\int_a^b \sum_{i=1}^k \lambda^i [f^i(t,x,\dot{x},y,\dot{y}) - f^i(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}p^i(t)^T A^i p^i(t) + \frac{1}{2}q^i(t)^T B^i q^i(t)]dt \geqq 0,$$

which contradicts (13). Hence, inequalities (11) and (12) can not hold.

In order to establish a strong duality theorem, we need the following Fritz John necessary optimality conditions [12] :

Theorem 4. *Let $\bar{x}(t)$ be an efficient solution of (P). Then there exist $\bar{\lambda}^i \in R,\ i \in K$ and a piecewise smooth function $\bar{y} : I \to R^m$ such that*

$$\sum_{i=1}^k \bar{\lambda}^i(\phi^i_x(t,\bar{x},\dot{\bar{x}}) - D\phi^i_{\dot{x}}(t,\bar{x},\dot{\bar{x}})) + g_x(t,\bar{x},\dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\bar{y}(t)) = 0, \quad t \in I,$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0, \quad t \in I,$$

$$(\bar{\lambda},\bar{y}(t)) \geq 0,\ t \in I.$$

In the following theorems, $(VP)_{\lambda^0}$ and $(VD)_{\lambda^0}$, respectively denote the problems (VP) and (VD) when λ is fixed to be λ^0.

Theorem 5. *(Strong duality). Assume that the assumptions of weak duality theorem are satisfied for all feasible solutions of (VP) and (VD). Fix $\lambda = \lambda^0$. Let*

(i) $(x^0(t), y^0(t), \lambda^0, p^0(t))$ be an efficient solution of (VP),
(ii) the matrices $A^i,\ t \in I,\ i \in K$, be nonsingular,
(iii) the set $\{f^i_y(t,x^0,\dot{x}^0,y^0,\dot{y}^0) - Df^i_{\dot{y}}(t,x^0,\dot{x}^0,y^0,\dot{y}^0) + A^i p^{i0}(t), t \in I, i \in K\}$ be linearly independent, and
(iv) the matrix

$$\sum_{i=1}^k \lambda^{i0}[(A^i p^{i0}(t))_y - D(A^i p^{i0}(t))_{\dot{y}} + D^2(A^i p^{i0}(t))_{\ddot{y}} - D^3(A^i p^{i0}(t))_{\dddot{y}} + D^4(A^i p^{i0}(t))_{\ddddot{y}}],\ t \in I,$$

be positive or negative definite.

Then $(x^0(t), y^0(t), \lambda^0, p^0(t) = 0)$ is an efficient solution of $(VD)_{\lambda^0}$.

Proof : Since $(x^0(t), y^0(t), \lambda^0, p^0(t))$ is an efficient solution of (VP), there exist $\alpha,\ \mu \in R^k$ and piecewise smooth functions $\beta : I \to R^m,\ \gamma : I \to R$, such that the following Fritz John conditions (Theorem 4.2) are satisfied at $(x^0(t), y^0(t), \lambda^0, p^0(t))$:

$$\sum_{i=1}^k \alpha^i[f^i_x - Df^i_{\dot{x}} - \tfrac{1}{2}(p^{i0}(t)^T A^i p^{i0}(t))_x + \tfrac{1}{2}D(p^{i0}(t)^T A^i p^{i0}(t))_{\dot{x}} - \tfrac{1}{2}D^2(p^{i0}(t)^T A^i p^{i0}(t))_{\ddot{x}}$$

$$+\tfrac{1}{2}D^3(p^{i0}(t)^T A^i p^{i0}(t))_{\dddot{x}} - \tfrac{1}{2}D^4(p^{i0}(t)^T A^i p^{i0}(t))_{\ddddot{x}}] + (\beta - \gamma y^0)\sum_{i=1}^k \lambda^{i0}[f^i_{yx} - Df^i_{y\dot{x}}$$

$$-Df^i_{\dot{y}x}+D^2f^i_{\dot{y}\dot{x}}-D^3f^i_{\dot{y}\ddot{x}}+(A^ip^{i0}(t))_x-D(A^ip^{i0}(t))_{\dot{x}}+D^2(A^ip^{i0}(t))_{\ddot{x}}-D^3(A^ip^{i0}(t))_{\dddot{x}}$$

$$+D^4(A^ip^{i0}(t))_{\ddddot{x}}]=0, \quad t\in I, \tag{19}$$

$$\sum_{i=1}^{k}\alpha^i[f^i_y-Df^i_{\dot{y}}-\tfrac{1}{2}(p^{i0}(t)^TA^ip^{i0}(t))_y+\tfrac{1}{2}D(p^{i0}(t)^TA^ip^{i0}(t))_{\dot{y}}-\tfrac{1}{2}D^2(p^{i0}(t)^TA^ip^{i0}(t))_{\ddot{y}}+$$
$$\tfrac{1}{2}D^3(p^{i0}(t)^TA^ip^{i0}(t))_{\dddot{y}}-\tfrac{1}{2}D^4(p^{i0}(t)^TA^ip^{i0}(t))_{\ddddot{y}}]+(\beta-\gamma y^0)\sum_{i=1}^{k}\lambda^{i0}[A^i+(A^ip^{i0}(t))_y$$
$$-D(A^ip^{i0}(t))_{\dot{y}}+D^2(A^ip^{i0}(t))_{\ddot{y}}-D^3(A^ip^{i0}(t))_{\dddot{y}}+D^4(A^ip^{i0}(t))_{\ddddot{y}}]$$
$$-\gamma(t)\sum_{i=1}^{k}\lambda^{i0}[f^i_y-Df^i_{\dot{y}}+A^ip^{i0}(t)]=0, \quad t\in I, \tag{20}$$

$$(\beta-\gamma y^0)^T[f^i_y-Df^i_{\dot{y}}+A^i(t)p^{i0}(t)]-\mu^i=0, \ t\in I, \ i\in K, \tag{21}$$

$$-\alpha^iA^ip^{i0}(t)+\lambda^{i0}A^i(\beta-\gamma y^0)=0, \ t\in I, \ i\in K, \tag{22}$$

$$\beta^T\sum_{i=1}^{k}\lambda^{i0}[f^i_y-Df^i_{\dot{y}}+A^ip^{i0}(t)]=0, \quad t\in I, \tag{23}$$

$$\gamma {y^0}^T\sum_{i=1}^{k}\lambda^{i0}[f^i_y-Df^i_{\dot{y}}+A^ip^{i0}(t)]=0, \quad t\in I, \tag{24}$$

$$\mu^T\lambda=0, \tag{25}$$

$$(\alpha,\beta(t),\gamma(t),\mu)\neq 0, \quad t\in I, \tag{26}$$

$$(\alpha,\beta(t),\gamma(t),\mu)\geqq 0, \quad t\in I. \tag{27}$$

Since $\lambda>0$, (25) implies $\mu=0$. Therefore from (21), we get

$$(\beta-\gamma y^0)^T(f^i_y-Df^i_{\dot{y}}+A^ip^{i0}(t))=0, \ t\in I, \quad i\in K. \tag{28}$$

As $A^i, \ t\in I, \ i\in K$ are nonsingular, from (22), it follows that

$$(\beta-\gamma y^0)\lambda^{i0}=\alpha^ip^{i0}(t), \ t\in I, \ i\in K. \tag{29}$$

Equation (20) can be written as

$$\sum_{i=1}^{k}(\alpha^i-\gamma\lambda^{i0})(f^i_y-Df^i_{\dot{y}})+\sum_{i=1}^{k}\lambda^{i0}A^i[(\beta-\gamma y^0)-\gamma p^{i0}(t)]+\sum_{i=1}^{k}[(A^ip^{i0}(t))_y-D(A^ip^{i0}(t))_{\dot{y}}$$
$$+D^2(A^ip^{i0}(t))_{\ddot{y}}-D^3(A^ip^{i0}(t))_{\dddot{y}}+D^4(A^ip^{i0}(t))_{\ddddot{y}}][(\beta-\gamma y^0)\lambda^{i0}-\frac{1}{2}\alpha^ip^{i0}(t)]=0$$

or using (29),

$$\sum_{i=1}^{k}(\alpha^i-\gamma\lambda^{i0})[f_y^i-Df_{\dot{y}}^i+A^ip^{i0}(t)]+\frac{1}{2}\sum_{i=1}^{k}\lambda^{i0}[(A^ip^{i0}(t))_y-D(A^ip^{i0}(t))_{\dot{y}}+D^2(A^ip^{i0}(t))_{\ddot{y}}$$
$$-D^3(A^ip^{i0}(t))_{\dddot{y}}+D^4(A^ip^{i0}(t))_{\ddddot{y}}](\beta-\gamma y^0)=0,\quad t\in I. \tag{30}$$

Premultiplying (30) by $(\beta-\gamma y^0)$ and using (28), we get

$$(\beta-\gamma y^0)^T\sum_{i=1}^{k}\lambda^{i0}[(A^ip^{i0}(t))_y-D(A^ip^{i0}(t))_{\dot{y}}+D^2(A^ip^{i0}(t))_{\ddot{y}}$$
$$-D^3(A^ip^{i0}(t))_{\dddot{y}}+D^4(A^ip^{i0}(t))_{\ddddot{y}}](\beta-\gamma y^0)=0,\quad t\in I,$$

which by hypothesis (iv) imply

$$\beta=\gamma y^0,\quad t\in I. \tag{31}$$

From (30) and (31),

$$\sum_{i=1}^{k}(\alpha^i-\gamma\lambda^{i0})[f_y^i-Df_{\dot{y}}^i+A^ip^{i0}(t)]=0,\quad t\in I.$$

Since the set $\{f_y^i-Df_{\dot{y}}^i+A^ip^{i0}(t),\ t\in I,\ \ i\in K\}$ is linearly independent,

$$\alpha^i=\gamma\lambda^{i0},\quad t\in I,\ i\in K. \tag{32}$$

Now, suppose $\gamma(t)=0$ for some $t=t_0$, i.e., $t_0\in I$ and $\gamma(t_0)=0$. Then relations (31) and (32) imply $\beta(t)=0$ and $\alpha^i=0,\ i\in K$, respectively. Hence $(\alpha,\beta(t_0),\gamma(t_0),\mu)=0$, which contradicts (27). Therefore

$$\gamma(t)>0,\ t\in I. \tag{33}$$

As $\lambda^{i0}>0,\ i\in K$, from (32) we conclude that

$$\alpha^i>0,\quad i\in K.$$

From (29) and (31),

$$\alpha^ip^{i0}(t)=0,\ t\in I,\quad i\in K,$$

and hence

$$p^{i0}(t)=0,\ t\in I,\quad i\in K.$$

Therefore (19) and (31) imply

$$\sum_{i=1}^{k}\alpha^i(f_x^i-Df_{\dot{x}}^i)=0,$$

which in view of (32) and (33) give

$$\sum_{i=1}^{k} \lambda^{i0}(f_x^i - Df_{\dot{x}}^i) = 0,$$

and so

$$x^{0T} \sum_{i=1}^{k} \lambda^{i0}(f_x^i - Df_{\dot{x}}^i) = 0.$$

Thus it follows that $(x^0(t), y^0(t), \lambda^0, p^0(t) = 0)$ is a feasible solution of $(VD)_{\lambda^0}$ and the objective function values of (VP) and $(VD)_{\lambda^0}$ are equal.

If $(x^0, y^0, \lambda^0, p^0 = 0)$ is not an efficient solution for $(VD)_{\lambda^0}$, then there exists a point $(u^0, v^0, \lambda^0, q^0) \in G$ such that

$$\Big(\int_a^b (f^1(t,u,\dot{u},v,\dot{v}) - \tfrac{1}{2}q^{10}(t)^T B^1 q^{10}(t))dt, ..., \int_a^b (f^k(t,u,\dot{u},v,\dot{v}) - \tfrac{1}{2}q^{k0}(t)^T B^k q^{k0}(t))dt\Big)$$
$$\geq \Big(\int_a^b (f^1(t,x,\dot{x},y,\dot{y}) - \tfrac{1}{2}p^{10}(t)^T A^1 p^{10}(t))dt, ..., \int_a^b (f^k(t,x,\dot{x},y,\dot{y}) - \tfrac{1}{2}p^{k0}(t)^T A^k p^{k0}(t))dt\Big),$$

which contradicts the conclusion of the weak duality theorem. Hence $(x^0, y^0, \lambda^0, p^0 = 0)$ is an efficient solution for $(VD)_{\lambda^0}$.

The converse duality theorem is stated below. Its proof is analogous to that of the strong duality theorem proved above.

Theorem 6. *(Converse duality). Assume that the assumptions of weak duality theorem are satisfied for all feasible solutions of (VP) and (VD). Fix $\lambda = \lambda^0$. Also, let*

(i) $(u^0(t), v^0(t), \lambda^0, q^0(t))$ be an efficient solution of (VD),
(ii) the matrices B^i, $t \in I$, $i \in K$, be nonsingular,
(iii) the set $\{f_x^i(t,u^0,\dot{u}^0,v^0,\dot{v}^0) - Df_{\dot{x}}^i(t,u^0,\dot{u}^0,v^0,\dot{v}^0) + B^i q^{i0}(t), t \in I, i \in K\}$ be linearly independent, and
(iv) the matrix $\sum_{i=1}^{k} \lambda^{i0}[(B^i q^{i0}(t))_x - D(B^i q^{i0}(t))_{\dot{x}} + D^2(B^i q^{i0}(t))_{\ddot{x}} - D^3(B^i q^{i0}(t))_{\dddot{x}} + D^4(B^i q^{i0}(t))_{\ddddot{x}}]$, $t \in I$, be positive or negative definite.

Then $(u^0(t), v^0(t), \lambda^0, q^0(t) = 0)$ is an efficient solution to $(VP)_{\lambda^0}$.

5. RELATED PROBLEMS

If the time dependency of (VP) and (VD) is removed, then these problems reduce to the following second-order symmetric multiobjective nonlinear problems studied by Suneja et al. [22], under the same hypotheses.

(SP)

Minimize $\quad (f^1(x,y) - \frac{1}{2}p^{1^T} f^1_{yy}(x,y)p^1, ..., f^k(x,y) - \frac{1}{2}p^{k^T} f^k_{yy}(x,y)p^k)$

Subject to

$$\sum_{i=1}^{k} \lambda^i (\nabla_y f^i(x,y) + \nabla_{yy} f^i(x,y)p^i) \leqq 0,$$

$$y^T \sum_{i=1}^{k} \lambda^i (\nabla_y f^i(x,y) + \nabla_{yy} f^i(x,y)p^i) \geqq 0,$$

$$\lambda > 0,$$

(SD)

Maximize $\quad (f^1(u,v) - \frac{1}{2}q^{1^T} f^1_{xx}(u,v)q^1, ..., f^k(u,v) - \frac{1}{2}q^{k^T} f^k_{xx}(u,v)q^k)$

Subject to

$$\sum_{i=1}^{k} \lambda^i (\nabla_x f^i(u,v) + \nabla_{xx} f^i(x,y)q^i) \geqq 0,$$

$$u^T \sum_{i=1}^{k} \lambda^i (\nabla_x f^i(u,v) + \nabla_{xx} f^i(u,v)q^i) \leqq 0,$$

$$\lambda > 0.$$

6. CONCLUSION

A pair of multiobjective second-order symmetric dual variational problems has been formulated and various daulity results have been proved assuming η-bonvexity on the functionals involved. It may be noted that these results can be extended to establish the duality relations for the second-order fractional variational programs and other related programming problems over cone constraints.

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