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LAGRANGE DUALITY AND SADDLE POINT OPTIMALITY CONDITIONS FOR SEMI-INFINITE MATHEMATICAL PROGRAMMING PROBLEMS WITH EQUILIBRIUM CONSTRAINTS

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Abstract: In this paper, we consider a special class of optimization problems which contains infinitely many inequality constraints and finitely many complementarity constraints known as the semi-infinite mathematical programming problem with equilibrium constraints (SIMPEC). We propose Lagrange type dual model for the SIMPEC and obtain their duality results using convexity assumptions. Further, we discuss the saddle point optimality conditions for the SIMPEC. Some examples are given to illustrate the obtained results.

Keywords: Mathematical Programming Problems With Equilibrium Constraints, Duality, Saddle Point, Semi-Infinite Programming.

MSC: 90C33, 90C34, 90C46.

1. INTRODUCTION

Mathematical programming problem with equilibrium constraints is a constrained optimization problem in which constraints include some complementarity conditions. We consider the following semi-infinite mathematical programming problem with equilibrium constraints:

$$
\begin{aligned}
(SIMPEC) \qquad & \min & & f(z) & \\
& \text{subject to} & & g(z,t) \leq 0, & \forall\, t \in T, \\
& & & h(z) = 0, & \\
& & & G(z) \geq 0, & \\
& & & H(z) \geq 0, & \\
& & & G(z)^T H(z) = 0, &
\end{aligned}
$$

where the index set T is an infinite compact subset of $\mathbb{R}^n$. $f : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}^m$, $H : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^q$ and $g : \mathbb{R}^n \times T \to \mathbb{R}$ are continuously differentiable functions on $\mathbb{R}^n$.

Mathematical programming problems with equilibrium constraints belong to a difficult class of nonlinear optimization problems. Since the feasible region of these problems are not necessarily convex, many constraint qualifications like Abadie constraint qualification, Mangasarian-Fromovitz constraint qualification, Slater constraint qualification do not hold (see [28]). Mathematical programming problems with equilibrium constraints are applicable in many fields as hydro-economic river-basin model [1], chemical engineering process [18], traffic and telecommunications networks [19], etc. For more details on mathematical programming problems with equilibrium constraints we refer to [2, 3, 4, 5, 6, 8, 12, 15]. Pandey and Mishra [16] formulated Wolfe and Mond-Weir type dual models and established duality results for mathematical programming problems with equilibrium constraints. Singh *et al.* [21] discussed the Lagrange duality for mathematical programming problems with equilibrium constraints for saddle point criteria.

A semi-infinite programming problem (SIP) is a mathematical programming problem with finitely many variables and infinitely many constraints. SIP has been widely applied in many fields, such as transportation problem [10], robot trajectory design problem [13], engineering design problem [17], disjunctive programming [22], robust optimization and design centering problem [23] , optimal power flow problems in power systems with transient stability constraints [24], air pollution control problem [25], lapidary cutting problems [27]. Shapiro [20] gave many results on Lagrangian duality for convex semi-infinite programming problem. Mishra and Jaiswal [14] obtained necessary and sufficient optimality conditions for the (SIMPEC), also formulated Wolfe and Mond-Weir type dual models and established weak, strong and converse duality results. For more details on SIP, we refer to [7, 9, 11, 26] and references therein.

Motivated by Mishra and Jaiswal [14] and Singh *et al.* [21], we construct Lagrange type dual model for (SIMPEC) in smooth case and present weak and strong duality along with the concepts of saddle point. Although in this paper, we apply the idea of [21] to the semi-infinite programming, the duality model and the technique of proof is differ from those in [21]. In particular, we present the relationships between saddle point of (SIMPEC), optimal solutions of (SIMPEC), and its dual, and M-stationary point for (SIMPEC), which are not given in [21].

The organization of the paper is as follows: in Section 2, we give some definitions and preliminary results to be used in the rest of the paper. In Section 3, we propose Lagrange type dual model and establish weak and strong duality results. In Section 4, we derive saddle point optimality criteria for semi-infinite mathematical programming problems with equilibrium constraints. We illustrate our results by suitable examples.

2. PRELIMINARIES

In this section, we present some basic definitions and results, which will be used in this article. Let P be defined as the feasible region of the (SIMPEC). Let $\bar{z} \in P$ be any feasible solution of (SIMPEC). The following index sets will be used in the sequel.

$$T_g = T_g(\bar{z}) = \{t \in T : g(\bar{z}, t) = 0\},$$
$$\alpha = \alpha(\bar{z}) = \{i : G_i(\bar{z}) = 0\ ,\ H_i(\bar{z}) > 0\},$$
$$\beta = \beta(\bar{z}) = \{i : G_i(\bar{z}) = 0\ ,\ H_i(\bar{z}) = 0\},$$
$$\gamma = \gamma(\bar{z}) = \{i : G_i(\bar{z}) > 0\ ,\ H_i(\bar{z}) = 0\}.$$

In the standard nonlinear programming problem, there is only one stationary point condition that is KKT- type condition; but the mathematical programming problem with equilibrium constraints has many stationary point conditions like S-stationary point, M-stationary point, A-stationary point, C-stationary point, W-stationary point(see, [28]). The following concept of M-stationary point was introduced by Mishra and Jaiswal [14] for semi-infinite programming.

Definition 1. *A feasible point $\bar{z} \in P$ of the (SIMPEC) is called the Mordukhovich stationary point (M-stationary point) if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$, such that the following conditions hold:*

$$\nabla f(\bar{z}) + \sum_{i=1}^{l} \lambda_i^g \nabla g(\bar{z}, t_i) + \sum_{j=1}^{q} \lambda_j^h \nabla h_j(\bar{z}) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(\bar{z}) + \lambda_i^H \nabla H_i(\bar{z})] = 0, \quad (1)$$

$$\lambda_{T_g}^g \geq 0,\ \lambda_\gamma^G = 0,\ \lambda_\alpha^H = 0, \quad (2)$$

$$\forall i \in \beta \text{ either } \lambda_i^G > 0,\ \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0. \quad (3)$$

The following No Nonzero Abnormal Multiplier Constraint Qualification for (SIMPEC) is given by Mishra and Jaiswal [14].

Definition 2. *Let $\bar{z} \in P$ be a feasible point of the (SIMPEC). We say that the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) is satisfied at $\bar{z}$ if there is no* *nonzero vector* $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{l+q+2m}$ *and indices* $t_1, t_2, ..., t_l \in T_g(\bar{z}), l \leq n+1$, *such that the following conditions hold:*

$$\sum_{i=1}^{l} \lambda_i^g \nabla g(\bar{z}, t_i) + \sum_{j=1}^{q} \lambda_j^h \nabla h_j(\bar{z}) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(\bar{z}) + \lambda_i^H \nabla H_i(\bar{z})] = 0, \tag{4}$$

$$\lambda_{T_g}^g \geq 0,\ \lambda_\gamma^G = 0,\ \lambda_\alpha^H = 0, \tag{5}$$

$$\forall i \in \beta \text{ either } \lambda_i^G > 0,\ \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0. \tag{6}$$

The following necessary optimality condition given by Mishra and Jaiswal [14] presents the relationship between a local optimal solution and M-stationary point for the (SIMPEC).

Theorem 3. *Let* $\bar{z} \in P$ *be a local optimal solution for the (SIMPEC) where all functions are continuously differentiable at* $\bar{z}$. *Suppose that NNAMCQ is satisfied at* $\bar{z}$ *then* $\bar{z}$ *is an M-stationary point.*

Definition 4. *A differentiable function* f *defined on a nonempty open convex subset* $X \subset \mathbb{R}^n$ *is convex on* X *if and only if for all* $z,\ \bar{z} \in X$, *we have*

$$f(z) - f(\bar{z}) \geq \langle \nabla f(\bar{z}), z - \bar{z} \rangle. \tag{7}$$

3. SIMPEC LAGRANGE DUALITY

In this section, we present the Lagrange type dual model and find out weak and strong duality relationship between the primal and their dual models.
Let

$$\varphi(\lambda) = \min_{z \in \mathbb{R}^n} L_{SIMPEC}(z, \lambda),$$

where

$$L_{SIMPEC}(z, \lambda) = f(z) + \sum_{i=1}^{l} \lambda_i^g g(z, t_i) + \sum_{j=1}^{q} \lambda_j^h h_j(z) - \sum_{i=1}^{m} [\lambda_i^G G_i(z) + \lambda_i^H H_i(z)],$$

is the SIMPEC Lagrangian, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{l+q+2m}$ and indices $t_1, t_2, ...,$ $t_l \in T_g(\bar{z}),\ l \leq n+1$.

We present the following Lagrange type dual model which depends on a feasible point $\bar{z} \in P$ for the (SIMPEC):

$$\begin{aligned} &\text{SIMPEC-LD}(\bar{z}) && \max\ \varphi(\lambda) \\ &\text{subject to} && \lambda_i^g \geq 0, t_i \notin T_g,\ \lambda_\gamma^G \geq 0,\ \lambda_\alpha^H \geq 0, \\ &&& \forall i \in \beta \text{ either } \lambda_i^G > 0,\ \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0. \end{aligned}$$

Let $P_D(\bar{z})$ be the feasible region of the SIMPEC-LD corresponding to a point $\bar{z} \in P$. The dual model SIMPEC-LD($\bar{z}$) depends on a feasible point $\bar{z} \in P$ of the

(SIMPEC). Now, we present the Lagrange type dual model which is independent of the (SIMPEC):

SIMPEC-LD	$\max \varphi(\lambda)$
subject to	$\lambda \in P_D = \cap_{z \in P} P_D(z),$

where $P_D = \cap_{z \in P} P_D(z) \neq \emptyset$ is feasible region of the SIMPEC-LD.

Remark 5. *The feasible solution of the SIMPEC-LD is also a feasible solution of the SIMPEC-LD(z) for all $z \in P$, since the feasible region of the SIMPEC-LD is the intersection of the feasible regions of the SIMPEC-LD(z) for all $z \in P$. Also, SIMPEC-LD is independent of the primal problem, therefore SIMPEC-LD will perform better dual model than SIMPEC-LD(z).*

Theorem 6. *(Weak Duality) Let z be feasible point for the (SIMPEC) and λ be the feasible point for the SIMPEC-LD(z), where $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$, then*

$$\varphi(\lambda) \leq f(z).$$

Proof. Since $z \in P$ and $\lambda \in P_D(z)$ are feasible points for the (SIMPEC) and the SIMPEC-LD(z) respectively, then we get

$$\begin{aligned} \varphi(\lambda) &= \min_{z \in \mathbb{R}^n} L_{SIMPEC}(z, \lambda) \\ &\leq f(z) + \sum_{i=1}^{l} \lambda_i^g g(z, t_i) + \sum_{j=1}^{q} \lambda_j^h h_j(z) - \sum_{i=1}^{m} \left[\lambda_i^G G_i(z) + \lambda_i^H H_i(z) \right]. \end{aligned}$$

Using the feasibilty conditions of the (SIMPEC) $g(z, t_i) \leq 0$, $h_j(z) = 0$, $-G_i(z) \leq 0$, $-H_i(z) \leq 0$, we have

$$\sum_{i=1}^{l} \lambda_i^g g(z, t_i) + \sum_{j=1}^{q} \lambda_j^h h_j(z) - \sum_{i=1}^{m} \left[\lambda_i^G G_i(z) + \lambda_i^H H_i(z) \right] \leq 0. \tag{8}$$

From (8), we have

$$\varphi(\lambda) \leq f(z).$$

The following results are direct consequences of the Theorem 6.

Corollary 7. *If z is a feasible solution for the (SIMPEC) and λ is a feasible solution for the SIMPEC-LD, where $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$, then*

$$\varphi(\lambda) \leq f(z).$$

Corollary 8. *If $\bar{z}$ and $\bar{\lambda}$ are the optimal solutions for the (SIMPEC) and the SIMPEC-LD, respectively, where $\bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$, then*

$$\varphi(\bar{\lambda}) \leq f(\bar{z}).$$

Corollary 9. *If $\bar{z}$ and $\bar{\lambda}$ are feasible solutions for the (SIMPEC) and the SIMPEC-LD, respectively, and $f(\bar{z}) = \varphi(\bar{\lambda})$, then $\bar{z}$ and $\bar{\lambda}$ are optimal solutions for the (SIMPEC) and the SIMPEC-LD, respectively where $\bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$.*

The following example illustrates the result of Theorem 6.

Example 10. *Consider the following (SIMPEC):*

$$\begin{aligned} \min \quad f(z) &= z_1^2 + z_2^2 \\ \text{subject to} \quad g(z,t) &= -z_1 - z_2 + 1 - t \leq 0, \ \forall t \in [0,1], \\ G_1(z) &= z_1 \geq 0, \\ H_1(z) &= z_2 \geq 0, \\ G_1(z).H_1(z) &= z_1.z_2 = 0. \end{aligned}$$

The feasible region of the above problem is given by

$$S = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 \geq 1, \ z_2 \geq 1, \ z_1.z_2 = 0\}.$$

The Lagrange function for the (SIMPEC) is given by

$$L_{SIMPEC}(z, \lambda^g, \lambda^G, \lambda^H) = f(z) + \lambda^g g(z,0) - \lambda^G G_1(z) - \lambda^H H_1(z).$$

Therefore, we have

$$\begin{aligned} \varphi(\lambda^g, \lambda^G, \lambda^H) &= \min_{z \in \mathbb{R}^2} L_{SIMPEC}(z, \lambda^g, \lambda^G, \lambda^H), \\ &= -\frac{(\lambda^g + \lambda^G)^2}{4} - \frac{(\lambda^g + \lambda^H)^2}{4}. \end{aligned}$$

Let $P = \cup_{i=1,2} P^i$ where,

$$\begin{aligned} P^1 &= \{(z_1, z_2) : z_1 \geq 1, z_2 = 0\}, \\ P^2 &= \{(z_1, z_2) : z_1 = 0, z_2 \geq 1\}. \end{aligned}$$

Now, we get the following two Lagrange type dual problems for any feasible $z \in P^1$,

$$SIMPECLD_1(z) \qquad \max \varphi(\lambda^g, \lambda^G, \lambda^H) = -\frac{(\lambda^g + \lambda^G)^2}{4} - \frac{(\lambda^g + \lambda^H)^2}{4},$$

subject to $\lambda^g \geq 0, \lambda^G = 0, \lambda^H \in \mathbb{R}$.
For any $z \in P^2$,

$$SIMPECLD_2(z) \qquad \max \varphi(\lambda^g, \lambda^G, \lambda^H) = -\frac{(\lambda^g + \lambda^G)^2}{4} - \frac{(\lambda^g + \lambda^H)^2}{4},$$

subject to $\lambda^g \geq 0, \lambda^G \in \mathbb{R}, \lambda^H = 0$.

Let P_D^1 and P_D^2 be the feasible region of SIMPECLD$_1(z)$ and SIMPECLD$_2(z)$ respectively. Then, we have

$$\text{SIMPECLD} \qquad \max \varphi(\lambda^g, \lambda^G, \lambda^H) = -\frac{(\lambda^g + \lambda^G)^2}{4} - \frac{(\lambda^g + \lambda^H)^2}{4},$$

subject to $(\lambda^g, \lambda^G, \lambda^H) \in P_D$, where $P_D = \cap_{z \in P_i, i=1,2} P_D^i(z) = \{(\lambda^g, \lambda^G, \lambda^H) : \lambda^g \geq 0, \lambda^G = 0, \lambda^H = 0\}$, is feasible region for SIMPECLD. So, it is easy to show that

$$f(z) = z_1^2 + z_2^2 \geq -\frac{(\lambda^g + \lambda^G)^2}{4} - \frac{(\lambda^g + \lambda^H)^2}{4} = \varphi(\lambda^g, \lambda^G, \lambda^H),$$

which implies the weak duality theorem holds.

Before going to next result, we define some index sets as follow:

$$\begin{aligned}
J^+ &= \{j : \lambda_j^h > 0\}, \quad J^- = \{j : \lambda_j^h < 0\}, \\
\alpha^+ &= \{i \in \alpha : \lambda_i^G > 0\}, \quad \alpha^- = \{i \in \alpha : \lambda_i^G < 0\}, \\
\gamma^+ &= \{i \in \gamma : \lambda_i^H > 0\}, \quad \gamma^- = \{i \in \gamma : \lambda_i^H < 0\}, \\
\beta^+ &= \{i \in \beta : \lambda_i^G > 0, \lambda_i^H > 0\}, \\
\beta_G^+ &= \{i \in \beta : \lambda_i^G = 0, \lambda_i^H > 0\}, \quad \beta_G^- = \{i \in \beta : \lambda_i^G = 0, \lambda_i^H < 0\}, \\
\beta_H^+ &= \{i \in \beta : \lambda_i^H = 0, \lambda_i^G > 0\}, \quad \beta_H^- = \{i \in \beta : \lambda_i^H = 0, \lambda_i^G < 0\}.
\end{aligned}$$

The following theorem establishes strong duality relationship between the (SIMPEC) and the SIMPEC-LD($\bar{z}$) at a local optimal solution $\bar{z}$ of the (SIMPEC).

Theorem 11. *(Strong Duality) Let $\bar{z}$ be a local optimal solution of the (SIMPEC) such that the SIMPEC-NNAMCQ holds at $\bar{z}$. Suppose that f, $g(.,t)$ $(t \in T_g)$, h_j $(j \in J^+)$, $-h_j$ $(j \in J^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta^+ \cup \beta_H^+)$, $-H_i$ $(i \in \gamma^+ \cup \beta^+ \cup \beta_G^+)$ are convex functions at $\bar{z}$ on P. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \emptyset$ then, there exists $\bar{\lambda} \in \mathbb{R}^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$, such that $\bar{\lambda}$ is an optimal solution of SIMPEC-LD($\bar{z}$) and the respective objective values are equal.*

Proof. Since $\bar{z}$ is a local optimal solution of (SIMPEC) and the NNAMCQ is satisfied at $\bar{z}$. From Theorem 3, M-stationary conditions for (SIMPEC) are satisfied, that is, there exist $\bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in R^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$, such that the following conditions hold:

$$\nabla f(\bar{z}) + \sum_{i=1}^{l} \bar{\lambda}_i^g \nabla g(\bar{z}, t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h \nabla h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G \nabla G_i(\bar{z}) + \bar{\lambda}_i^H \nabla H_i(\bar{z})] = 0, \tag{9}$$

$$\bar{\lambda}_{T_g}^g \geq 0, \ \bar{\lambda}_\gamma^G = 0, \ \bar{\lambda}_\alpha^H = 0, \tag{10}$$

$$\forall i \in \beta \text{ either } \bar{\lambda}_i^G > 0, \ \bar{\lambda}_i^H > 0 \text{ or } \bar{\lambda}_i^G \bar{\lambda}_i^H = 0. \tag{11}$$

Therefore, $\bar{\lambda}$ is a feasible solution for SIMPEC-LD($\bar{z}$).
By the convexity of f at $\bar{z}$ on P, we obtain the following inequality for any $z \in P$

$$f(z) - f(\bar{z}) \geq \langle \nabla f(\bar{z}), z - \bar{z} \rangle. \tag{12}$$

Similarly, we have

$$g(z,t_i) - g(\bar{z},t_i) \geq \langle \nabla g(\bar{z},t_i), z-\bar{z}\rangle, \quad \forall t_i \in T_g(\bar{z}), \tag{13}$$

$$h_j(z) - h_j(\bar{z}) \geq \langle \nabla h_j(\bar{z}), z-\bar{z}\rangle, \quad \forall j \in J^+, \tag{14}$$

$$-h_j(z) + h_j(\bar{z}) \geq -\langle \nabla h_j(\bar{z}), z-\bar{z}\rangle, \quad \forall j \in J^-, \tag{15}$$

$$-G_i(z) + G_i(\bar{z}) \geq -\langle \nabla G_i(\bar{z}), z-\bar{z}\rangle, \quad \forall i \in \alpha^+ \cup \beta^+ \cup \beta_H^+, \tag{16}$$

$$-H_i(z) + H_i(\bar{z}) \geq -\langle \nabla H_i(\bar{z}), z-\bar{z}\rangle, \quad \forall i \in \alpha^+ \cup \beta^+ \cup \beta_G^+. \tag{17}$$

If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \emptyset$, multiplying (13)-(17), by $\bar{\lambda}_i^g \geq 0 \ \ (t_i \in T_g(\bar{z}))$, $\bar{\lambda}_j^h > 0 \ \ (j \in j^+)$, $-\bar{\lambda}_j^h > 0 \ \ (j \in j^-)$, $\bar{\lambda}_i^G > 0 \ \ (i \in \alpha^+ \cup \beta^+ \cup \beta_H^+)$, $\bar{\lambda}_i^H > 0$ $(i \in \gamma^+ \cup \beta^+ \cup \beta_G^+)$, respectively and adding (12)-(17), we get

$$\begin{aligned} f(z) \quad & -f(\bar{z}) + \sum_{i=1}^{l} \bar{\lambda}_i^g g(z,t_i) - \sum_{i=1}^{l} \bar{\lambda}_i^g g_i(\bar{z},t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(z) - \sum_{i=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) \\ & - \sum_{i=1}^{m} \bar{\lambda}_i^G G_i(z) + \sum_{i=1}^{m} \bar{\lambda}_i^G G_i(\bar{z}) - \sum_{i=1}^{m} \bar{\lambda}_i^H H_i(z) + \sum_{i=1}^{m} \bar{\lambda}_i^H H_i(\bar{z}) \\ \geq \Big\langle \nabla f(\bar{z}) & + \sum_{i=1}^{l} \bar{\lambda}_i^g \nabla g(\bar{z},t_i) \quad + \quad \sum_{j=1}^{q} \bar{\lambda}_j^h \nabla h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G \nabla G_i(\bar{z}) + \\ & \bar{\lambda}_i^H \nabla H_i(\bar{z})], z-\bar{z} \Big\rangle. \end{aligned} \tag{18}$$

From (9) and (18), for all $z \in P$, we get

$$\begin{aligned} f(z) \quad & -f(\bar{z}) + \sum_{i=1}^{l} \bar{\lambda}_i^g g(z,t_i) - \sum_{i=1}^{l} \bar{\lambda}_i^g g_i(\bar{z},t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(z) - \sum_{i=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) \\ & - \sum_{i=1}^{m} \bar{\lambda}_i^G G_i(z) + \sum_{i=1}^{m} \bar{\lambda}_i^G G_i(\bar{z}) - \sum_{i=1}^{m} \bar{\lambda}_i^H H_i(z) + \sum_{i=1}^{m} \bar{\lambda}_i^H H_i(\bar{z}) \geq 0. \end{aligned}$$

That is,

$$L_{SIMPEC}(z,\bar{\lambda}) \geq L_{SIMPEC}(\bar{z},\bar{\lambda}), \quad \forall\, z \in P. \tag{19}$$

Using the index sets and M-stationary condition (10) and (11) the following equality holds:

$$\sum_{i=1}^{l} \bar{\lambda}_i^g g(\bar{z},t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})] = 0.$$

Then, we get

$$f(\bar{z}) = L_{SIMPEC}(\bar{z}, \bar{\lambda}) = \varphi(\bar{\lambda}), \tag{20}$$

by weak duality theorem, we get

$$\varphi(\lambda) \leq f(\bar{z}), \quad \forall\ \lambda\ \in P_D(\bar{z}). \tag{21}$$

From (20) and (21), we get

$$\varphi(\lambda) \leq \varphi(\bar{\lambda}), \quad \forall\ \lambda\ \in P_D(z). \tag{22}$$

Thus, $\bar{\lambda} \in P_D(\bar{z})$ is a global optimal solution of SIMPEC-LD ($\bar{z}$). Optimal values of (SIMPEC) and the SIMPEC-LD($\bar{z}$) are equal. □

The following example illustrates the result of Theorem 11.

Example 12. *Consider the following (SIMPEC):*

$$\begin{array}{rcl} min \quad f(z) & = & z_1^2 + z_2^2 \\ subject\ to \quad g(z,t) & = & -z_1 - z_2 + t \leq 0,\ \forall t \in [-1, 0], \\ G_1(z) & = & z_1 \geq 0, \\ H_1(z) & = & z_2 \geq 0, \\ G_1(z).H_1(z) & = & z_1.z_2 = 0. \end{array}$$

The feasible region of the above problem is given by

$$P = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 \geq 0,\ z_2 \geq 0,\ z_1.z_2 = 0\}.$$

It is clear that $\bar{z} = (0, 0)$ *is local optimal solution of the above problem. Since there are no nonnegative* $(\lambda^g, \lambda^G, \lambda^H)$*, different from zero such that*

$$\lambda^g \nabla g(z, 0) - \lambda^G \nabla G_1(z) - \lambda^H H_1(z) = \lambda^g \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \lambda^G \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \lambda^H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

That is, NNAMCQ is satisfied at $\bar{z} = (0, 0)$*. We define the Lagrangian as follows*

$$L_{SIMPEC}(\bar{z}) = f(z) + \lambda^g g(z, 0) - \lambda^G G_1(z) - \lambda^H H_1(z).$$

Therefore, we have

$$\begin{array}{rcl} \varphi(\lambda^g, \lambda^G, \lambda^H) & = & min_{z \in \mathbb{R}^2} L_{SIMPEC}(z, \lambda^g, \lambda^G, \lambda^H) \\ & = & -\dfrac{(\lambda^g + \lambda^G)^2}{4} - \dfrac{(\lambda^g + \lambda^H)^2}{4}. \end{array}$$

Let $P = \cup_{i=1,2,3} P^i$,

$$\begin{array}{rcl} where\ P^1 & = & \{(z_1, z_2) : z_1 > 0, z_2 = 0\}, \\ P^2 & = & \{(z_1, z_2) : z_1 = 0, z_2 = 0\}, \\ P^3 & = & \{(z_1, z_2) : z_1 = 0, z_2 > 0\}. \end{array}$$

We get the following three Lagrange type problems:
For any $z \in P^1$,

$$SIMPECLD_1(\bar{z}) \qquad max\varphi(\lambda^g, \lambda^G, \lambda^H)$$
$$\text{subject to} \quad \lambda^g \geq 0, \ \lambda^G = 0, \ \lambda^H \in \mathbb{R}.$$

For any $z \in P^2$,

$$SIMPECLD_2(\bar{z}) \qquad max\varphi(\lambda^g, \lambda^G, \lambda^H)$$
$$\text{subject to} \quad \lambda^g \geq 0, \ \text{either } \lambda^G > 0, \ \lambda^H > 0 \ \text{or } \lambda^G\lambda^H = 0.$$

For any $z \in P^3$,

$$SIMPECLD_3(\bar{z}) \qquad max\varphi(\lambda^g, \lambda^G, \lambda^H)$$
$$\text{subject to} \quad \lambda^g \geq 0, \ \lambda^G \in \mathbb{R}, \ \lambda^H = 0.$$

Let P_D^1, P_D^2 *and* P_D^3 *be the feasible regions of the* $SIMPECLD_1(\bar{z})$, $SIMPECLD_2(\bar{z})$ *and* $SIMPECLD_3(\bar{z})$, *respectively. Also, we have*

$$SIMPECLD \qquad max\varphi(\lambda^g, \lambda^G, \lambda^H) = -\frac{(\lambda^g + \lambda^G)^2}{4} - \frac{(\lambda^g + \lambda^H)^2}{4}$$
$$\text{subject to} \quad (\lambda^g, \lambda^G, \lambda^H) \in P_D,$$

where $P_D = \cap_{z \in P^i, i=1,2,3} P_D^i(z) = \{(\lambda^g, \lambda^G, \lambda^H) \in \mathbb{R}^3 \mid \lambda^g \geq 0, \lambda^G = 0, \lambda^H = 0\}$, *is the feasible region of the above SIMPECLD. It is easy to see that for any* $(z_1, z_2) \in P$, *and* $(\lambda^g, \lambda^G, \lambda^H) \in P_D$,

$$z_1^2 + z_2^2 = -\frac{(\lambda^g + \lambda^G)^2}{4} - \frac{(\lambda^g + \lambda^H)^2}{4}, \tag{23}$$

is only possible for $z_1 = 0, z_2 = 0$ *and* $\lambda^g = 0, \lambda^G = 0, \lambda^H = 0$. *Since* $\bar{z} = (0, 0)$ *is a local (global) solution for (SIMPEC) and all assumptions of Theorem 3 hold at* $\bar{z}$ *therefore by Theorem 3, there exists* $(\bar{\lambda}^g, \bar{\lambda}^G, \bar{\lambda}^H)$ *with* $\bar{\lambda}^g = 0, \bar{\lambda}^G = 0, \bar{\lambda}^H = 0$, *such that strong duality hold.*

4. SADDLE POINT OPTIMALITY CONDITIONS FOR SIMPEC

In this section, we define saddle point condition for (SIMPEC) and establish the relationships between strong duality and saddle point conditions.

Definition 13. *A point* $(\bar{z}, \bar{\lambda})$ *with* $\bar{z} \in P$ *and* $\bar{\lambda} \in P_D(\bar{z})$ *is said to be saddle point for the Lagrange function* L_{SIMPEC}, *if*

$$L_{SIMPEC}(\bar{z}, \lambda) \leq L_{SIMPEC}(\bar{z}, \bar{\lambda}) \leq L_{SIMPEC}(z, \bar{\lambda}),$$

holds for all $z \in P$ *and* $\lambda \in P_D(\bar{z})$.

Theorem 14. *Let $\bar{z}$ be a local optimal solution of the (SIMPEC) and the conditions of strong duality theorem hold, then there exists $\bar{\lambda} \in \mathbb{R}^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$, such that $(\bar{z}, \bar{\lambda})$ is a saddle point of $L_{SIMPEC}(z, \lambda)$. Moreover, if $(\bar{z}, \bar{\lambda}) \in P \times P_D(\bar{z})$ is (SIMPEC) Lagrange saddle point, then $\varphi(\bar{\lambda}) = f(\bar{z})$, where $\bar{z}$ and $\bar{\lambda}$ are optimal solutions to the primal (SIMPEC) and dual SIMPEC-LD $(\bar{z})$, respectively.*

Proof. From (19), we have for all $z \in P$,

$$L_{SIMPEC}(\bar{z}, \bar{\lambda}) \leq L_{SIMPEC}(z, \bar{\lambda}). \tag{24}$$

From (20) the following inequality holds:

$$\begin{aligned} L_{SIMPEC}(\bar{z}, \bar{\lambda}) &= f(\bar{z}) \\ &\geq f(\bar{z}) + \sum_{i=1}^{l} \lambda_i^g g(\bar{z}, t_i) + \sum_{j=1}^{q} \lambda_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\lambda_i^G G_i(\bar{z}) + \lambda_i^H H_i(\bar{z})], \end{aligned}$$

that is, for all $\lambda \in P_D(\bar{z})$, we have

$$L_{SIMPEC}(\bar{z}, \lambda) \leq L_{SIMPEC}(\bar{z}, \bar{\lambda}). \tag{25}$$

From (24) and (25), $(\bar{z}, \bar{\lambda})$ is a saddle point of $L_{SIMPEC}(z, \lambda)$.
Let $(\bar{z}, \lambda) \in P \times P_D(\bar{z})$ is (SIMPEC) saddle point, hence

$$\begin{aligned} &L_{SIMPEC}(\bar{z}, \lambda) \leq L_{SIMPEC}(\bar{z}, \bar{\lambda}), \quad \forall \lambda \in P_D(\bar{z}), \\ &f(\bar{z}) + \sum_{i=1}^{l} \lambda_i^g g(\bar{z}, t_i) + \sum_{j=1}^{q} \lambda_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\lambda_i^G G_i(\bar{z}) + \lambda_i^H H_i(\bar{z})] \\ &\quad \leq f(\bar{z}) + \sum_{i=1}^{l} \bar{\lambda}_i^g g(\bar{z}, t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})]. \end{aligned} \tag{26}$$

Setting $\lambda = 0$ in (26), we get

$$\sum_{i=1}^{l} \bar{\lambda}_i^g g(\bar{z}, t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})] \geq 0. \tag{27}$$

Since $(\bar{z}, \bar{\lambda}) \in P \times P_D(\bar{z})$, hence

$$\sum_{i=1}^{l} \bar{\lambda}_i^g g(\bar{z}, t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})] \leq 0. \tag{28}$$

From (27) and (28), we get

$$\sum_{i=1}^{l} \bar{\lambda}_i^g g(\bar{z}, t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})] = 0. \tag{29}$$

Also, we have $L_{SIMPEC}(\bar{z}, \bar{\lambda}) \leq L_{SIMPEC}(z, \bar{\lambda})$, therefore

$$\begin{aligned} \varphi(\bar{\lambda}) &= \min_{z \in P} L_{SIMPEC}(z, \bar{\lambda}) \\ &= L_{SIMPEC}(\bar{z}, \bar{\lambda}) \\ &= f(\bar{z}). \end{aligned}$$

By Corollary 9 of Theorem 6, $\bar{z}$ and $\bar{\lambda}$ are, respectively, optimal solutions to the primal (SIMPEC) and dual SIMPECLD($\bar{z}$). □

The following theorem shows the relationship between Lagrange saddle point for (SIMPEC) and M-stationary point for (SIMPEC).

Theorem 15. *Let $\bar{z}$ be a feasible point of the (SIMPEC) and the M-stationary condition holds at $\bar{z}$. Suppose that $f, g(., t_i)$ $(t_i \in T_g)$, h_j $(j \in J^+)$, G_i $(i \in \alpha^- \cup \beta_H^-)$, H_i $(i \in \gamma^- \cup \beta_G^-)$, $-h_j$ $(j \in J^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta^+ \cup \beta_H^+)$, $-H_i$ $(i \in \gamma^+ \cup \beta^+ \cup \beta_G^+)$ are convex at $\bar{z}$ on P. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \emptyset$, then there exists $\bar{\lambda} \in \mathbb{R}^{l+q+2m}$ and indices $t_1, t_2, ..., t_l \in T_g(\bar{z})$, $l \leq n+1$, such that $(\bar{z}, \bar{\lambda})$ such that $(\bar{z}, \bar{\lambda})$ is a saddle point of $L_{SIMPEC}(\bar{z})$. Further, let $(\bar{z}, \bar{\lambda}) \in P \times P_D(\bar{z})$ and if $(\bar{z}, \bar{\lambda})$ is saddle point of $L_{SIMPEC}(z, \lambda)$, then $\bar{z}$ is M-stationary point.*

Proof. We have

$$\begin{aligned} L_{SIMPEC}(z, \bar{\lambda}) - L_{SIMPEC}(\bar{z}, \bar{\lambda}) = f(z) + \sum_{i=1}^{l} \bar{\lambda}_i^g g(z, t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(z) \\ - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(z) + \bar{\lambda}_i^H H_i(z)] - f(\bar{z}) - \sum_{i=1}^{l} \bar{\lambda}_i^g g(\bar{z}, t_i) - \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) \\ + \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})]. \end{aligned}$$

Since f, $g(., t_i)$ $(t_i \in T_g)$, h_j $(j \in J^+)$, G_i $(i \in \alpha^- \cup \beta_H^-)$, H_i $(i \in \gamma^- \cup \beta_G^-)$, $-h_j$ $(j \in J^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta^+ \cup \beta_H^+)$, $-H_i$ $(i \in \gamma^+ \cup \beta^+ \cup \beta_G^+)$ are convex functions at $\bar{z}$ on P and $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \emptyset$. Then, we get

$$\begin{aligned} L_{SIMPEC}(z, \bar{\lambda}) - L_{SIMPEC}(\bar{z}, \bar{\lambda}) \geq \Big\langle \nabla f(\bar{z}) + \sum_{i=1}^{l} \bar{\lambda}_i^g \nabla g(\bar{z}, t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h \nabla h_j(\bar{z}) \\ - \sum_{i=1}^{m} [\bar{\lambda}_i^G \nabla G_i(\bar{z}) + \bar{\lambda}_i^H \nabla H_i(\bar{z})], z - \bar{z} \Big\rangle. \end{aligned}$$

Since, M-stationary condition holds at $\bar{z}$, then from (1), we get

$$L_{SIMPEC}(z, \bar{\lambda}) \geq L_{SIMPEC}(\bar{z}, \bar{\lambda}). \tag{30}$$

$$\begin{aligned} L_{SIMPEC}(\bar{z},\lambda) &= f(\bar{z}) + \sum_{i=1}^{l} \lambda_i^g g(\bar{z},t_i) + \sum_{j=1}^{q} \lambda_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\lambda_i^G G_i(\bar{z}) + \lambda_i^H H_i(\bar{z})] \\ &\leq f(\bar{z}) \\ &= f(\bar{z}) + \sum_{i=1}^{l} \bar{\lambda}_i^g g(\bar{z},t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})] \\ &= L_{SIMPEC}(\bar{z},\bar{\lambda}). \end{aligned}$$

Then,

$$L_{SIMPEC}(\bar{z},\bar{\lambda}) \geq L_{SIMPEC}(\bar{z},\lambda), \quad \forall \lambda \in P_D(\bar{z}). \tag{31}$$

Hence, $(\bar{z},\bar{\lambda})$ is a saddle point of $L_{SIMPEC}(z,\lambda)$.
Since $(\bar{z},\bar{\lambda}) \in P \times P_D(\bar{z})$ is (SIMPEC) saddle point, hence

$$L_{SIMPEC}(\bar{z},\lambda) \leq L_{SIMPEC}(\bar{z},\bar{\lambda}), \quad \forall \lambda \in P_D(\bar{z}). \tag{32}$$

Therefore, from (27), we get

$$\sum_{i=1}^{l} \bar{\lambda}_i^g g(\bar{z},t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})] = 0. \tag{33}$$

From (33) and feasibility of $\bar{\lambda} \in P_D(\bar{z})$, we get

$$\bar{\lambda}_j^g \geq 0, j \notin T_g, \ \ \bar{\lambda}_\gamma^G = 0, \ \bar{\lambda}_\alpha^H = 0, \tag{34}$$

$$\forall i \in \beta \ either \ \bar{\lambda}_i^G > 0, \ \bar{\lambda}_i^H > 0 \ or \ \bar{\lambda}_i^G \bar{\lambda}_i^H = 0. \tag{35}$$

Also, we have $L_{SIMPEC}(\bar{z},\bar{\lambda}) \leq L_{SIMPEC}(z,\bar{\lambda}), \quad \forall z \in P$.
Then,

$$\varphi(\bar{\lambda}) = \min_{z \in P} L_{SIMPEC}(z,\bar{\lambda}) = L_{SIMPEC}(\bar{z},\bar{\lambda}).$$

Therefore, we get

$$\nabla_z L_{SIMPEC}(\bar{z},\bar{\lambda}) = 0,$$

that is,

$$\nabla f(\bar{z}) + \sum_{i=1}^{l} \bar{\lambda}_i^g \nabla g(\bar{z},t_i) + \sum_{j=1}^{q} \bar{\lambda}_j^h \nabla h_j(\bar{z}) - \sum_{i=1}^{m} [\bar{\lambda}_i^G \nabla G_i(\bar{z}) + \bar{\lambda}_i^H \nabla H_i(\bar{z})] = 0. \tag{36}$$

Therefore, from (34), (35) and (36), $\bar{z}$ is M-stationary point. □

The following example illustrates the result of Theorem 15.

Example 16. *Consider the (SIMPEC) problem*

$$\begin{aligned} \textit{min} \quad f(z) &= z_1 + z_2 \\ \textit{subject to} \;\; g(z,t) &= z_1 - 2z_2 - 1 - t \leq 0, \; \forall t \in [0,1], \\ G_1(z) &= z_1 \geq 0, \\ H_1(z) &= -z_1^2 - z_2^2 + 1 \geq 0, \\ G_1(z).H_1(z) &= z_1.(-z_1^2 - z_2^2 + 1) = 0. \end{aligned}$$

The feasible region is given as:

$$P = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 - 2z_2 \leq 1, \; z_1 \geq 0, \; -z_1^2 - z_2^2 + 1 \geq 0, \; z_1.(-z_1^2 - z_2^2 + 1) = 0\}.$$

Since $\bar{z} = (0, -\frac{1}{2})$ is feasible (global optimal) point and for $\bar{\lambda} = (\lambda^g, \lambda^G, \lambda^H) = (\frac{1}{2}, \frac{3}{2}, 0)$ M-stationary condition holds and satisfies the assumption of Theorem 5 at $\bar{z}$. In addition, for

$$L_{SIMPEC}(z, \lambda) = f(z) + \lambda^g g(z, 0) - \lambda^G G_1(z) - \lambda^H H_1(z),$$

the following inequality holds

$$L_{SIMPEC}(\bar{z}, \lambda) \leq L_{SIMPEC}(\bar{z}, \bar{\lambda}) \leq L_{SIMPEC}(z, \bar{\lambda}),$$

for any (λ, z). Hence, Theorem 15 is verified.

5. CONCLUSIONS

We proposed a Lagrange type dual model for the semi-infinite mathematical programming problems with equilibrium constraints. We established weak and strong duality results under convexity assumptions. Further, we discussed the saddle point optimality conditions for the (SIMPEC). We illustrated the obtained results with the help of suitable examples. The Lagrange type dual model for (SIMPEC) is easier to deal than the Wolfe and Mond-Weir type dual models for (SIMPEC) since the weak duality results in Lagrange type model for (SIMPEC) do not require any convexity assumptions and the constraints set do not involve nonlinear stationarity conditions.

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