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EXPONENTIAL TYPE DUALITY FOR η-APPROXIMATED VARIATIONAL PROBLEMS

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Abstract: In this article, we use the so-called η-approximation method for solving a new class of nonconvex variational problems with exponential (p, r)-invex functionals. In this approach, we construct η- approximated variational problem and η-approximated Mond-Weir dual variational problem for the considered variational problem and its Mond-Weir dual variational problem. Then several duality results for considered variational problem and its Mond-Weir dual variational problem are proved by the help of duality results established between η-approximated variational problems mentioned above.

Keywords: Variational Problem, η-Approximated Variational Problem, Mond-Weir Dual Variational Problem, (p, r)-Invexity.

MSC: 49N15, 90C26, 90C30, 90C46.

1. INTRODUCTION

Calculus of variation is a technique used to solve the problems appearing in diverse fields, eg., dynamics of rigid body, optimization of orbit [10], flight design [9, 13], etc. Optimality conditions and duality results for various classes of nonconvex variational problems have been of much interest in recent years, and several contributions have been made to its development (see, for example, [1, 6, 12, 14, 15, 20]). Mond and Hanson [18] established the duality results for variational problems under convexity assumptions. Later on, Mond et al. [17] extended the work of Mond and Hanson to variational problems involving invex functions.

On the other hand, Mond and Husain [19] derived Kuhn-Tucker type sufficient optimality criteria and duality for variational problems under a variety of generalized invexity assumptions. Arana et al. [7] established necessary and sufficient optimality conditions for variational problems under L-FJ and L-KT-pseudoinvexity. Moreover, he also established the weak, strong and converse duality theorems. Thereafter, Husain and Ahmad [11] formulated a mixed type dual for the considered variational problem and established various duality results under strong pseudo-invexity assumptions.

Very recently, Mandal et al. [16] derived Mond-Weir type duality theorems for variational problems involving (p, r)-ρ-(η, θ)-invexity. He also introduced l_1 exact penalty method to convert the considered constrained variational into a unconstrained one and established the equivalence between the optimal solutions of original variational problem and its exponential type problem.

The approach, called the η-approximated method, for solving the nonlinear mathematical programming problems involving r-invex functions was introduced by Antczak [4]. Under r-invexity hypotheses, he proved that an optimal solution in the original extremum problem is equivalent to a minimizer in its η-approximated optimization problem. Thereafter, in [5], he established various duality results in the sense of Mond-Weir for r-invex optimization problems by using the η-approximation method. It turns out that, under r-invexity assumptions, the η-approximation method was also used for such nonconvex optimization problems, in which involved functions are not invex with respect to the same function η and/or it is difficult to show that there exists the same function η with respect to which the involved function would be invex.

Motivated by the research works mentioned above, we use the η-approximation method in proving duality results in the sense of Mond-Weir for a new class of nonconvex variational problems. In this method, for the original variational problem and its original Mond-Weir variational problem, we construct their η- approximated variational problems. Then, we prove Mond-Weir weak, strong and converse duality results between the aforesaid η-approximated variational problems under exponential type invexity hypotheses. Hence, using these duality results established between the above mentioned η-approximated variational problems, we prove Mond-Weir duality results between the original variational problem and its original Mond-Weir dual problem. We illustrate Mond-Weir weak duality theorem established in the paper by an example of a nonconvex variational problem with $(p;r)$-invex functionals. The concept of exponential invexity, named (p,r)-invexity, was introduced by Antczak [2].

The outline of this paper is as follows: in Section 2, we recall some definitions and notations used in the paper. Further, we formulate a variational problem which is considered in the paper and its original Mond- Weir variational dual problem. Moreover, we re-call the necessary optimality conditions for the considered variational problem. In Section 3, we formulate a pair of η-approximated variational problems by modifying both objective and constraint functions in the original variational problems. Namely, for the original variational problem, we construct its original Mond-Weir dual, its η-approximated variational problem and its η-approximated Mond-Weir dual variational problem are constructed. Further, using the duality results proved between the η-approximated variational problem and the η-approximated Mond-Weir dual variational problem, we establish several duality results between the considered variational problem and its Mond-Weir dual problem. Finally, in Section 4, we conclude the results established in the paper.

2. NOTATIONS and PRELIMINARIES

Throughout this paper, consider a real interval $I = [a, b]$. Let X denote the space of continuously differentiable functions $x : I \mapsto R^n$. Let $\phi : I \times R^n \times R^n \mapsto R$ and $g : I \times R^n \times R^n \mapsto R^m$ be continuously differentiable functions. For $\phi(t, x(t), \dot{x}(t))$, where $t \in I$ is an independent variable, we denote the partial derivatives of ϕ with respect to x and $\dot{x}$ by

$$\phi_x = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \ldots, \frac{\partial\phi}{\partial x_n}\right)^T \text{ and } \phi_{\dot{x}} = \left(\frac{\partial\phi}{\partial \dot{x}_1}, \frac{\partial\phi}{\partial \dot{x}_2}, \ldots, \frac{\partial\phi}{\partial \dot{x}_n}\right)^T,$$

respectively. Similarly, we write the partial derivatives of g with respect to x and $\dot{x}$. The symbol T denotes the transpose of a vector and, for convenience, we write $\phi(t, x, \dot{x})$ in place of $\phi(t, x(t), \dot{x}(t))$. Further, let $\eta : I \times X \times X \mapsto R^n$ be a vector-valued function.

In [2], Antczak introduced the definition of the concept of exponential invexity, named (p, r)-invexity, for differentiable optimization problems. Now, we generalize this concept of generalized convexity to the case of variational problems.

Definition 1. *The functional $\int_a^b \phi(t, x, \dot{x})dt$ is said to be (p, r)-invex at the given point $\bar{x} \in X$ on X with respect to η satisfying the condition $\eta(t, \bar{x}, \bar{x}) = 0$ if, for* *all* $x \in X$,

$$p \neq 0, r \neq 0$$
$$\frac{1}{r}[e^{r[\int_a^b (\phi(t,x,\dot{x})-\phi(t,\bar{x},\dot{\bar{x}}))dt]} - 1]$$
$$\geq \int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,x,\bar{x})} - \boldsymbol{1}) + \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,x,\bar{x})}(\frac{d}{dt}\eta(t,x,\bar{x})) \cdot \boldsymbol{1}\}dt,$$
$$p = 0, r \neq 0$$
$$\frac{1}{r}[e^{r[\int_a^b (\phi(t,x,\dot{x})-\phi(t,\bar{x},\dot{\bar{x}}))dt]} - 1]$$
$$\geq \int_a^b \{\eta(t,x,\bar{x})\phi_x(t,\bar{x},\dot{\bar{x}}) + (\frac{d}{dt}\eta(t,x,\bar{x}))\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\}dt,$$
$$p \neq 0, r = 0$$
$$\int_a^b (\phi(t,x,\dot{x}) - \phi(t,\bar{x},\dot{\bar{x}}))dt$$
$$\geq \int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,x,\bar{x})} - \boldsymbol{1}) + \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,x,\bar{x})}(\frac{d}{dt}\eta(t,x,\bar{x})) \cdot \boldsymbol{1}\}dt,$$
$$p = 0, r = 0$$
$$\int_a^b (\phi(t,x,\dot{x}) - \phi(t,\bar{x},\dot{\bar{x}}))dt$$
$$\geq \int_a^b \{\eta(t,x,\bar{x})\phi_x(t,\bar{x},\dot{\bar{x}}) + (\frac{d}{dt}\eta(t,x,\bar{x}))\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\}dt,$$

where $\boldsymbol{1} = (1,1,\ldots,1) \in R^n$ *and the exponential terms are taken as componentwise.*

Let us consider the following pair of primal and dual variational problems:
Primal problem:

$$(\mathbf{P}) \qquad \text{minimize } \int_a^b \phi(t,x,\dot{x})\,dt$$
$$\text{subject to } \quad g(t,x,\dot{x}) \leq 0,\ t \in I,$$
$$x(a) = \alpha,\ x(b) = \beta,$$

where $\phi : I \times R^n \times R^n \mapsto R$ and $g : I \times R^n \times R^n \mapsto R^m$ are continuously differentiable functions.
Let Ω denote the set of all feasible solutions of the variational problem (P), i.e.,

$$\Omega = \{x \in X :\ x(a) = \alpha,\ x(b) = \beta \ \text{ and } g(t,x,\dot{x}) \leq 0,\ t \in I\}.$$

Mond-Weir dual problem:

$$(\mathbf{MWD}) \qquad \text{maximize } \int_a^b \phi(t,u,\dot{u})\,dt$$
$$\text{subject to}$$
$$u(a) = \alpha,\ u(b) = \beta,$$
$$\phi_x(t,u,\dot{u}) + \bar{y}(t)g_x(t,u,\dot{u}) = \tfrac{d}{dt}[\phi_{\dot{x}}(t,u,\dot{u}) + \bar{y}(t)g_{\dot{x}}(t,u,\dot{u})],$$
$$\int_a^b \bar{y}(t)g(t,u,\dot{u})dt \geq 0,$$

$$\bar{y}(t) \geq 0, t \in I.$$

Let S denote the set of all feasible solutions of (MWD). Further, let $U = \{u \in X : (u, \bar{y}) \in S\}$.

Definition 2. *A point $\bar{x} \in \Omega$ is said to be an optimal solution of the variational problem (P) if, for all $x \in \Omega$,*

$$\int_a^b \phi(t, x, \dot{x})\, dt \geq \int_a^b \phi(t, \bar{x}, \dot{\bar{x}})\, dt.$$

Now, we re-call the necessary optimality conditions for the considered variational problem (P) established by Bector and Husain [8].

Theorem 3. *(Necessary optimality conditions). If $\bar{x}$ is a normal optimal solution [18] of the variational problem (P), then there exists a piecewise smooth function $\bar{y} : I \mapsto R^m$ such that, for all $t \in I$,*

$$\phi_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t) g_x(t, \bar{x}, \dot{\bar{x}}) = \frac{d}{dt}[\phi_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t) g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})], \tag{1}$$

$$\bar{y}(t) g(t, \bar{x}, \dot{\bar{x}}) = 0, \tag{2}$$

$$\bar{y}(t) \geq 0. \tag{3}$$

3. A PAIR of PRIMAL and η-APPROXIMATED VARIATIONAL PROBLEMS

Antczak [3] introduced the η-approximation method for solving the considered nonconvex optimization problem and he established the equivalence between the sets of optimal solutions in the original extremum problem and its associated η-approximated optimization problem constructed in this approach. In view of the importance of the applications of a variational problem in real world and engineering problems, in the present paper, we extend the aforesaid approach for the considered variational problem (P) and its associated Mond-Weir type dual program.

Let $(\bar{u}, \tilde{y})$ be any given feasible solution of variational Mond-Weir type dual problem (MWD). Now, we construct a pair of primal and dual η-approximated variational problems ($\mathrm{P}_\eta(\bar{u})$) and ($\mathrm{MWD}_\eta(\bar{u})$) as follows:

The η-approximated primal problem:

$p \neq 0,\ r \neq 0$

minimize

$\frac{1}{r} e^{r(\int_a^b \phi(t, \bar{u}, \dot{\bar{u}})dt)} + \int_a^b \{\phi_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p}(e^{p\eta(t,x,\bar{u})} - \mathbf{1}) + \phi_{\dot{x}}(t, \bar{x}, \dot{\bar{u}}) e^{p\eta(t,x,\bar{u})} (\frac{d}{dt}\eta(t, x, \bar{u})) \cdot \mathbf{1}\} dt$

subject to

$\frac{1}{r} e^{r(\int_a^b \bar{y}(t) g(t, \bar{u}, \dot{\bar{u}})dt)} \Big[1 + r\Big(\int_a^b \{\bar{y}(t) g_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p}(e^{p\eta(t,x,\bar{u})} - \mathbf{1})$

$$+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt\Big)\Big]\leq\frac{1}{r},$$
$$x(a)=\alpha,\ x(b)=\beta,$$

$p=0,\ r\neq 0$

minimize

$$\frac{1}{r}e^{r(\int_a^b\phi(t,\bar{u},\dot{\bar{u}})dt)}+\int_a^b\{\phi_x(t,\bar{u},\dot{\bar{u}})\eta(t,x,\bar{u})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})(\frac{d}{dt}\eta(t,x,\bar{u}))\}dt$$

subject to

$$\frac{1}{r}e^{r(\int_a^b\bar{y}(t)g(t,\bar{u},\dot{\bar{u}})dt)}\Big[1+r\Big(\int_a^b\{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\eta(t,x,\bar{u})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})(\frac{d}{dt}\eta(t,x,\bar{u}))\}dt\Big)\Big]\leq\frac{1}{r},$$
$$x(a)=\alpha,\ x(b)=\beta,$$

$p\neq 0,\ r=0$

minimize

$$\int_a^b\phi(t,\bar{u},\dot{\bar{u}})dt+\int_a^b\{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt$$

subject to

$$\int_a^b\bar{y}(t)g(t,\bar{u},\dot{\bar{u}})dt+\int_a^b\{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt\leq 0,$$
$$x(a)=\alpha,\ x(b)=\beta,$$

$p=0,\ r=0$

minimize

$$\int_a^b\phi(t,\bar{u},\dot{\bar{u}})dt+\int_a^b\{\phi_x(t,\bar{u},\dot{\bar{u}})\eta(t,x,\bar{u})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})(\frac{d}{dt}\eta(t,x,\bar{u}))\}dt$$

subject to

$$\int_a^b\bar{y}(t)g(t,\bar{u},\dot{\bar{u}})dt+\int_a^b\{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\eta(t,x,\bar{u})+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})(\frac{d}{dt}\eta(t,x,\bar{u}))\}dt\leq 0,$$
$$x(a)=\alpha,\ x(b)=\beta,$$

$$(\mathbf{P}_\eta(\bar{u}))$$

where ϕ and g are defined as in the variational problem (P).

The η-approximated dual problem:

$p\neq 0, r\neq 0$

maximize

$$\frac{1}{r}e^{r(\int_a^b\phi(t,\bar{u},\dot{\bar{u}})dt)}+\int_a^b\{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt$$

subject to

$$u(a)=\alpha,\ u(b)=\beta,$$
$$\phi_x(t,\bar{u},\dot{\bar{u}})+\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})=\tfrac{d}{dt}[\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})],$$
$$\frac{1}{r}\Big[e^{r(\int_a^b\bar{y}(t)g(t,\bar{u},\dot{\bar{u}})dt)}\Big\{1+r\Big(\int_a^b\{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt\Big)\Big\}-1\Big]\geq 0,$$

$$\bar{y}(t) \geq 0, \ t \in I,$$

$p = 0, \ r \neq 0$

maximize

$$\frac{1}{r} e^{r(\int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt)} + \int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\eta(t,u,\bar{u}) + \phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})(\frac{d}{dt}\eta(t,u,\bar{u}))\}dt$$

subject to

$$u(a) = \alpha, \ u(b) = \beta,$$
$$\phi_x(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}}) = \tfrac{d}{dt}[\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})],$$
$$\frac{1}{r}\Big[e^{r(\int_a^b \bar{y}(t)g(t,\bar{u},\dot{\bar{u}})dt)}\Big\{1 + r\Big(\int_a^b \{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\eta(t,u,\bar{u}) + \bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})(\frac{d}{dt}\eta(t,u,\bar{u}))\}dt\Big)\Big\} - 1\Big] \geq 0,$$
$$\bar{y}(t) \geq 0, \ t \in I,$$

$p \neq 0, \ r = 0$

maximize

$$\int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt + \int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})} - \mathbf{1}) + \phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u})) \cdot \mathbf{1}\}dt$$

subject to

$$u(a) = \alpha, \ u(b) = \beta,$$
$$\phi_x(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}}) = \tfrac{d}{dt}[\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})],$$
$$\int_a^b \bar{y}(t)g(t,\bar{u},\dot{\bar{u}})dt + \int_a^b \{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})} - \mathbf{1}) + \bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u})) \cdot \mathbf{1}\}dt \geq 0,$$
$$\bar{y}(t) \geq 0, \ t \in I,$$

$p = 0, \ r = 0$

maximize

$$\int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt + \int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\eta(t,u,\bar{u}) + \phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})(\frac{d}{dt}\eta(t,u,\bar{u}))\}dt$$

subject to

$$u(a) = \alpha, \ u(b) = \beta,$$
$$\phi_x(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}}) = \tfrac{d}{dt}[\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})],$$
$$\int_a^b \bar{y}(t)g(t,\bar{u},\dot{\bar{u}})dt + \int_a^b \{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\eta(t,u,\bar{u}) + \bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})(\frac{d}{dt}\eta(t,u,\bar{u}))\}dt \geq 0,$$
$$\bar{y}(t) \geq 0, \ t \in I. \qquad \textbf{(MWD}_\eta(\bar{u})\textbf{)}$$

Let $\Omega(\bar{u})$ and $S(\bar{u})$ denote the sets of all feasible solutions of $(\mathrm{P}_\eta(\bar{u}))$ and $(\mathrm{MWD}_\eta(\bar{u}))$, respectively. Further, let $U(\bar{u}) = \{u \in X : \ (u,\bar{y}) \in S(\bar{u})\}$.

Remark 4. *All theorems will be proved only in the case when $p \neq 0, \ r \neq 0$ (other cases can be dealt with likewise since the only changes arise from form of inequality). Moreover, without loss of generality, we shall assume that $r > 0$ (in the cases when $r < 0$, the direction, some of the inequalities in the proofs of the given theorems should be changed to the opposite one).*

Throughout the paper, we shall use the condition $\eta(t,x,x) = 0, \ \forall \ x \in X$.

Proposition 5. *(Weak duality for the η-approximated problems). Let x and $(u, \bar{y})$ be any feasible solutions of ($P_\eta(\bar{u})$) and* $(\mathrm{MWD}_\eta(\bar{u}))$*, respectively. Then,*

$$\int_a^b \{\phi_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p}(e^{p\eta(t,x,\bar{u})} - \boldsymbol{1}) + \phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,x,\bar{u})} (\frac{d}{dt}\eta(t, x, \bar{u})) \cdot \boldsymbol{1}\} dt$$
$$\geq \int_a^b \{\phi_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p}(e^{p\eta(t,u,\bar{u})} - \boldsymbol{1}) + \phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,u,\bar{u})} (\frac{d}{dt}\eta(t, u, \bar{u})) \cdot \boldsymbol{1}\} dt.$$

Proof. Since x and $(u, \bar{y})$ are any feasible solutions of $(\mathrm{P}_\eta(\bar{u}))$ and $(\mathrm{MWD}_\eta(\bar{u}))$, respectively, we have

$$\frac{1}{r} e^{r(\int_a^b \bar{y}(t) g(t,\bar{u},\dot{\bar{u}}) dt)} \Big[1 + r\Big(\int_a^b \{\bar{y}(t) g_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p}(e^{p\eta(t,x,\bar{u})} - \mathbf{1})$$
$$+ \bar{y}(t) g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,x,\bar{u})} (\frac{d}{dt}\eta(t, x, \bar{u})) \cdot \mathbf{1}\} dt\Big)\Big] \leq \frac{1}{r}. \tag{4}$$

$$\frac{1}{r} \Big[e^{r(\int_a^b \bar{y}(t) g(t,\bar{u},\dot{\bar{u}}) dt)} \Big\{1 + r\Big(\int_a^b \{\bar{y}(t) g_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p}(e^{p\eta(t,u,\bar{u})} - \mathbf{1})$$
$$+ \bar{y}(t) g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,u,\bar{u})} (\frac{d}{dt}\eta(t, u, \bar{u})) \cdot \mathbf{1}\} dt\Big)\Big\} - 1\Big] \geq 0. \tag{5}$$

On combining (4) and (5), we get

$$\int_a^b \{\bar{y}(t) g_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p}(e^{p\eta(t,x,\bar{u})} - \mathbf{1})$$
$$+ \bar{y}(t) g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,x,\bar{u})} (\frac{d}{dt}\eta(t, x, \bar{u})) \cdot \mathbf{1}\} dt$$
$$- \int_a^b \{\bar{y}(t) g_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p}(e^{p\eta(t,u,\bar{u})} - \mathbf{1})$$
$$+ \bar{y}(t) g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,u,\bar{u})} (\frac{d}{dt}\eta(t, u, \bar{u})) \cdot \mathbf{1}\} dt \leq 0. \tag{6}$$

On the other hand, multiplying both sides of the second constraint of Mond-Weir dual problem $(\mathrm{MWD}_\eta(\bar{u}))$ by $(e^{p\eta(t,u,\bar{u})} - \mathbf{1})$ and then integrating between a and b, we get

$$\int_a^b \Big[\{\phi_x(t, \bar{u}, \dot{\bar{u}}) + \bar{y}(t) g_x(t, \bar{u}, \dot{\bar{u}})\}(e^{p\eta(t,u,\bar{u})} - \mathbf{1})\Big] dt$$
$$= \int_a^b \Big[\frac{d}{dt}[\phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) + \bar{y}(t) g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}})](e^{p\eta(t,u,\bar{u})} - \mathbf{1})\Big] dt.$$

On integration by parts to the right hand side of the above equation and using the condition $\eta(t,\bar{u},\bar{u})=0$, we get

$$\int_a^b \Big[\{\phi_x(t,\bar{u},\dot{\bar{u}})+\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})\Big]dt$$
$$=-\int_a^b \Big[\{\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})\}pe^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\Big]dt.$$

Hence,

$$\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})$$
$$+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt$$
$$=-\int_a^b \{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt. \qquad (7)$$

Proceed in the similar way for the feasible point $(x,\bar{y})$ in $(\text{MWD}_\eta(\bar{u}))$, we obtain

$$\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt$$
$$=-\int_a^b \{\bar{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt. \qquad (8)$$

On combining (6),(7) and (8), we obtain

$$\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt$$
$$\geq\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt.$$

This completes the proof of theorem. □

Theorem 6. *(Weak duality for the original problems). Let $\bar{x}$ and $(\bar{u},\tilde{y})$ be any feasible solutions of* (P) *and* (MWD)*, respectively. Assume that the functions $\int_a^b \phi(t,x,\dot{x})dt$ and $\int_a^b \tilde{y}(t)g(t,x,\dot{x})dt$ are (p,r)-invex at $\bar{u}$ on X with respect to η. Then,*

$$\int_a^b \phi(t,\bar{x},\dot{\bar{x}})\geq\int_a^b \phi(t,\bar{u},\dot{\bar{u}}).$$

Proof. Firstly, we prove that $\bar{x}$ and $(\bar{u}, \tilde{y})$ are the feasible solutions of $(\mathrm{P}_\eta(\bar{u}))$ and $(\mathrm{MWD}_\eta(\bar{u}))$, respectively. By the feasibility of $\bar{x}$ in the problem (P), it follows that

$$g(t, \bar{x}, \dot{\bar{x}}) \leq 0.$$

Since $\tilde{y}(t) \in R^m_+$, using the exponential property, the above inequality gives

$$\frac{1}{r} e^{r(\int_a^b \tilde{y}(t) g(t,\bar{x},\dot{\bar{x}})dt)} \leq \frac{1}{r}. \tag{9}$$

By assumption, $\int_a^b \tilde{y}(t) g(t, x, \dot{x}) dt$ is (p, r)-invex at $\bar{u}$ on X with respect to η. Hence, we have

$$\begin{aligned} &\frac{1}{r} e^{r(\int_a^b \tilde{y}(t) g(t,\bar{x},\dot{\bar{x}})dt)} \\ &\geq \frac{1}{r} e^{r(\int_a^b \tilde{y}(t) g(t,\bar{u},\dot{\bar{u}})dt)} \Big[1 + r \Big(\int_a^b \{ \tilde{y}(t) g_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p} (e^{p\eta(t,\bar{x},\bar{u})} - \mathbf{1}) \\ &\qquad + \tilde{y}(t) g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,\bar{x},\bar{u})} (\frac{d}{dt} \eta(t, \bar{x}, \bar{u})) \cdot \mathbf{1} \} dt \Big) \Big]. \end{aligned}$$

Using (9), the above inequality can be rewritten as

$$\begin{aligned} &\frac{1}{r} e^{r(\int_a^b \tilde{y}(t) g(t,\bar{u},\dot{\bar{u}})dt)} \Big[1 + r \Big(\int_a^b \{ \tilde{y}(t) g_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p} (e^{p\eta(t,\bar{x},\bar{u})} - \mathbf{1}) + \\ &\qquad \tilde{y}(t) g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,\bar{x},\bar{u})} (\frac{d}{dt} \eta(t, \bar{x}, \bar{u})) \cdot \mathbf{1} \} dt \Big) \Big] \leq \frac{1}{r}, \end{aligned}$$

which shows that $\bar{x}$ is a feasible solution of $(\mathrm{P}_\eta(\bar{u}))$.
On the other hand, since $(\bar{u}, \tilde{y})$ is a feasible solution of (MWD), therefore

$$\int_a^b \tilde{y}(t) g(t, \bar{u}, \dot{\bar{u}}) dt \geq 0.$$

Again using the exponential property, the above inequality gives

$$\frac{1}{r} \Big[e^{r(\int_a^b \tilde{y}(t) g(t,\bar{u},\dot{\bar{u}})dt)} - 1 \Big] \geq 0.$$

Using the condition $\eta(t, \bar{u}, \bar{u}) = 0$, the inequality above implies that

$$\begin{aligned} &\frac{1}{r} \Big[e^{r(\int_a^b \tilde{y}(t) g(t,\bar{u},\dot{\bar{u}})dt)} \Big\{ 1 + r \Big(\int_a^b \{ \tilde{y}(t) g_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p} (e^{p\eta(t,\bar{u},\bar{u})} - \mathbf{1}) + \\ &\qquad \tilde{y}(t) g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,\bar{u},\bar{u})} (\frac{d}{dt} \eta(t, \bar{u}, \bar{u})) \cdot \mathbf{1} \} dt \Big) \Big\} - 1 \Big] \geq 0, \end{aligned}$$

which shows that $(\bar{u}, \tilde{y})$ is a feasible solution of $(\mathrm{MWD}_\eta(\bar{u}))$. Thus, by Proposition 5, we have

$$\begin{aligned} &\int_a^b \{ \phi_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p} (e^{p\eta(t,\bar{x},\bar{u})} - \mathbf{1}) + \phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,\bar{x},\bar{u})} (\frac{d}{dt} \eta(t, \bar{x}, \bar{u})) \cdot \mathbf{1} \} dt \\ &\quad \geq \int_a^b \{ \phi_x(t, \bar{u}, \dot{\bar{u}}) \frac{1}{p} (e^{p\eta(t,\bar{u},\bar{u})} - \mathbf{1}) + \phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) e^{p\eta(t,\bar{u},\bar{u})} (\frac{d}{dt} \eta(t, \bar{u}, \bar{u})) \cdot \mathbf{1} \} dt. \end{aligned}$$

Again using the condition $\eta(t,\bar{u},\bar{u})=0$, the above inequality reduces to

$$\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,\bar{x},\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,\bar{x},\bar{u})}(\frac{d}{dt}\eta(t,\bar{x},\bar{u}))\cdot\mathbf{1}\}dt\geq 0,$$

which, by assumption that $\phi(t,x,\dot{x})$ is (p,r)-invex at $\bar{u}$ on X with respect to η, gives

$$\frac{1}{r}[e^{r[\int_a^b(\phi(t,\bar{x},\dot{\bar{x}})-\phi(t,\bar{u},\dot{\bar{u}}))dt]}-1]\geq 0.$$

By using the exponential property, the above inequality gives

$$\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt\geq \int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt.$$

This completes the proof of theorem. $\square$

Now, we give an example of a nonconvex variational problem and its Mond-Weir dual to illustrate the results established in Theorem 6.

Example 7. *Let $I=[0,1]$ and X be the space of continuously differentiable functions*
$x:I\mapsto[0,1]$. Let us consider the following pair of primal and dual variational problems.
Primal problem:

$$\text{(P1)}\qquad \text{minimize}\quad \int_0^1\phi(t,x,\dot{x})dt=\int_0^1\{\ln(x^2+1)+tx^2+x\}dt$$
$$\text{subject to}\qquad g(t,x,\dot{x})=x^2-x\leq 0,$$

$$x(0)=0,x(1)=1.$$

The set of all feasible solutions of (P1) is given by $\Omega=\{x\in X: x(0)=0,x(1)=1 \text{ and } x^2-x\leq 0,\ t\in I\}$. Let $\eta: I\times X\times X\mapsto R$ be defined as

$$\eta(t,x,\bar{x})=x(t)-\bar{x}(t).$$

Let $\tilde{y}(t)=1$, where $\tilde{y}: I\mapsto R_+$ and $t\in I$ and consider a feasible point $\bar{x}(t)=0$. Obviously,

$$\phi_x(t,x,\dot{x})=\frac{2x}{1+x^2}+2tx+1,\ \phi_{\dot{x}}(t,x,\dot{x})=0,$$
$$g_x(t,x,\dot{x})=2x-1,\ g_{\dot{x}}(t,x,\dot{x})=0.$$

Dual problem:

$$\text{(MWD1)}\qquad \text{maximize}\quad \int_0^1\{\ln(u^2+1)+tu^2+u\}dt$$
$$\text{subject to}\qquad u(0)=0,u(1)=1,$$

$$\frac{2u}{1+u^2}+2tu+1+\tilde{y}(t)(2u-1)=0,$$

$$\tilde{y}(t)(u^2 - u) \geq 0,$$

$$\tilde{y}(t) \geq 0.$$

Consider a feasible point $(\bar{u}(t), \tilde{y}(t)) = (0, 1)$.
For the considered primal and dual pair of (P1) and (MWD1), the corresponding η-approximated variational problems at the feasible point $(\bar{u}(t), \tilde{y}(t)) = (0, 1)$ are as follows.
The η-approximated primal problem:

$$\begin{array}{lll} (P1_\eta(\bar{u})) & \text{minimize} & 1 + \int_0^1 x dt \\ & \text{subject to} & \int_0^1 \tilde{y}(t)(-x)dt \leq 0, \end{array}$$

$$x(0) = 0, x(1) = 1.$$

The η-approximated dual problem:

$$\begin{array}{lll} (MWD1_\eta(\bar{u})) & \text{maximize} & 1 + \int_0^1 u dt \\ & \text{subject to} & u(0) = 0, u(1) = 1, \end{array}$$

$$1 - \tilde{y}(t) = 0,$$

$$\int_0^2 \tilde{y}(t)(-x)dt \geq 0,$$

$$\tilde{y}(t) \geq 0.$$

Clearly, $\bar{x}(t) = 0$ and $(\bar{u}(t), \tilde{y}(t)) = (0, 1)$ are feasible solutions of $(P1_\eta(\bar{u}))$ and $(MWD1_\eta(\bar{u}))$, respectively. Further, it one can easily verify that the conditions of Proposition 5 are satisfied at the above feasible points.
Now, if we take $p = 0,\ r = 1$, then, by Definition 1, we have

$$e^{\int_0^1 \{\ln(x^2+1)+tx^2+x\}dt} - 1 \geq \int_0^1 x dt.$$

Thus,

$$e^{\int_0^1 \{\ln(x^2+1)+tx^2+x\}dt} \geq 1 + \int_0^1 x dt. \tag{10}$$

If $\int_0^1 xdt \leq -1$, then (10) holds. Thus, we consider the case $\int_0^1 xdt > -1$. Hence, (10) yields

$$\int_0^1 \{\ln(x^2+1) + tx^2 + x\}dt \geq \ln\Big(1 + \int_0^1 xdt\Big).$$

or,

$$\int_0^1 (\ln(x^2+1) + tx^2)dt + \int_0^1 xdt \geq \ln\Big(1 + \int_0^1 xdt\Big). \tag{11}$$

We shall use the following inequality:

$$a \geq \ln(1 + a),\ \forall\ a > -1. \tag{12}$$

If we set $\int_0^1 xdt = a$, *the (11) implies*

$$\int_0^1 (\ln(x^2+1)+tx^2)dt + a \geq \ln(1+a). \tag{13}$$

Since $\int_0^1 (\ln(x^2+1)+tx^2)dt \geq 0$, *for all* $x \in X$, *by (12), (13) is satisfied for all* $x \in \Omega$. *Hence,* $\int_0^1 \phi(t,x,\dot{x})dt$ *is* $(0,1)$*-invex at* $\bar{x}=0$ *with respect to* η.
Now, we shall show that $\int_0^1 \tilde{y}(t)g(t,x,\dot{x})dt$ *is* $(0,1)$*-invex at* $\bar{x}=0$. *Again, by Definition 1, we have*

$$e^{\int_0^1 (x^2-x)dt} - 1 \geq \int_0^1 (-x)dt,$$

and, therefore,

$$e^{\int_0^1 (x^2-x)dt} \geq 1 + \int_0^1 (-x)dt. \tag{14}$$

If $\int_0^1 xdt \leq -1$, *then (14) holds. Thus, we consider the case when* $\int_0^1 xdt > -1$. *Hence, (14) yields*

$$\int_0^1 x^2 dt + \int_0^1 (-x)dt \geq \ln\Big(1+\int_0^1 (-x)dt\Big).$$

Since, $\int_0^1 x^2dt \geq 0 \ \forall \ x \in X$, *therefore, by (12), the above inequality is satisfied for all* $x \in \Omega$. *Therefore,* $\int_0^1 \tilde{y}(t)g(t,x,\dot{x})dt$ *is* $(0,1)$*-invex at* $\bar{x}=0$ *with respect to* η. *Hence, by Theorem 6,* $\int_0^1 \phi(t,\bar{x},\dot{\bar{x}})dt \geq \int_0^1 \phi(t,\bar{u},\dot{\bar{u}})dt$, *what it can be verified.*

Remark 8. *It follows from the above example that the* η*-approximated variational problem* $(\mathrm{P1}_\eta(\bar{x}))$ *constructed for the original nonlinear variational problem* (P1) *can be a linear variational problem. This property is of great importance since we are in a position to find an optimal solution in the nonlinear variational problem by the help of an optimal solution in a linear variational one, that is, in its* η*-approximated variational problem.*

For a feasible point $\bar{x} \in \Omega$, we construct the following pair of $(\mathrm{P}_\eta(\bar{x}))$ and $(\mathrm{MWD}_\eta(\bar{x}))$ as follows:

Primal problem:
$p \neq 0, \ r \neq 0$
minimize
$\frac{1}{r}e^{r(\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt)} + \int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,x,\bar{x})} - \mathbf{1}) + \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,x,\bar{x})}(\frac{d}{dt}\eta(t,x,\bar{x}))\cdot \mathbf{1}\}dt$
subject to

$$\frac{1}{r}e^{r(\int_a^b \bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt)}\Big[1+r\Big(\int_a^b\{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,x,\bar{x})}-\mathbf{1})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,x,\bar{x})}(\frac{d}{dt}\eta(t,x,\bar{x}))\cdot\mathbf{1}\}dt\Big)\Big]\leq\frac{1}{r},$$
$$x(a)=\alpha,\ x(b)=\beta,$$

$p=0,\ r\neq 0$

minimize

$$\frac{1}{r}e^{r(\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt)}+\int_a^b\{\phi_x(t,\bar{x},\dot{\bar{x}})\eta(t,x,\bar{x})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})(\frac{d}{dt}\eta(t,x,\bar{x}))\}dt$$

subject to

$$\frac{1}{r}e^{r(\int_a^b \bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt)}\Big[1+r\Big(\int_a^b\{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\eta(t,x,\bar{x})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})(\frac{d}{dt}\eta(t,x,\bar{x}))\}dt\Big)\Big]\leq\frac{1}{r},$$
$$x(a)=\alpha,\ x(b)=\beta,$$

$p\neq 0,\ r=0$

minimize

$$\int_a^b\phi(t,\bar{x},\dot{\bar{x}})dt+\int_a^b\{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,x,\bar{x})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,x,\bar{x})}(\frac{d}{dt}\eta(t,x,\bar{x}))\cdot\mathbf{1}\}dt$$

subject to

$$\int_a^b\bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt+\int_a^b\{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,x,\bar{x})}-\mathbf{1})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,x,\bar{x})}(\frac{d}{dt}\eta(t,x,\bar{x}))\cdot\mathbf{1}\}dt\leq 0,$$
$$x(a)=\alpha,\ x(b)=\beta,$$

$p=0,\ r=0$

minimize

$$\int_a^b\phi(t,\bar{x},\dot{\bar{x}})dt+\int_a^b\{\phi_x(t,\bar{x},\dot{\bar{x}})\eta(t,x,\bar{x})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})(\frac{d}{dt}\eta(t,x,\bar{x}))\}dt$$

subject to

$$\int_a^b\bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt+\int_a^b\{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\eta(t,x,\bar{x})+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})(\frac{d}{dt}\eta(t,x,\bar{x}))\}dt\leq 0,$$
$$x(a)=\alpha,\ x(b)=\beta,$$

$$\mathbf{(P}_\eta(\bar{x})\mathbf{)}$$

Dual problem:

$p\neq 0,\ r\neq 0$

maximize

$$\frac{1}{r}e^{r(\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt)}+\int_a^b\{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,u,\bar{x})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,u,\bar{x})}(\frac{d}{dt}\eta(t,u,\bar{x})).\mathbf{1}\}dt$$

subject to

$$u(a)=\alpha,\ u(b)=\beta,$$
$$\phi_x(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})=\tfrac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})],$$
$$\frac{1}{r}\Big[e^{r(\int_a^b \bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt)}\Big\{1+r\Big(\int_a^b\{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,u,\bar{x})}-\mathbf{1})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,u,\bar{x})}(\frac{d}{dt}\eta(t,u,\bar{x})).\mathbf{1}\}dt\Big)\Big\}-1\Big]\geq 0,$$
$$\bar{y}(t)\geq 0,\ t\in I,$$

$p=0,\ r\neq 0$

maximize

$$\frac{1}{r}e^{r(\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt)}+\int_a^b\{\phi_x(t,\bar{x},\dot{\bar{x}})\eta(t,u,\bar{x})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})(\frac{d}{dt}\eta(t,u,\bar{x}))\}dt$$

subject to

$$u(a)=\alpha,\ u(b)=\beta,$$
$$\phi_x(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})=\tfrac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})],$$
$$\frac{1}{r}\Big[e^{r(\int_a^b \bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt)}\Big\{1+r\Big(\int_a^b\{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\eta(t,u,\bar{x})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})(\frac{d}{dt}\eta(t,u,\bar{x}))\}dt\Big)\Big\}-1\Big]\geq 0,$$
$$\bar{y}(t)\geq 0,\ t\in I,$$

$p\neq 0,\ r=0$

maximize

$$\int_a^b\phi(t,\bar{x},\dot{\bar{x}})dt+\int_a^b\{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,u,\bar{x})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,u,\bar{x})}(\frac{d}{dt}\eta(t,u,\bar{x})).\mathbf{1}\}dt$$

subject to

$$u(a)=\alpha,\ u(b)=\beta,$$
$$\phi_x(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})=\tfrac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})],$$
$$\int_a^b\bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt+\int_a^b\{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,u,\bar{x})}-\mathbf{1})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,u,\bar{x})}(\frac{d}{dt}\eta(t,u,\bar{x})).\mathbf{1}\}dt\geq 0,\text{if}\,p\neq 0,\ r=0,$$
$$\bar{y}(t)\geq 0,\ t\in I,$$

$p=0,\ r=0$

maximize

$$\int_a^b\phi(t,\bar{x},\dot{\bar{x}})dt+\int_a^b\{\phi_x(t,\bar{x},\dot{\bar{x}})\eta(t,u,\bar{x})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})(\frac{d}{dt}\eta(t,u,\bar{x}))\}dt$$

subject to

$$u(a)=\alpha,\ u(b)=\beta,$$
$$\phi_x(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})=\tfrac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})],$$
$$\int_a^b\bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt+\int_a^b\{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\eta(t,u,\bar{x})$$
$$+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})(\frac{d}{dt}\eta(t,u,\bar{x}))\}dt\geq 0,$$
$$\bar{y}(t)\geq 0,\ t\in I. \qquad (\mathbf{MWD}_\eta(\bar{x}))$$

Theorem 9. *Let $\bar{x}$ be a normal optimal solution of the variational problem $(\mathrm{P}_\eta(\bar{x}))$. Then there exists a piecewise smooth function $\bar{y}:I\mapsto R^m$ such that, for all $t\in I$,*

$$\phi_x(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})=\frac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})], \tag{15}$$
$$\bar{y}(t)g(t,\bar{x},\dot{\bar{x}})=0,\ t\in I, \tag{16}$$
$$\bar{y}(t)\geq 0,\ t\in I, \tag{17}$$

which can be easily verified by assuming the conditions $\eta(t,\bar{x},\bar{x})=0,\ \eta_x(t,\bar{x},\bar{x})=\gamma\cdot\mathbf{1}$, γ is a positive real number.

Theorem 10. *Let $\bar{x}$ be a normal optimal solution of the variational problem* (P)*. Then, $\bar{x}$ is also an optimal solution of* $(\mathrm{P}_\eta(\bar{x}))$*.*

Proof. Since $\bar{x}$ is a normal optimal solution of (P), therefore, by Theorem 3, there exists $\bar{y} : I \mapsto R_+^m$ such that

$$\phi_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}}) = \frac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})], \tag{18}$$
$$\bar{y}(t)g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I, \tag{19}$$
$$\bar{y}(t) \geq 0,\ t \in I \tag{20}$$

holds.
Suppose, contrary to the result, that $\bar{x}$ is not an optimal solution of $(\mathrm{P}_\eta(\bar{x}))$. Then there exists $y \in \Omega(\bar{x})$ such that

$$\begin{aligned}&\frac{1}{r}e^{r(\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt)}\\ &+ \int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,y,\bar{x})} - \mathbf{1}) + \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,y,\bar{x})}(\frac{d}{dt}\eta(t,y,\bar{x}))\cdot\mathbf{1}\}dt\\ &< \frac{1}{r}e^{r(\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt)}\\ &+ \int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{x},\bar{x})} - \mathbf{1}) + \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{x},\bar{x})}(\frac{d}{dt}\eta(t,\bar{x},\bar{x}))\cdot\mathbf{1}\}dt.\end{aligned}$$

Using the condition $\eta(t,\bar{x},\bar{x}) = 0$, we obtain

$$\int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,y,\bar{x})} - \mathbf{1}) + \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,y,\bar{x})}(\frac{d}{dt}\eta(t,y,\bar{x}))\cdot\mathbf{1}\}dt < 0. \tag{21}$$

Since y is a feasible solution of $(\mathrm{P}_\eta(\bar{x}))$, we have

$$\begin{aligned}\frac{1}{r}\Big[e^{r(\int_a^b \bar{y}(t)g(t,\bar{x},\dot{\bar{x}})dt)}\Big\{1 + r\Big(\int_a^b \{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,y,\bar{x})} - \mathbf{1})\\ + \bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,y,\bar{x})}(\frac{d}{dt}\eta(t,y,\bar{x}))\cdot\mathbf{1}\}dt\Big)\Big\} - 1\Big] \leq 0.\end{aligned}$$

Using (19), the above inequality yields

$$\begin{aligned}\int_a^b \{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,y,\bar{x})} - \mathbf{1})\\ + \bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,y,\bar{x})}(\frac{d}{dt}\eta(t,y,\bar{x}))\cdot\mathbf{1}\}dt \leq 0.\end{aligned} \tag{22}$$

On the other hand, multiplying both sides of (18) by $(e^{p\eta(t,y,\bar{x})} - \mathbf{1})$ and then integrating between a and b, we get

$$\int_a^b \Big[\{\phi_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\}(e^{p\eta(t,y,\bar{x})} - \mathbf{1})\Big]dt$$
$$= \int_a^b \Big[\frac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})](e^{p\eta(t,y,\bar{x})} - \mathbf{1})\Big]dt.$$

On integration by parts to the right hand side of the above equation and using the condition $\eta(t,\bar{x},\bar{x}) = 0$, we get

$$\int_a^b \Big[\{\phi_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\}(e^{p\eta(t,y,\bar{x})} - \mathbf{1})\Big]dt$$
$$= -\int_a^b \Big[\{\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\}pe^{p\eta(t,y,\bar{x})}(\frac{d}{dt}\eta(t,y,\bar{x}))\cdot\mathbf{1}\Big]dt.$$

Hence,

$$\int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,y,\bar{x})} - \mathbf{1}) + \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,y,\bar{x})}(\frac{d}{dt}\eta(t,y,\bar{x}))\cdot\mathbf{1}\}dt$$
$$= -\int_a^b \{\bar{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,y,\bar{x})} - \mathbf{1})$$
$$+ \bar{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,y,\bar{x})}(\frac{d}{dt}\eta(t,y,\bar{x}))\cdot\mathbf{1}\}dt.$$

Using (22), we obtain

$$\int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,y,\bar{x})} - \mathbf{1}) + \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,y,\bar{x})}(\frac{d}{dt}\eta(t,y,\bar{x}))\cdot\mathbf{1}\}dt \geq 0,$$

which contradicts (21). Hence, $\bar{x}$ is an optimal solution of $(\mathrm{P}_\eta(\bar{x}))$. This completes the proof of theorem. □

Proposition 11. *(Strong duality for the η-approximated problems). Let $\bar{x}$ be a normal optimal solution of* $(\mathrm{P}_\eta(\bar{x}))$*. Then there exists a piecewise smooth function $\tilde{y} : I \mapsto R^m_+$ such that $(\bar{x}, \tilde{y})$ is an optimal solution of* $(\mathrm{MWD}_\eta(\bar{x}))$*.*

Proof. Since $\bar{x}$ is a normal optimal solution of $(\mathrm{P}_\eta(\bar{x}))$, there exists a piecewise smooth function $\tilde{y} : I \mapsto R^m_+$ such that conditions (15)-(17) hold. Now, using conditions (15), (16) together with $\eta(t,\bar{x},\bar{x}) = 0$ and $\eta_x(t,\bar{x},\bar{x}) = \gamma\cdot\mathbf{1}$, we obtain

$$\phi_x(t,\bar{x},\dot{\bar{x}}) + \tilde{y}(t)g_x(t,\bar{x},\dot{\bar{x}}) = \frac{d}{dt}\Big\{\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \tilde{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\Big\}.$$

$$\frac{1}{r}\Big[e^{r(\int_a^b \tilde{y}(t)g(t,\bar{x},\dot{\bar{x}})dt)}\Big\{1 + r\Big(\int_a^b \{\tilde{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{x},\bar{x})} - \mathbf{1})$$
$$+ \tilde{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{x},\bar{x})}(\frac{d}{dt}\eta(t,\bar{x},\bar{x}))\cdot\mathbf{1}\}dt\Big)\Big\} - 1\Big] = 0,$$

which shows that $(\bar{x}, \tilde{y})$ is a feasible solution of $(\mathrm{MWD}_\eta(\bar{x}))$.
Now, suppose contrary to the result, that $(\bar{x}, \tilde{y})$ is not an optimal solution of $(\mathrm{MWD}_\eta(\bar{x}))$. Then there exists $(\bar{u}, \bar{\xi}) \in S(\bar{x})$ such that

$$\begin{aligned}&\frac{1}{r}e^{r(\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt)}\\
&+\int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{u},\bar{x})}(\frac{d}{dt}\eta(t,\bar{u},\bar{x}))\cdot\mathbf{1}\}dt\\
&>\frac{1}{r}e^{r(\int_a^b \phi(t,\bar{x},\dot{\bar{x}})dt)}\\
&+\int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{x},\bar{x})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{x},\bar{x})}(\frac{d}{dt}\eta(t,\bar{x},\bar{x}))\cdot\mathbf{1}\}dt.\end{aligned}$$

Using the condition $\eta(t, \bar{x}, \bar{x}) = 0$, we obtain

$$\int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{u},\bar{x})}(\frac{d}{dt}\eta(t,\bar{u},\bar{x}))\cdot\mathbf{1}\}dt>0. \tag{23}$$

Now, by the feasibility of $(\bar{u}, \bar{\xi})$ in $(\mathrm{MWD}_\eta(\bar{x}))$, we have

$$\phi_x(t,\bar{x},\dot{\bar{x}})+\bar{\xi}(t)g_x(t,\bar{x},\dot{\bar{x}})=\frac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{\xi}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})].$$

Multiplying both sides by $(e^{p\eta(t,\bar{u},\bar{x})} - \mathbf{1})$ and then integrating between a and b, we get

$$\begin{aligned}&\int_a^b \Big[\{\phi_x(t,\bar{x},\dot{\bar{x}})+\bar{\xi}(t)g_x(t,\bar{x},\dot{\bar{x}})\}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})\Big]dt\\
&\qquad=\int_a^b \Big[\frac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{\xi}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})](e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})\Big]dt.\end{aligned}$$

On integration by parts to the right hand side of the above equation and using the condition $\eta(t,\bar{u},\bar{u})=0$, we get

$$\int_a^b \Big[\{\phi_x(t,\bar{x},\dot{\bar{x}})+\bar{\xi}(t)g_x(t,\bar{x},\dot{\bar{x}})\}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})\Big]dt$$
$$=-\int_a^b \Big[\{\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{\xi}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\}pe^{p\eta(t,\bar{u},\bar{x})}(\frac{d}{dt}\eta(t,\bar{u},\bar{x}))\cdot\mathbf{1}\Big]dt.$$

Hence,

$$\int_a^b \{\phi_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})$$
$$+\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{u},\bar{x})}(\frac{d}{dt}\eta(t,\bar{u},\bar{x}))\cdot\mathbf{1}\}dt$$
$$=-\int_a^b \{\bar{\xi}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})$$
$$+\bar{\xi}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{u},\bar{x})}(\frac{d}{dt}\eta(t,\bar{u},\bar{x}))\cdot\mathbf{1}\}dt.$$

Using (23), we get

$$\int_a^b \{\bar{\xi}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})$$
$$+\bar{\xi}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{u},\bar{x})}(\frac{d}{dt}\eta(t,\bar{u},\bar{x}))\cdot\mathbf{1}\}dt<0. \tag{24}$$

Again, by the feasibility of $(\bar{u},\bar{\xi})$ in $(\mathrm{MWD}_\eta(\bar{x}))$, we get

$$\frac{1}{r}\Big[e^{r(\int_a^b\bar{\xi}(t)g(t,\bar{x},\dot{\bar{x}})dt)}\Big\{1+r\Big(\int_a^b\{\bar{\xi}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})$$
$$+\bar{\xi}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{u},\bar{x})}(\frac{d}{dt}\eta(t,\bar{u},\bar{x}))\cdot\mathbf{1}\}dt\Big)\Big\}-1\Big]\geq 0.$$

Using (16), then the above inequality reduces to

$$\int_a^b \{\bar{\xi}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{x})}-\mathbf{1})$$
$$+\bar{\xi}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{u},\bar{x})}(\frac{d}{dt}\eta(t,\bar{u},\bar{x}))\cdot\mathbf{1}\}dt\geq 0,$$

which contradicts (24). Hence, $(\bar{u},\tilde{y})$ is an optimal solution of $(\mathrm{MWD}_\eta(\bar{x}))$. This completes the proof of theorem. □

Theorem 12. *(Strong duality for the original problems). Let $\bar{x}$ be a normal optimal solution of* (P). *If all hypotheses of Theorem 6 hold, then $(\bar{x},\tilde{y})$ is an optimal solution of* (MWD)*, and the corresponding optimal values are equal.*

Proof. By assumption, $\bar{x}$ is a normal optimal solution of (P). Hence, by Theorem 10, it follows that $\bar{x}$ is also an optimal solution of $(\mathrm{P}_\eta(\bar{x}))$. Now, using Proposition

11, we have $(\bar{x},\tilde{y})$ is an optimal solution of $(\mathrm{MWD}_\eta(\bar{x}))$. Therefore,

$$\phi_x(t,\bar{x},\dot{\bar{x}})+\tilde{y}(t)g_x(t,\bar{x},\dot{\bar{x}})=\frac{d}{dt}[\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\tilde{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})],$$
$$\frac{1}{r}\Big[e^{r(\int_a^b\tilde{y}(t)g(t,\bar{x},\dot{\bar{x}})dt)}\Big\{1+r\Big(\int_a^b\{\tilde{y}(t)g_x(t,\bar{x},\dot{\bar{x}})\frac{1}{p}(e^{p\eta(t,\bar{x},\bar{x})}-\mathbf{1})$$
$$+\tilde{y}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})e^{p\eta(t,\bar{x},\bar{x})}(\frac{d}{dt}\eta(t,\bar{x},\bar{x}))\cdot\mathbf{1}\}dt\Big)\Big\}-1\Big]\geq 0,$$
$$\tilde{y}(t)\geq 0,\ t\in I,$$

which shows that $(\bar{x},\tilde{y})$ is a feasible solution of (MWD). Since all hypotheses of weak duality (Theorem 6) are satisfied at $(\bar{x},\tilde{y})$, therefore, $(\bar{x},\tilde{y})$ is an optimal solution of (MWD) and the optimal values of (P) and (MWD) are the same. This completes the proof of theorem. □

Proposition 13. *(Converse duality for the η-approximated problems). Let $(\bar{u},\tilde{y})$ be an optimal solution of* $(\mathrm{MWD}_\eta(\bar{u}))$. *Then $\bar{u}$ is an optimal solution of* $(\mathrm{P}_\eta(\bar{u}))$.

Proof. Let $\bar{u}$ is feasible but not an optimal solution of $(\mathrm{P}_\eta(\bar{u}))$, then there exists a feasible point $x\in\Omega(\bar{u})$ such that

$$\frac{1}{r}e^{r(\int_a^b\phi(t,\bar{u},\dot{\bar{u}})dt)}$$
$$+\int_a^b\{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt$$
$$<\frac{1}{r}e^{r(\int_a^b\phi(t,\bar{u},\dot{\bar{u}})dt)}$$
$$+\int_a^b\{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,\bar{u},\bar{u})}(\frac{d}{dt}\eta(t,\bar{u},\bar{u}))\cdot\mathbf{1}\}dt.$$

Using the condition $\eta(t,\bar{u},\bar{u})=0$, the above inequality reduces to

$$\int_a^b\{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt<0. \tag{25}$$

Since $(\bar{u},\tilde{y})$ is a feasible solution of $(\mathrm{MWD}_\eta(\bar{u}))$, we have

$$\frac{1}{r}\Big[e^{r(\int_a^b\tilde{y}(t)g(t,\bar{u},\dot{\bar{u}})dt)}\Big\{1+r\Big(\int_a^b\{\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{u})}-\mathbf{1})$$
$$+\tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,\bar{u},\bar{u})}(\frac{d}{dt}\eta(t,\bar{u},\bar{u}))\cdot\mathbf{1}\}dt\Big)\Big\}-1\Big]\geq 0.$$

Using the exponential property and the condition $\eta(t,\bar{u},\bar{u})=0$, above inequality reduces to

$$\int_a^b\tilde{y}(t)g(t,\bar{u},\dot{\bar{u}})dt\geq 0. \tag{26}$$

Now, by the feasibility of $\bar{u}$ in $(\mathrm{P}_\eta(\bar{u}))$, we have

$$\frac{1}{r}e^{r(\int_a^b \tilde{y}(t)g(t,\bar{u},\dot{\bar{u}})dt)}\Big[1+r\Big(\int_a^b \{\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1}) \\ +\tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt\Big)\Big] \leq \frac{1}{r},$$

which, by using (26), reduces to

$$\int_a^b \{\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1}) \\ +\tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\}dt \leq 0. \quad (27)$$

On combining inequalities (25) and (27), we get

$$\int_a^b \Big[\{\phi_x(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\}\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})\Big]dt \\ +\int_a^b \Big[\{\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})\}e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\Big]dt < 0. \quad (28)$$

On the other hand, again using the feasibility of $(\bar{u},\tilde{y})$ in $(\mathrm{MWD}_\eta(\bar{u}))$, we have

$$\phi_x(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})=\frac{d}{dt}[\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})].$$

Multiplying both sides by $(e^{p\eta(t,x,\bar{u})}-\mathbf{1})$ and then integrating between a and b, we get

$$\int_a^b \Big[\{\phi_x(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})\Big]dt \\ =\int_a^b \Big[\frac{d}{dt}[\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})](e^{p\eta(t,x,\bar{u})}-\mathbf{1})\Big]dt.$$

On integration by parts to the right hand side of the above equation and using the condition $\eta(t,\bar{u},\bar{u})=0$, it follows that

$$\int_a^b \Big[\{\phi_x(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})\Big]dt \\ =-\int_a^b \Big[\{\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})\}pe^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\Big]dt.$$

Hence,

$$\int_a^b \Big[\{\phi_x(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\}\frac{1}{p}(e^{p\eta(t,x,\bar{u})}-\mathbf{1})\Big]dt \\ +\int_a^b \Big[\{\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})+\tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})\}e^{p\eta(t,x,\bar{u})}(\frac{d}{dt}\eta(t,x,\bar{u}))\cdot\mathbf{1}\Big]dt=0,$$

which contradicts (28). Thus, $\bar{u}$ is an optimal solution of $(\mathrm{P}_{\eta}(\bar{u}))$. This completes the proof of the theorem. □

Theorem 14. *(Converse duality for the original problems). Let* $(\bar{u}, \tilde{y})$ *be an optimal solution of the original dual problem* (MWD) *such that* $g(t, \bar{u}, \dot{\bar{u}}) = 0$. *Further, assume that* $\int_a^b \phi(t, x, \dot{x})dt$ *and* $\int_a^b \tilde{y}(t)g(t, x, \dot{x})dt$ *are* (p, r)*-invex at* $\bar{u}$ *on* X *with respect to* η. *Then* $\bar{u}$ *is an optimal solution of* (P).

Proof. Firstly, we show that $(\bar{u}, \tilde{y})$ is an optimal solution of $(\mathrm{MWD}_{\eta}(\bar{u}))$. Assume that $(\bar{u}, \tilde{y})$ is not an optimal solution of $(\mathrm{MWD}_{\eta}(\bar{u}))$. Then, there exists a feasible point $(u, y) \in S(\bar{u})$ such that

$$\begin{aligned}
&\frac{1}{r} e^{r(\int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt)} \\
&\quad + \int_a^b \{\phi_x(t, \bar{u}, \dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{u})} - \mathbf{1}) + \phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}})e^{p\eta(t,\bar{u},\bar{u})}(\frac{d}{dt}\eta(t, \bar{u}, \bar{u})) \cdot \mathbf{1}\}dt \\
&< \frac{1}{r} e^{r(\int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt)} \\
&\quad + \int_a^b \{\phi_x(t, \bar{u}, \dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})} - \mathbf{1}) + \phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t, u, \bar{u})) \cdot \mathbf{1}\}dt.
\end{aligned}$$

Using the condition $\eta(t, \bar{u}, \bar{u}) = 0$, above inequality reduces to

$$\int_a^b \{\phi_x(t, \bar{u}, \dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})} - \mathbf{1}) + \phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t, u, \bar{u})) \cdot \mathbf{1}\}dt > 0. \tag{29}$$

Also, by the feasibility of (u, y) in $(\mathrm{MWD}_{\eta}(\bar{u}))$, we have

$$\phi_x(t, \bar{u}, \dot{\bar{u}}) + y(t)g_x(t, \bar{u}, \dot{\bar{u}}) = \frac{d}{dt}[\phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) + y(t)g_x(t, \bar{u}, \dot{\bar{u}})]$$

Multiplying both sides by $(e^{p\eta(t,u,\bar{u})} - \mathbf{1})$ and integrating between a and b, we get

$$\begin{aligned}
&\int_a^b \Big[\{\phi_x(t, \bar{u}, \dot{\bar{u}}) + y(t)g_x(t, \bar{u}, \dot{\bar{u}})\}(e^{p\eta(t,u,\bar{u})} - \mathbf{1})\Big]dt \\
&\qquad = \int_a^b \Big[\frac{d}{dt}[\phi_{\dot{x}}(t, \bar{u}, \dot{\bar{u}}) + y(t)g_{\dot{x}}(t, \bar{u}, \dot{\bar{u}})](e^{p\eta(t,u,\bar{u})} - \mathbf{1})\Big]dt.
\end{aligned}$$

Applying integration by parts to the right hand side of the above equation and using the condition $\eta(t,\bar{u},\bar{u})=0$, we get

$$\int_a^b \Big[\{\phi_x(t,\bar{u},\dot{\bar{u}})+y(t)g_x(t,\bar{u},\dot{\bar{u}})\}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})\Big]dt$$
$$=-\int_a^b \Big[\{\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})+y(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})\}pe^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\Big]dt.$$

Hence,

$$\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt$$
$$=-\int_a^b \{y(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+y(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt,$$

which in turn, by using (29), we get

$$\int_a^b \{y(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+y(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt<0.$$

Using the hypothesis $g(t,\bar{u},\dot{\bar{u}})=0$, the above inequality can be rewritten as

$$\frac{1}{r}\Big[e^{r(\int_a^b y(t)g(t,\bar{u},\dot{\bar{u}})dt)}\Big\{1+r\Big(\int_a^b \{y(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})$$
$$+y(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt\Big)\Big\}-1\Big]<0,$$

which contradicts that (u,y) is a feasible solution of $(\mathrm{MWD}_\eta(\bar{u}))$ and, thus $(\bar{u},\tilde{y})$ is an optimal solution of $(\mathrm{MWD}_\eta(\bar{u}))$. Hence, by Proposition 13, $\bar{u}$ is an optimal solution of $(\mathrm{P}_\eta(\bar{u}))$.

Now, we shall show that $\bar{u}$ is an optimal solution of (P). We proceed by contradiction. Suppose, contrary to the result, that $\bar{u}$ is not an optimal solution of (P). Then, there exists a feasible point $u\in\Omega$ such that

$$\int_a^b \phi(t,u,\dot{u})dt<\int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt. \tag{30}$$

Since $\int_a^b \phi(t,x,\dot{x})dt$ is (p,r)-invex at $\bar{u}$ with respect to η, therefore, by Definition 2.1, we have

$$\frac{1}{r}[e^{r[\int_a^b (\phi(t,u,\dot{u})-\phi(t,\bar{u},\dot{\bar{u}}))dt]}-1]$$
$$\geq\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt.$$

Using (30), the above inequality reduces to

$$\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt<0. \tag{31}$$

Also, by assumption, $\int_a^b \tilde{y}(t)g(t,x,\dot{x})dt$ is (p,r)-invex at $\bar{u}$ on X with respect to η, we have

$$\frac{1}{r}e^{r(\int_a^b \tilde{y}(t)g(t,u,\dot{u})dt)} \geq \frac{1}{r}e^{r(\int_a^b \tilde{y}(t)g(t,\bar{u},\dot{\bar{u}})dt)}\Big[1+r\Big(\int_a^b \{\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})$$

Using the feasibility of $u \in \Omega$ with exponential property, the above inequality can be written as

$$\frac{1}{r}e^{r(\int_a^b \tilde{y}(t)g(t,\bar{u},\dot{\bar{u}})dt)}\Big[1+r\Big(\int_a^b \{\tilde{y}(t)g_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+ \\ \tilde{y}(t)g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt\Big)\Big] \leq \frac{1}{r},$$

which shows that u is a feasible solution of $(\mathrm{P}_\eta(\bar{u}))$.
Now, since $\bar{u}$ is an optimal solution of $(\mathrm{P}_\eta(\bar{u}))$, therefore

$$\frac{1}{r}e^{r(\int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt)} \\ +\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt \\ \geq \frac{1}{r}e^{r(\int_a^b \phi(t,\bar{u},\dot{\bar{u}})dt)} \\ +\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,\bar{u},\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,\bar{u},\bar{u})}(\frac{d}{dt}\eta(t,\bar{u},\bar{u}))\cdot\mathbf{1}\}dt.$$

Using the condition $\eta(t,\bar{u},\bar{u})=0$, the above inequality reduces to

$$\int_a^b \{\phi_x(t,\bar{u},\dot{\bar{u}})\frac{1}{p}(e^{p\eta(t,u,\bar{u})}-\mathbf{1})+\phi_{\dot{x}}(t,\bar{u},\dot{\bar{u}})e^{p\eta(t,u,\bar{u})}(\frac{d}{dt}\eta(t,u,\bar{u}))\cdot\mathbf{1}\}dt \geq 0,$$

which contradicts (31). Hence, $\bar{u}$ is an optimal solution of (P). This completes the proof of theorem. □

4. CONCLUSIONS

In this paper, the η-approximated method has been used to prove several duality results between the considered nonconvex variational problem (P) and its Mond-Weir dual variational problem (MWD) by the help of the similar duality results established between η-approximated problems constructed for the problems (P) and (MWD), respectively. In order to illustrate the results proved in the paper, an example of nonconvex variational problems has been presented. It seems that the results established in the paper for a class of nonconvex scalar variational problems with (p,r)-invex functionals can be extended to various classes of nonconvex multiobjective variational problems. This will orient the future task of the authors.

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