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ON NONSMOOTH MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS USING GENERALIZED CONVEXITY

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Abstract: In this paper, we derive the sufficient condition for global optimality for a nonsmooth mathematical program with equilibrium constraints involving generalized invexity.We formulate the Wolfe and Mond-Weir type dual models for the problem using convexificators.We establish weak and strong duality theorems to relate the mathematical program with equilibrium constraints and the dual models in the framework of convexificators.

Keywords: Duality, Stationary point, Generalized invexity, Convexificators.

MSC: 90C46,49J52.

1. INTRODUCTION

The concept of convexificators was introduced by Demyanov [6]. Convexificators has been employed to extend the results in optimization and nonsmooth analysis [14, 15, 30, 19]. It has been shown in [19] that the Clarke subdifferentials, Michel-Penot subdifferentials, and Treiman subdifferentials of a locally Lipschitz real-valued function are convexificators. For recent developments and results on convexificators, we refer to [16, 18, 17, 1] and the references therein.

A mathematical program with equilibrium constraints (MPEC) usually refers to an optimization problem in which the essential constraints are defined by complementarity system or a parametric variational inequality. There are many equilibrium phenomena that arise from economics and engineering, characterized by either a variational inequality or an optimization problem, which justifies the name mathematical program with equilibrium constraints (MPEC) for the smooth case [35, 10] and for the nonsmooth case [29, 28, 36]. Luo *et al.* [20] presented a comprehensive study of MPEC. By using the standard Fritz-John conditions, Flegel and Kanzow [8] obtained the optimality conditions for MPEC. Moreover, Flegel and Kanzow [9] introduced a new Slater type constraint qualification and a new Abadie type constraint qualification for the MPEC, and proved that the new Slater type constraint qualification implied a new Abadie type constraint qualification.

The class of MPEC is an extension of the class of bi-level programming problems, also known as the mathematical program with optimization constraints. By using the notion of convexificators, Ardali *et al.* [2] derived optimality conditions for MPEC. There are numerous real-world applications of MPEC, such as hydro-economic river basin model [4], chemical process engineering [31], design of transportation networks [12], and shape optimization [13].

It is well known that convexity and generalized convexity of a function play a significant role in optimization theory. One of the important generalization of a convex function is invex (invariant convex) function, which was introduced by Hanson [11] and later named by Craven [5]. For the last three decades, duality and optimality conditions in invex optimization theory have been discussed by several authors (see [3, 25, 22, 23]). Duality results are very useful and fruitful in the development of numerical algorithms for solving certain classes of the optimization problems. The existence of duality theory in the nonlinear programming problem helps to develop numerical algorithm as it provides suitable stopping rules for primal and dual problems. Also, duality theory is very important subject in the study of mathematical programming problems as weak duality gives a lower bound to the objective function of the primal problem. Wolfe [34], and Mond and Weir [27] dual models are most popular in nonlinear programming problems. Furthermore, these dual models have been abundantly studies for bi-level problems [33], semi- infinite programming problems [24], and mathematical programs with vanishing constraints (MPVC) [26].

In this paper, we derive the sufficient condition for global optimality for a mathematical program with equilibrium constraints using generalized invexity assumptions. We introduce Wolfe and Mond-Weir type dual programs to the MPEC and establish weak and strong duality theorems. The organization of this paper is as follows: in Section 2, we provide some preliminary definitions and results. In Section 3, we derive the sufficient optimality condition for MPEC, under generalized invexity assumptions. In Section 4, we establish weak and strong duality theorems relating to the MPEC and two dual models using invex function and generalized invex functions in the framework of convexificators. In Section 5, we conclude the results of this paper.

2. PRELIMINARIES

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with inner product $\langle .,. \rangle$ and C is a nonempty subset of $\mathbb{R}^n$. The convex hull of C is denoted by *co* C.

We consider the MPEC in the following form:

$$\begin{aligned} \text{MPEC} \quad & \min \quad F(u) \\ & \text{subject to}: \quad g(u) \leq 0, \ \ h(u) = 0, \\ & \qquad \theta(u) \geq 0, \ \psi(u) \geq 0, \ \langle \theta(u), \psi(u) \rangle = 0, \end{aligned}$$

where $F : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^k$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $\theta : \mathbb{R}^n \to \mathbb{R}^l$ and $\psi : \mathbb{R}^n \to \mathbb{R}^l$ are given functions. If we take $h(u) := 0$, $\theta(u) := 0$, $\psi(u) := 0$, then, the optimization problem with equilibrium constraint coincides with the standard nonlinear programming problem, which is well studied in the literature, see e.g., Mangasarian [21].

The feasible set of the problem MPEC is denoted by X and defined by

$$X := \{u \in \mathbb{R}^n : g(u) \leq 0, \ h(u) = 0, \ \theta(u) \geq 0, \ \psi(u) \geq 0, \ \langle \theta(u), \psi(u) \rangle = 0\}.$$

The following index sets will be used throughout the paper:

$$\begin{aligned} & I_g := I_g(\tilde{u}) := \{i = 1, 2, \ldots, k : g_i(\tilde{u}) = 0\}, \\ & \delta := \delta(\tilde{u}) := \{i = 1, 2, \ldots, l : \theta_i(\tilde{u}) = 0, \psi_i(\tilde{u}) > 0\}, \\ & \omega := \omega(\tilde{u}) := \{i = 1, 2, \ldots, l : \theta_i(\tilde{u}) = 0, \psi_i(\tilde{u}) = 0\}, \\ & \kappa := \kappa(\tilde{u}) := \{i = 1, 2, \ldots, l : \theta_i(\tilde{u}) > 0, \psi_i(\tilde{u}) = 0\}, \end{aligned}$$

where $\tilde{u} \in X$ is a feasible vector for the problem MPEC and the set ω denotes the degenerate set.

Definition 2.1. *Let $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, $u \in \mathbb{R}^n$, and let $F(u)$ be finite. Then, the lower and upper Dini directional derivatives of F at u in the direction y are defined, respectively, by*

$$F_d^-(u, y) := \liminf_{t \to 0^+} \frac{F(u + ty) - F(u)}{t},$$

and

$$F_d^+(u, y) := \limsup_{t \to 0^+} \frac{F(u + ty) - F(u)}{t}.$$

Definition 2.2. *(see [14]) A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have upper convexificators, $\partial^* F(u)$ at $u \in \mathbb{R}^n$ if $\partial^* F(u) \subseteq \mathbb{R}^n$ is a closed set and, for each $y \in \mathbb{R}^n$,*

$$F_d^-(u, y) \leq \sup_{\xi \in \partial^* F(u)} \langle \xi, y \rangle.$$

Definition 2.3. *(see [14]) A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have lower convexificators, $\partial_* F(u)$ at $u \in \mathbb{R}^n$ if $\partial_* F(u) \subseteq \mathbb{R}^n$ is a closed set and, for each $y \in \mathbb{R}^n$,*

$$F_d^+(u, y) \geq \inf_{\xi \in \partial_* F(u)} \langle \xi, y \rangle .$$

The function F is said to have a convexificator $\partial^* F(u) \subseteq \mathbb{R}^n$ at $u \in \mathbb{R}^n$, iff $\partial^* F(u)$ is both upper and lower convexificators of F at u.

Definition 2.4. *(see [7]) A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have upper semi-regular convexificators, $\partial^* F(u)$ at $u \in \mathbb{R}^n$ if $\partial^* F(u) \subseteq \mathbb{R}^n$ is a closed set and, for each $y \in \mathbb{R}^n$*

$$F_d^+(u, y) \leq \sup_{\xi \in \partial^* F(u)} \langle \xi, y \rangle . \tag{1}$$

Based on the definitions of an invex function [23] and generalized invex functions [32], we are introducing the definition of invex function and generalized invex functions in terms of convexificators.

Definition 2.5. *Let $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real valued function, which admit convexificator at $\tilde{u} \in \mathbb{R}^n$ and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a kernel function then, f is said to be*
(i) ∂^*-invex at $\tilde{u}$ with respect to η if for every $u \in \mathbb{R}^n$,

$$F(u) \geq F(\tilde{u}) + \langle \xi, \eta(u, \tilde{u}) \rangle, \forall\ \xi \in \partial^* F(\tilde{u}).$$

(ii) ∂^*-pseudoinvex at $\tilde{u}$ with respect to η if for every $u \in \mathbb{R}^n$,

$$\exists\ \xi \in \partial^* F(\tilde{u}),\ \langle \xi, \eta(u, \tilde{u}) \rangle \geq 0 \Rightarrow F(u) \geq F(\tilde{u}).$$

(iii) ∂^*-quasiinvex at $\tilde{u}$ with respect to η if for every $u \in \mathbb{R}^n$,

$$F(u) \leq F(\tilde{u}) \Rightarrow \langle \xi, \eta(u, \tilde{u}) \rangle \leq 0, \forall\ \xi \in \partial^* F(\tilde{u}).$$

We provide following examples in support of the definition of ∂^*-invex function and generalized ∂^*-invex functions respectively.
Example 2.1 Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by $F(u) = |u|$, if we take point $\tilde{u} = 0$, then the function becomes ∂^*-invex function at $\tilde{u} = 0$ with respect to the kernel function, $\eta(u, \tilde{u}) = \cos u \sin \tilde{u}$ and $\partial^* F(0) = \{-1, 1\}$.

Example 2.2 Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by $F(u) = |u|$, if we take point $\tilde{u} = 0$, then the function becomes ∂^*-pseudoinvex function at $\tilde{u} = 0$ with respect to the kernel function, $\eta(u, \tilde{u}) = \sin u\tilde{u}$ and $\partial^* F(0) = \{-1, 1\}$.

Example 2.3 Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by $F(u) = \sin u$, if we take point $\tilde{u} = \frac{\pi}{2}$, then the function becomes ∂^*-quasiinvex function at $\tilde{u} = \frac{\pi}{2}$ with respect to the kernel function, $\eta(u, \tilde{u}) = \cos u \sin \tilde{u}$ and $\partial^* F(\frac{\pi}{2}) = \{0\}$.

The following definitions of a generalized alternatively stationary point and a generalized strong stationary point are taken from Ardali *et.al.* [2].

Definition 2.6. *A feasible point $\tilde{u}$ of MPEC is called a generalized alternatively stationary (GA-stationary) point if there are vectors $\tau = (\tau^g, \tau^h, \tau^\theta, \tau^\psi) \in \mathbb{R}^{k+p+2l}$ and $\gamma = (\gamma^h, \gamma^\theta, \gamma^\psi) \in \mathbb{R}^{p+2l}$ satisfying the following conditions*

$$\begin{aligned} 0 \in co\partial^* F(\tilde{u}) &+ \sum_{i\in I_g} \tau_i^g co\partial^* g_i(\tilde{u}) + \sum_{m=1}^{p} \left[\tau_m^h co\partial^* h_m(\tilde{u}) + \gamma_m^h co\partial^*(-h_m)(\tilde{u})\right] \\ &+ \sum_{i=1}^{l} \left[\tau_i^\theta co\partial^*(-\theta_i)(\tilde{u}) + \tau_i^\psi co\partial^*(-\psi_i)(\tilde{u})\right] \\ &+ \sum_{i=1}^{l} \left[\gamma_i^\theta co\partial^*(\theta_i)(\tilde{u}) + \gamma_i^\psi co\partial^*(\psi_i)(\tilde{u})\right], \qquad (2) \\ &\tau_{I_g}^g \geq 0,\ \tau_m^h,\ \gamma_m^h \geq 0,\ m = 1, 2, \ldots, p, \qquad (3) \\ &\tau_i^\theta, \tau_i^\psi, \gamma_i^\theta, \gamma_i^\psi \geq 0,\ i = 1, 2, \ldots, l, \qquad (4) \\ &\tau_\kappa^\theta = \tau_\delta^\psi = \gamma_\kappa^\theta = \gamma_\delta^\psi = 0, \qquad (5) \\ &\forall\ i \in \omega, \gamma_i^\theta = 0 \text{ or } \gamma_i^\Psi = 0. \qquad (6) \end{aligned}$$

Definition 2.7. *A feasible point $\tilde{u}$ of MPEC is called a generalized strong stationary (GS-stationary) point if there are vectors $\tau = (\tau^g, \tau^h, \tau^\theta, \tau^\psi) \in \mathbb{R}^{k+p+2l}$ and $\gamma = (\gamma^h, \gamma^\theta, \gamma^\psi) \in \mathbb{R}^{p+2l}$ satisfying* (2)-(5) *together with the following condition*

$$\forall\ i \in \omega,\ \gamma_i^\theta = 0,\ \gamma_i^\Psi = 0.$$

In the next section, we show that under certain MPEC generalized invexity assumptions, generalized alternatively (GA) -stationarity turns into a global sufficient optimality condition.

3. OPTIMALITY CONDITION

We consider the following index sets:

$$\begin{aligned} \omega_\gamma^\theta &:= \{i \in \omega : \gamma_i^\psi = 0, \gamma_i^\theta > 0\}, \\ \omega_\gamma^\psi &:= \{i \in \omega : \gamma_i^\psi > 0, \gamma_i^\theta = 0\}, \\ \delta_\gamma^+ &:= \{i \in \delta : \gamma_i^\theta > 0\}, \\ \kappa_\gamma^+ &:= \{i \in \kappa : \gamma_i^\psi > 0\}. \end{aligned}$$

Theorem 3.1. *Let $\tilde{u}$ be a feasible GA-stationary point of MPEC, assume that F is ∂^*-pseudoinvex at $\tilde{u}$ with respect to the kernel η and $g_i(i \in I_g), \pm h_m(m = 1, 2, \ldots, p), -\theta_i(i \in \delta \cup \omega), -\psi_i(i \in \omega \cup \kappa)$ are ∂^*-quasiinvex at $\tilde{u}$ with respect to the common kernel η. If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then $\tilde{u}$ is a global optimal solution of MPEC.*

Proof. Let u be any arbitrary feasible point of MPEC, i.e.,

$$g_i(u) \leq 0 = g_i(\tilde{u}),\ \forall\ i \in I_g.$$

By ∂^*-quasiinvexity of g_i at $\tilde{u}$, we get

$$\langle \xi_i^g, \eta(u,\tilde{u})\rangle \leq 0,\ \forall\ \xi_i^g \in \partial^* g_i(\tilde{u}),\ \forall\ i \in I_g. \tag{7}$$

Similarly, we have

$$\Big\langle \zeta_m, \eta(u,\tilde{u})\Big\rangle \leq 0,\ \forall\ \zeta_m \in \partial^* h_m(\tilde{u}), \forall\ m = \{1,2,\ldots,p\}, \tag{8}$$

$$\Big\langle \nu_m, \eta(u,\tilde{u})\Big\rangle \leq 0,\ \forall\ \nu_m \in \partial^*(-h_m)(\tilde{u}), \forall\ m = \{1,2,\ldots,p\}, \tag{9}$$

$$\Big\langle \xi_i^\theta, \eta(u,\tilde{u})\Big\rangle \leq 0, \forall\ \xi_i^\theta \in \partial^*(-\theta_i)(\tilde{u}), \forall\ i \in \delta \cup \omega, \tag{10}$$

$$\Big\langle \xi_i^\Psi, \eta(u,\tilde{u})\Big\rangle \leq 0, \forall\ \xi_i^\Psi \in \partial^*(-\Psi_i)(\tilde{u}), \forall\ i \in \omega \cup \kappa. \tag{11}$$

If $\omega_\gamma^\theta \cup \omega_\gamma^\Psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, multiplying (7)-(11) by $\tau_i^g \geq 0\ (i \in I_g)$, $\tau_m^h > 0\ (m = 1,2,\ldots,p)$, $\gamma_m^h > 0\ (m = 1,2,\ldots,p)$, $\tau_i^\theta > 0\ (i \in \delta \cup \omega)$, $\tau_i^\Psi > 0\ (i \in \omega \cup \kappa)$, respectively and adding, we obtain

$$\left\langle \left(\sum_{i \in I_g} \tau_i^g \xi_i^g + \sum_{m=1}^{p} \left[\tau_m^h \zeta_m + \gamma_m^h \nu_m \right] + \sum_{i=1}^{l} \tau_i^\theta \xi_i^\theta + \sum_{i=1}^{l} \tau_i^\Psi \xi_i^\Psi \right), \eta(u,\tilde{u}) \right\rangle \leq 0,$$

for all $\xi_i^g \in co\partial^* g_i(\tilde{u})$, $\zeta_m \in co\partial^* h_m(\tilde{u})$, $\nu_m \in co\partial^*(-h_m)(\tilde{u})$, $\xi_i^\theta \in co\partial^*(-\theta_i)(\tilde{u})$ and $\xi_i^\Psi \in co\partial^*(-\Psi_i)(\tilde{u})$. Thus by GA-stationarity of $\tilde{u}$, we can select $\xi \in co\partial^* F(\tilde{u})$, so that,

$$\langle \xi, \eta(u,\tilde{u})\rangle \geq 0.$$

By ∂^*-pseudoinvexity of F at $\tilde{u}$ with respect to the common kernel η, we get $F(u) \geq F(\tilde{u})$ for all feasible points u. Hence $\tilde{u}$ is a global optimal solution of MPEC. □

The following example illustrates Theorem 3.1.

Example 3.1 Consider the following MPEC problem

$$\begin{array}{lll} \text{MPEC} & \min & F(u) = |u| \\ & \text{subject to}: & g(u) = -u^2 \leq 0, \\ & & \theta(u) = u^2 \geq 0, \\ & & \Psi(u) = |u| \geq 0, \\ & & \langle \theta(u), \Psi(u)\rangle = \langle u^2, |u|\rangle = 0. \end{array}$$

Here $F(u) = |u|$ is ∂^*-pseudoinvex at $\tilde{u} = 0$ with respect to the kernel, $\eta(u.\tilde{u}) = e^u\tilde{u}$. Further, $g, -\theta$ and $-\ \Psi$ are ∂^*-quasiinvex at $\tilde{u} = 0$ with respect to the common kernel, $\eta(u,\tilde{u}) = e^u\tilde{u}$. The feasible point for the given MPEC is $\tilde{u} = 0$. We have $co\partial^* F(0) = [-1,1]$, $co\partial^* g(0) = \{0\}$, $co\partial^*(-\theta)(0) = \{0\}$ and $co\partial^*(-\Psi)(0) = [-1,1]$. One can easily verify that there exist $\tau^g = 1, \tau^\theta = 1$, and $\tau^\Psi = 1$ such that $\tilde{u} = 0$ is a GA- stationary point, and $\tilde{u} = 0$ is a global optimal solution for the given primal problem MPEC. Hence, the assumptions of the Theorem 3.1 are satisfied.

Remark 3.2. *Based on the Definition 2.5, the definitions of an invex function and generalized invex functions can also be given in terms of upper semi-regular convexificators.*

4. DUALITY

In this section, we formulate and study a Wolfe type dual problem for the problem MPEC using the ∂^*-invexity. We also formulate Mond-Weir type dual problem and study the problem MPEC using ∂^*-invexity and generalized ∂^*-invexity assumptions.

The formulation of Wolfe type dual problem for the problem MPEC is as follows:

$$\text{WD} \qquad \max_{v,\tau} \left\{ F(v) + \sum_{i\in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v) \right] \right\}$$

subject to :

$$\begin{aligned} 0 \in co\partial^* F(v) &+ \sum_{i\in I_g} \tau_i^g co\partial^* g_i(v) + \sum_{m=1}^{p} \left[\tau_m^h co\partial^* h_m(v) + \gamma_m^h co\partial^*(-h_m)(v) \right] \\ &+ \sum_{i=1}^{l} \left[\tau_i^\theta co\partial^*(-\theta_i)(v) + \tau_i^\psi co\partial^*(-\psi_i)(v) \right], \\ &\tau_{I_g}^g \geq 0,\ \tau_m^h,\ \gamma_m^h \geq 0,\ m = 1,2,\ldots,p, \\ &\tau_i^\theta, \tau_i^\psi, \gamma_i^\theta, \gamma_i^\psi \geq 0,\ i = 1,2,\ldots,l, \\ &\tau_\kappa^\theta = \tau_\delta^\psi = \gamma_\kappa^\theta = \gamma_\delta^\psi = 0,\ \forall\, i \in \omega,\ \gamma_i^\theta = 0,\ \gamma_i^\psi = 0, \end{aligned} \tag{12}$$

where,

$$\rho_m^h = \tau_m^h - \gamma_m^h, \tau = (\tau^g, \tau^h, \tau^\theta, \tau^\psi) \in \mathbb{R}^{k+p+2l} \text{ and } \gamma = (\gamma^h, \gamma^\theta, \gamma^\psi) \in \mathbb{R}^{p+2l}.$$

Theorem 4.1. *(Weak Duality) Let $\tilde{u}$ be feasible for the problem MPEC, (v,τ) be feasible for the dual WD and the index sets $I_g, \delta, \omega, \kappa$ are defined accordingly. Suppose that F, g_i $(i \in I_g)$, $\pm h_m$ $(m = 1,2,\ldots,p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-invex functions at v, with respect to the common kernel η. If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then for any u feasible for the problem MPEC, we have*

$$F(u) \geq F(v) + \sum_{i\in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v) \right].$$

Proof. Let us suppose that u is any feasible point for the problem MPEC. Then, we have

$$g_i(u) \leq 0,\ \forall\, i \in I_g \text{ and } h_m(u) = 0,\ \ \forall\, m = \{1,2,\ldots,p\}.$$

Since F is invex at v, with respect to the kernel η, then, it follows that

$$F(u) - F(v) \geq \langle \xi, \eta(u,v) \rangle, \forall\, \xi \in \partial^* F(v). \tag{13}$$

Similarly, we have

$$g_i(u) - g_i(v) \geq \langle \xi_i^g, \eta(u,v)\rangle, \quad \forall\, \xi_i^g \in \partial^* g_i(v), \forall\, i \in I_g, \tag{14}$$

$$h_m(u) - h_m(v) \geq \langle \zeta_m, \eta(u,v)\rangle, \quad \forall\, \zeta_m \in \partial^* h_m(v), \forall\, m = \{1,2,\ldots,p\}, \tag{15}$$

$$-h_m(u) + h_m(v) \geq \langle \nu_m, \eta(u,v)\rangle, \quad \forall\, \nu_m \in \partial^*(-h_m)(v), \forall\, m = \{1,2,\ldots,p\}, \tag{16}$$

$$-\theta_i(u) + \theta_i(v) \geq \langle \xi_i^\theta, \eta(u,v)\rangle, \quad \forall\, \xi_i^\theta \in \partial^*(-\theta_i)(v), \forall\, i \in \delta \cup \omega, \tag{17}$$

$$-\psi_i(u) + \psi_i(v) \geq \langle \xi_i^\psi, \eta(u,v)\rangle, \quad \forall\, \xi_i^\psi \in \partial^*(-\psi_i)(v), \forall\, i \in \omega \cup \kappa. \tag{18}$$

If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then multiplying (14)-(18) by $\tau_i^g \geq 0\ (i \in I_g)$, $\tau_m^h > 0\ (m = 1,2,\ldots,p)$, $\gamma_m^h > 0\ (m = 1,2,\ldots,p)$, $\tau_i^\theta > 0\ (i \in \delta \cup \omega)$, $\tau_i^\psi > 0\ (i \in \omega \cup \kappa)$, respectively and adding (13)- (18), we obtain

$$\begin{aligned} F(u) - F(v) &+ \sum_{i\in I_g} \tau_i^g g_i(u) - \sum_{i\in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\ &+ \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^\theta \theta_i(u) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) - \sum_{i=1}^{l} \tau_i^\psi \psi_i(u) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \\ &\geq \left\langle \xi + \sum_{i\in I_g} \tau_i^g \xi_i^g + \sum_{m=1}^{p} \left[\tau_m^h \zeta_m + \gamma_m^h \nu_m\right] + \sum_{i=1}^{l} \left[\tau_i^\theta \xi_i^\theta + \tau_i^\psi \xi_i^\psi\right], \eta(u,v) \right\rangle. \end{aligned}$$

From (2), $\exists\ \tilde{\xi} \in co\partial^* F(v)$, $\tilde{\xi}_i^g \in co\partial^* g_i(v)$, $\tilde{\zeta}_m \in co\partial^* h_m(v)$, $\tilde{\nu}_m \in co\partial^*(-h_m)(v)$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(v)$ and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(v)$, such that

$$\tilde{\xi} + \sum_{i\in I_g} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] + \sum_{i=1}^{l} \left[\tau_i^\theta \tilde{\xi}_i^\theta + \tau_i^\psi \tilde{\xi}_i^\psi\right] = 0.$$

Therefore,

$$\begin{aligned} F(u) - F(v) &+ \sum_{i\in I_g} \tau_i^g g_i(u) - \sum_{i\in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\ &+ \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^\theta \theta_i(u) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) - \sum_{i=1}^{l} \tau_i^\psi \psi_i(u) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \geq 0. \end{aligned}$$

Now, using feasibility condition of MPEC, i.e, $g_i(u) \leq 0$, $h_m(u) = 0$, $\theta_i(u) \geq 0$, $\psi_i(u) \geq 0$, it follows that

$$F(u) - F(v) - \sum_{i\in I_g} \tau_i^g g_i(v) - \sum_{m=1}^{p} \tau_m^h h_m(v) + \sum_{m=1}^{p} \gamma_m^h h_m(v) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \geq 0.$$

Hence,

$$F(u) \geq F(v) + \sum_{i \in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v) \right],$$

where, $\rho_m^h = \tau_m^h - \gamma_m^h$, and the proof is completed. □

Theorem 4.2. *(Strong Duality) Let $\tilde{u}$ be a local optimal solution of the problem MPEC and assume that F is locally Lipschitz near $\tilde{u}$. Suppose that F, g_i $(i \in I_g)$, $\pm h_m$ $(m = 1, 2, \ldots, p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-invex functions at $\tilde{u}$ with respect to the common kernel η. If GS-ACQ [2] holds at $\tilde{u}$, then, $\exists\, \tilde{\tau} = (\tilde{\tau}^g, \tilde{\tau}^h, \tilde{\tau}^\theta, \tilde{\tau}^\psi) \in \mathbb{R}^{k+p+2l}$, such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of the dual WD and the corresponding objective values of MPEC and WD are equal.*

Proof. Since $\tilde{u}$ is a local optimal solution of the problem MPEC and the GS-ACQ is satisfied at $\tilde{u}$, now, using Corollary 4.6[2], i.e., $\exists\, \tilde{\tau} = (\tilde{\tau}^g, \tilde{\tau}^h, \tilde{\tau}^\theta, \tilde{\tau}^\psi) \in \mathbb{R}^{k+p+2l}$ and $\tilde{\gamma} \in (\tilde{\gamma}^h, \tilde{\gamma}^\theta, \tilde{\gamma}^\psi) \in \mathbb{R}^{p+2l}$, such that the GS-stationarity conditions for the problem MPEC are satisfied, it follows that $\exists\, \tilde{\xi} \in co\partial^* F(\tilde{u})$, $\tilde{\xi}_i^g \in co\partial^* g_i(\tilde{u})$, $\tilde{\zeta}_m \in co\partial^* h_m(\tilde{u})$, $\tilde{\nu}_m \in co\partial^*(-h_m)(\tilde{u})$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(\tilde{u})$ and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(\tilde{u})$, such that

$$\tilde{\xi} + \sum_{i \in I_g} \tilde{\tau}_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tilde{\tau}_m^h \tilde{\zeta}_m + \tilde{\gamma}_m^h \tilde{\nu}_m \right] + \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \tilde{\xi}_i^\theta + \tilde{\tau}_i^\psi \tilde{\xi}_i^\psi \right] = 0,$$
$$\tilde{\tau}_{I_g}^g \geq 0,\ \tilde{\tau}_m^h,\ \tilde{\gamma}_m^h \geq 0,\ \ m = 1, 2, \ldots, p,$$
$$\tilde{\tau}_i^\theta, \tilde{\tau}_i^\psi, \tilde{\gamma}_i^\theta, \tilde{\gamma}_i^\psi \geq 0,\ i = 1, 2, \ldots, l,$$
$$\tilde{\tau}_\kappa^\theta = \tilde{\tau}_\delta^\psi = \tilde{\gamma}_\kappa^\theta = \tilde{\gamma}_\delta^\psi = 0, \forall\, i \in \omega,\ \tilde{\gamma}_i^\theta = 0,\ \tilde{\gamma}_i^\psi = 0.$$

Therefore $(\tilde{u}, \tilde{\tau})$ is feasible for the dual WD. Now, using Theorem 4.1, we obtain

$$F(\tilde{u}) \geq F(v) + \sum_{i \in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v) \right], \tag{19}$$

where, $\rho_m^h = \tau_m^h - \gamma_m^h$, for any feasible solution (v, τ) for the dual WD. Using the feasibility condition of MPEC and its dual WD, i.e., for $i \in I_g(\tilde{u})$, $g_i(\tilde{u}) = 0$, $h_m(\tilde{u}) = 0$, $(m = 1, 2, \ldots, p)$, $\theta_i(\tilde{u}) = 0, \forall i \in \delta \cup \omega$, and $\psi_i(\tilde{u}) = 0, \forall i \in \omega \cup \kappa$, we get

$$F(\tilde{u}) = F(\tilde{u}) + \sum_{i \in I_g} \tilde{\tau}_i^g g_i(\tilde{u}) + \sum_{m=1}^{p} \tilde{\rho}_m^h h_m(\tilde{u}) - \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \theta_i(\tilde{u}) + \tilde{\tau}_i^\psi \psi_i(\tilde{u}) \right], \tag{20}$$

where, $\tilde{\rho}^h_m = \tilde{\tau}^h_m - \tilde{\gamma}^h_m$. Using (19) and (20), we obtain

$$F(\tilde{u}) + \sum_{i\in I_g} \tilde{\tau}^g_i g_i(\tilde{u}) + \sum_{m=1}^{p} \tilde{\rho}^h_m h_m(\tilde{u}) - \sum_{i=1}^{l} \left[\tilde{\tau}^\theta_i \theta_i(\tilde{u}) + \tilde{\tau}^\psi_i \psi_i(\tilde{u})\right]$$

$$\geq F(v) + \sum_{i\in I_g} \tau^g_i g_i(v) + \sum_{m=1}^{p} \rho^h_m h_m(v) - \sum_{i=1}^{l} \left[\tau^\theta_i \theta_i(v) + \tau^\psi_i \psi_i(v)\right].$$

Hence, $(\tilde{u}, \tilde{\tau})$ is an optimal solution for the dual WD and the corresponding objective values of MPEC and WD are equal. □

Now, we formulate the Mond-Weir type dual problem (MWD) for the problem MPEC and establish duality theorems using convexificators.

$$\text{MWD} \qquad \max_{v,\tau}\{F(v)\}$$

subject to:

$$0 \in co\partial^* F(v) + \sum_{i\in I_g} \tau^g_i co\partial^* g_i(v) + \sum_{m=1}^{p} \left[\tau^h_m co\partial^* h_m(v) + \gamma^h_m co\partial^*(-h_m)(v)\right]$$

$$+ \sum_{i=1}^{l} \left[\tau^\theta_i co\partial^*(-\theta_i)(v) + \tau^\psi_i co\partial^*(-\psi_i)(v)\right],$$

$$g_i(v) \geq 0 \ (i \in I_g), \ \ h_m(v) = 0 \ (m = 1, 2, \ldots, p),$$

$$\theta_i(v) \leq 0 \ (i \in \delta \cup \omega), \ \ \psi_i(v) \leq 0 \ (i \in \omega \cup \kappa),$$

$$\tau^g_{I_g} \geq 0, \ \tau^h_m, \gamma^h_m \geq 0, \ m = 1, 2, \ldots, p,$$

$$\tau^\theta_i, \tau^\psi_i, \gamma^\theta_i, \gamma^\psi_i \geq 0, \ i = 1, 2, \ldots, l,$$

$$\tau^\theta_\kappa = \tau^\psi_\delta = \gamma^\theta_\kappa = \gamma^\psi_\delta = 0, \forall\, i \in \omega, \gamma^\theta_i = 0, \gamma^\psi_i = 0, \tag{21}$$

where, $\tau = (\tau^g, \tau^h, \tau^\theta, \tau^\psi) \in \mathbb{R}^{k+p+2l}$ and $\gamma = (\gamma^h, \gamma^\theta, \gamma^\psi) \in \mathbb{R}^{p+2l}$.

Theorem 4.3. *(Weak Duality) Let $\tilde{u}$ be feasible for the problem MPEC, (v, τ) be feasible for the dual MWD and the index sets $I_g, \delta, \omega, \kappa$ be defined accordingly. Suppose that F, g_i $(i \in I_g)$, $\pm h_m$ $(m = 1, 2, \ldots, p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-invex functions at v, with respect to the common kernel η. If $\omega^\theta_\gamma \cup \omega^\psi_\gamma \cup \delta^+_\gamma \cup \kappa^+_\gamma = \phi$, then for any u feasible for the problem MPEC, we have*

$$F(u) \geq F(v).$$

Proof. Since f is invex at v, with respect to the kernel η, then, we have

$$F(u) - F(v) \geq \left\langle \xi, \eta(u, v)\right\rangle, \forall\, \xi \in \partial^* F(v). \tag{22}$$

Similarly, we have

$$g_i(u) - g_i(v) \geq \langle \xi_i^g, \eta(u,v)\rangle, \qquad \forall\ \xi_i^g \in \partial^* g_i(v), \forall\ i \in I_g, \tag{23}$$

$$h_m(u) - h_m(v) \geq \langle \zeta_m, \eta(u,v)\rangle, \quad \forall\ \zeta_m \in \partial^* h_m(v), \forall\ m = \{1,2,\ldots,p\}, \tag{24}$$

$$-h_m(u) + h_m(v) \geq \langle \nu_m, \eta(u,v)\rangle, \ \ \forall\ \nu_m \in \partial^*(-h_m)(v), \forall\ m = \{1,2,\ldots,p\}, \tag{25}$$

$$-\theta_i(u) + \theta_i(v) \geq \langle \xi_i^\theta, \eta(u,v)\rangle, \quad \forall\ \xi_i^\theta \in \partial^*(-\theta_i)(v), \forall\ i \in \delta \cup \omega, \tag{26}$$

$$-\psi_i(u) + \psi_i(v) \geq \langle \xi_i^\psi, \eta(u,v)\rangle, \quad \forall\ \xi_i^\psi \in \partial^*(-\psi_i)(v), \forall\ i \in \omega \cup \kappa. \tag{27}$$

If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, multiplying (23)-(27) by $\tau_i^g \geq 0$ $(i \in I_g)$, $\tau_m^h > 0$ $(m = 1,2,\ldots,p)$, $\gamma_m^h > 0$ $(m = 1,2,\ldots,p)$, $\tau_i^\theta > 0$ $(i \in \delta\cup\omega)$, $\tau_i^\psi > 0$ $(i \in \omega\cup\kappa)$, respectively and adding (22)-(27), we obtain

$$\begin{aligned}
F(u) - F(v) &+ \sum_{i\in I_g} \tau_i^g g_i(u) - \sum_{i\in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\
&+ \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^\theta \theta_i(u) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) - \sum_{i=1}^{l} \tau_i^\psi \psi_i(u) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \\
&\geq \left\langle \xi + \sum_{i\in I_g} \tau_i^g \xi_i^g + \sum_{m=1}^{p} \left[\tau_m^h \zeta_m + \gamma_m^h \nu_m\right] + \sum_{i=1}^{l} \left[\tau_i^\theta \xi_i^\theta + \tau_i^\psi \xi_i^\psi\right], \eta(u,v)\right\rangle.
\end{aligned}$$

From (21), $\exists\ \tilde{\xi} \in co\partial^* F(v)$, $\tilde{\xi}_i^g \in co\partial^* g_i(v), \tilde{\zeta}_m \in co\partial^* h_m(v)$, $\tilde{\nu}_m \in co\partial^*(-h_m)(v)$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(v)$ and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(v)$, such that

$$\tilde{\xi} + \sum_{i\in I_g} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] + \sum_{i=1}^{l} \left[\tau_i^\theta \tilde{\xi}_i^\theta + \tau_i^\psi \tilde{\xi}_i^\psi\right] = 0.$$

Therefore,

$$\begin{aligned}
F(u) - F(v) &+ \sum_{i\in I_g} \tau_i^g g_i(u) - \sum_{i\in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\
&+ \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^\theta \theta_i(u) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) - \sum_{i=1}^{l} \tau_i^\psi \psi_i(u) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \geq 0.
\end{aligned}$$

Now using the feasibility of u and v for MPEC and MWD, it follows that

$$F(u) \geq F(v).$$

Hence, the proof is completed. □

Theorem 4.4. *(Strong Duality) Let $\tilde{u}$ be a local optimal solution of the problem MPEC and let F be locally Lipschitz near $\tilde{u}$. Suppose that F, g_i $(i \in I_g)$, $\pm h_m$ $(m = 1, 2, \ldots, p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-invex functions at $\tilde{u}$ with respect to the common kernel η. If GS-ACQ [2] holds at $\tilde{u}$, then there exists $\tilde{\tau}$, such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of the dual MWD and the corresponding objective values of MPEC and MWD are equal.*

Proof. The proof can be done similar to the proof of Theorem 4.2 by invoking Theorem 4.3. □

Next, we establish weak duality and strong duality theorems for MPEC and its Mond-Weir type dual problem (MWD) under the assumptions of generalized ∂^*-invexity.

Theorem 4.5. *(Weak Duality) Let $\tilde{u}$ be feasible for the problem MPEC, (v, τ) be feasible for the dual MWD and the index sets $I_g, \delta, \omega, \kappa$ are defined accordingly. Suppose that F is ∂^*-pseudoinvex at v, with respect to the kernel η and g_i $(i \in I_g)$, $\pm h_m$ $(m = 1, 2, \ldots, p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-quasiinvex functions at v, with respect to the common kernel η. If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then for any u feasible for the problem MPEC, we have*

$$F(u) \geq F(v).$$

Proof. Assume that, for some feasible point u, such that $F(u) < F(v)$, then by ∂^*-pseudoinvexity of F at v, with respect to the kernel η, we get

$$\langle \xi, \eta(u, v) \rangle < 0, \forall\, \xi \in \partial^* F(v). \tag{28}$$

From (21), $\exists\, \tilde{\xi} \in co\partial^* F(v)$, $\tilde{\xi}_i^g \in co\partial^* g_i(v)$, $\tilde{\zeta}_m \in co\partial^* h_m(v)$, $\tilde{\nu}_m \in co\partial^*(-h_m)(v)$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(v)$ and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(v)$, such that

$$-\sum_{i \in I_g} \tau_i^g \tilde{\xi}_i^g - \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m \right] - \sum_{\delta \cup \omega} \tau_i^\theta \tilde{\xi}_i^\theta - \sum_{\omega \cup \kappa} \tau_i^\psi \tilde{\xi}_i^\psi \in \partial^* F(v). \tag{29}$$

By (28), we have

$$\left\langle \left(\sum_{i \in I_g} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m \right] + \sum_{\delta \cup \omega} \tau_i^\theta \tilde{\xi}_i^\theta + \sum_{\omega \cup \kappa} \tau_i^\psi \tilde{\xi}_i^\psi \right), \eta(u, v) \right\rangle > 0. \tag{30}$$

For each $i \in I_g$, $g_i(u) \leq 0 \leq g_i(v)$. Hence, by ∂^*-quasiinvexity, we obtain

$$\left\langle \xi_i^g, \eta(u, v) \right\rangle \leq 0, \forall\, \xi_i^g \in \partial^* g_i(v), \quad \forall\, i \in I_g. \tag{31}$$

Similarly, we have

$$\left\langle \zeta_m, \eta(u, v) \right\rangle \leq 0, \ \forall\, \zeta_m \in \partial^* h_m(v), \forall\, m = \{1, 2, \ldots, p\}, \tag{32}$$

for any feasible point v of the dual MWD, and for every $m, -h_m(v) = -h_m(u) = 0$. On the other hand, $-\theta_i(u) \leq -\theta_i(v), \forall i \in \delta \cup \omega$, and $-\psi_i(u) \leq -\psi_i(v), \forall i \in \omega \cup \kappa$. By ∂^*- quasiinvexity, we obtain

$$\left\langle \nu_m, \eta(u,v) \right\rangle \leq 0, \ \forall \ \nu_m \in \partial^*(-h_m)(v), \forall \ m = \{1,2,\ldots,p\}, \tag{33}$$

$$\left\langle \xi_i^{\theta}, \eta(u,v) \right\rangle \leq 0, \forall \ \xi_i^{\theta} \in \partial^*(-\theta_i)(v), \forall \ i \in \delta \cup \omega, \tag{34}$$

$$\left\langle \xi_i^{\psi}, \eta(u,v) \right\rangle \leq 0, \forall \ \xi_i^{\psi} \in \partial^*(-\psi_i)(v), \forall \ i \in \omega \cup \kappa. \tag{35}$$

From Eqs, (31)-(35), we have

$$\left\langle \tilde{\xi}_i^g, \eta(u,v) \right\rangle \leq 0 \ (i \in I_g), \ \left\langle \tilde{\zeta}_m, \eta(u,v) \right\rangle \leq 0, \ \left\langle \tilde{\nu}_m, \eta(u,v) \right\rangle \leq 0 \ (m = \{1,2,\ldots,p\}),$$
$$\left\langle \tilde{\xi}_i^{\theta}, \eta(u,v) \right\rangle \leq 0, \forall \ i \in \delta \cup \omega, \ \left\langle \tilde{\xi}_i^{\psi}, \eta(u,v) \right\rangle \leq 0, \forall \ i \in \omega \cup \kappa.$$

Since $\omega_{\gamma}^{\theta} \cup \omega_{\gamma}^{\psi} \cup \delta_{\gamma}^{+} \cup \kappa_{\gamma}^{+} = \phi$, we have

$$\left\langle \sum_{i \in I_g} \tau_i^g \tilde{\xi}_i^g, \eta(u,v) \right\rangle \leq 0, \ \left\langle \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\gamma}_m \right], \eta(u,v) \right\rangle \leq 0,$$
$$\left\langle \sum_{\delta \cup \omega} \tau_i^{\theta} \tilde{\xi}_i^{\theta}, \eta(u,v) \right\rangle \leq 0, \ \left\langle \sum_{\omega \cup \kappa} \tau_i^{\psi} \tilde{\xi}_i^{\psi}, \eta(u,v) \right\rangle \leq 0.$$

Therefore,

$$\left\langle \left(\sum_{i \in I_g} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m \right] + \sum_{\delta \cup \omega} \tau_i^{\theta} \tilde{\xi}_i^{\theta} + \sum_{\omega \cup \kappa} \tau_i^{\psi} \tilde{\xi}_i^{\psi} \right), \eta(u,v) \right\rangle \leq 0.$$

which contradicts (30). Therefore $F(u) \geq F(v)$. Hence the proof is completed. □

Theorem 4.6. *(Strong Duality) Let $\tilde{u}$ be a local optimal solution of the problem MPEC and let F be locally Lipschitz near $\tilde{u}$. Suppose that F is ∂^*-pseudoinvex at $\tilde{u}$, with respect to the kernel η, g_i $(i \in I_g)$, $\pm h_m$ $(m = 1,2,\ldots,p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-quasiinvex functions at $\tilde{u}$ with respect to the common kernel η. If GS-ACQ [2] holds at $\tilde{u}$, then there exists $\tilde{\tau}$, such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of the dual MWD and the respective objective values are equal.*

Proof. The proof can be done similar to the proof of Theorem 4.2 by invoking Theorem 4.5. □

5. CONCLUSIONS

We studied a mathematical program with equilibrium constraints (MPEC) and derived the sufficient conditions for global optimality for MPEC using generalized invexity assumptions. We formulated the Wolfe type and Mond-Weir type dual models for the problem MPEC in the framework of convexificators, and established weak and strong duality theorems relating to the problem MPEC and two dual models using ∂^*-invexity and generalized ∂^*-invexity assumptions.

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