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SOME ASPECTS ON SOLVING TRANSPORTATION PROBLEM

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Abstract: In this paper, we consider a class of transportation problems which arises in sample surveys and other areas of statistics. The associated cost matrices to these transportation problems are of special structure. We observe that the optimality of North West corner solution holds for the problem where cost component is replaced by a convex function. We revisit assignment problem and present a weighted version of König-Egerváry theorem. Finally, we propose weighted Hungarian method to solve the transportation problem.

Keywords: Transportation Problem, North West Corner Solution, Weighted König-Egerváry theorem, Assignment Problem, Weighted Hungarian Method, Sample Survey.

MSC: 65K10, 90B06.

1. INTRODUCTION

A transportation model is a bipartite graph $G = (A \cup B, E)$, where $A = \{O_1, \cdots, O_m\}$ is the set of source vertices, $B = \{D_1, \cdots, D_n\}$ is the set of destination vertices and E is the set of edges from A to B. Each edge $(i, j) \in E$ has an associated cost c_{ij}. The problem is to find out a flow of least costs regarding shipment that ships from supply sources O_i, $i = 1, \cdots, m$ to consumer destinations D_j, $j = 1, \cdots, n$. Suppose a_i is the supply at the i^{th} source O_i and b_j is the demand at the j^{th} destination D_j. In a balanced transportation problem, we assume that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Let x_{ij} be the quantity to be shipped from origin O_i to destination D_j with cost c_{ij}. The transportation problem can be formulated as a linear programming problem to determine a shipping schedule that minimizes the total cost of shipment which is given below.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i \in A} \sum_{j \in B} c_{ij} x_{ij} \\
\text{subject to} \quad & \\
& \sum_{j \in B} x_{ij} = a_i, \ \forall\, i \in A \\
& \sum_{i \in A} x_{ij} = b_j, \ \forall\, j \in B \\
& x_{ij} \geq 0, \ i \in A, \ j \in B.
\end{aligned}
$$

Transportation problems have wide range of applications in logistic systems [50], human resources management [12], production planning [41], routing problems [52] and many other related areas. In real life situations, the traditional transportation problems deal with several issues such as selection of sources and delivery routes given the destinations, handling and packing, financing and insurance, duty and taxes, and is related to other operations such as selection of production place and capacity, decision on outsourcing of production and hiring human capital.

Transportation problems uniformly follow a special mathematical structure in their constraints and objective functions. Because of their special structure, they are solved by an approach different from simplex method. The approach consists of two phases. In phase one, a feasible solution of the transportation problem is found. A well known phase one method is North West corner rule. In phase two, optimal solution based on the identified feasible solution is found. Phase two is basically a primal-dual method.

The motivation of this study is to address the following two questions:

(i) Is there any instance of transportation problem for which the feasible solution obtained by North West corner rule becomes the optimal solution?

(ii) Is there any single phase method which can solve transportation problem?

The paper is organized as follows. In section 2, we present a literature review of the wide algorithms used for solving various transportation problems. In section 3, we consider a class of transportation problems and its application in statistics. We discuss various structures and solution methods of this class. We present an elegant proof of the result that the North West corner rule provides an optimal solution to the transportation problems under some conditions. In section 4, we consider the assignment problem and present a weighted version of König-Egerváry theorem and Hungarian method. We present minimum cut - maximum flow theorem of Ford and Fulkerson [22] in a different way so that in a bipartite graph, finding a minimum weight vertex cover is equivalent with finding a minimum cut if the capacity of an edge is given by the minimum weight of its end nodes. We establish a connection between transportation problem and assignment problem, and propose a weighted Hungarian method to solve the transportation problem. We show that the weighted Hungarian method is formulated based on primal-dual method for solving minimum cost network flow problems. We provide an illustration to demonstrate our result. Section 5 presents the conclusion of the paper.

2. LITERATURE REVIEW

The section presents a survey of the computational methods used to solve various transportation problems. There are diverse types of transportation models documented in the literature, for details, see [3]. Øvstebø et al. [43] present an optimization model for RoRo ship stowage problem which is closely related to maritime transportation problems. Fagerholt [20] provides optimal fleet design in a ship routing problems. Routing problems are very well known class of transportation problems. Oil tanker routing and scheduling problems are two kinds of important transportation problems, for details, see [13]. Air transportation is another branch of transportation problems which include air traffic flow management problem, helicopter routing problem, airline crew scheduling problem and oil platform transport problem, for details, see [27], [21], [4] and [15]. The variants of the routing problems include convoy routing problem [25], inventory routing problem [37], school bus routing problem [42], truck and trailer routing problem [48], bus scheduling problem [32] and vehicle routing problem [52], for further details, see [14]. Pamucar and Ćirović [44] have used an adaptive neuro fuzzy inference system in uncertainty conditions on selection of vehicle routing problem. Among many facets of research in transportation problems, one that has received extensive attention in recent years is the development of efficient algorithms. Souffriau [49] proposed a greedy randomized adaptive search procedure (GRASP) for solving the team orienteering problem (TOP) which is a particular routing problem to earn a score for visiting a location. Tabu search technique is used to solve team orienting problem as well as truck and trailer routing problem, details in [23]. Archetti et al. [6] used branch and cut algorithm to solve inventory routing problem. Popović et al. [45] proposed a variable neighborhood search (VNS) algorithm to solve a multi-product multi-period inventory routing problem in which fuel delivery with multi-compartment homogeneous vehicles and deterministic consumption vary with each petrol station and each fuel type. Hoffman and Padberg [30] showed that branch and cut technique can be used to solve airline crew scheduling problem. Emden-Weinert and Proksch [18] proposed a simulated annealing algorithm for the airline crew pairing problem based on a run-cutting formulation. Ghanbari and Mahdavi-Amiri [24] proposed an evolutionary algorithm to solve bus terminal location problem. Vignaux and Michalewicz [53] proposed a genetic algorithm for solving the linear transportation problem. Michalewicz et al. [39] described a nonstandard genetic algorithm approach to solve the nonlinear transportation problem. The D-R model is a technique to minimize the cost and a variety of risk criteria in hazmat routing and minimizes the number of risks on potential routes. Ebrahimi and Tadic [16] proposed a model based on a multi-criteria risk analysis and the traditional Dijkstra algorithm (D-R model). Roy et al. [47] proposed a memetic genetic algorithm for solving traveling salesman problem (TSP) which is related to different type of transportation problem. Barma et al. [8] studied multi depot vehicle routing problem (MDVRP) and presented a discrete antli-on optimization algorithm (DALO) to minimize the total routing distance of the MDVRP. Adhikary et al. [1] presented newsboy problem with birandom demand. Anholcer [2] considered stochastic generalized transportation problem with random discretely distributed demand. Kuhn [34] presented variants of the Hungarian method for solving assignment problems. In this paper we propose a weighted Hungarian method to solve linear transportation problems.

3. TRANSPORTATION PROBLEM AND ITS APPLICATIONS in STATISTICS

Transportation problem arises in various applications of Sample Surveys and Statistics, details in [7]. The cost matrices associated with these transportation problems are of special structure. Now, we raise the following question. What are the structure of the cost matrix for which North West corner solution produces an optimal solution? We consider some of the structures of the cost matrix which arise in some of the applications in the literature. Hoffman [28] studied transportation problem in the context of North West Corner Rule. Burkard et al. [9] mentioned Monge properties in connection with the transportation problem. Szwarc [51] developed direct methods for solving transportation problems with cost coefficients of the form $c_{ij} = x_i + x_j$, having applications in shop loading and aggregate scheduling.

Evans [19] studied factored transportation problem in which cost coefficients are factorable, i.e., $c_{ij} = x_i x_j$. It is shown that the rows and columns can easily be ordered so that the North West corner rule provides an optimal solution of the transportation problem. We state some of the results of Evans [19] which are needed in the sequel.

Lemma 3.1. *[19] The North West corner rule produces an optimal solution of the balanced transportation problem whenever $c_{ij} + c_{rs} \leq c_{rj} + c_{is}$ for all i, j, r, s such that $i < r$ and $j < s$.*

Szwarc [51] showed that if $c_{ij} = x_i + y_j$ then any feasible solution is optimal. Raj [46] studied the problem of integration of surveys, i.e., the problem of designing a sampling program for two or more surveys which maximizes the overlap between observed samples as a transportation problem and shows that the solution is just the North West corner solution of the transportation problem and this is optimal when $c_{ij} = |i - j|$. For the connection between integration of surveys and the transportation problem see Matei and Tillé [38], Aragon and Pathak [5], Causey et al. [10] and Raj [46]. In this context, Burkard et al. [9] studied several perspectives of Monge properties in optimization. Mitra and Mohan [40] observed that the North West corner solution is optimal for the following problem.

Suppose X and Y are two discrete random variables which assume values $x_1 \leq x_2 \leq \cdots \leq x_m$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ respectively. Let $p_{i\cdot} = \text{Prob}(X = x_i)$ and $p_{\cdot j} = \text{Prob}(Y = y_j)$. The problem is to find out the joint probabilities $p_{ij} = Prob(X = x_i, Y = y_j)$ so that $\text{Cov}(X, Y)$ is maximum. This problem can be formulated as a transportation problem as follows.

Given values of random variables X, Y, $p_{i\cdot}$, $p_{\cdot j}$, the problem is to find p_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ which

$$
\begin{aligned}
\text{minimize} \quad & \sum_i \sum_j (x_i - y_j)^2 \, p_{ij} \\
\text{subject to} \quad & \\
& \sum_j p_{ij} = p_{i\cdot}, \; 1 \leq i \leq m \\
& \sum_i p_{ij} = p_{\cdot j}, \; 1 \leq j \leq n \\
& p_{ij} \geq 0, \; 1 \leq i \leq m, \; 1 \leq j \leq n.
\end{aligned}
$$

In this article, we consider a more general problem and show that the optimality of North West corner solution holds.

Suppose $f : R \to R$ is a convex function. Let $c_{ij} = f(x_i - y_j)$. Given values of random variables X, Y, $p_{i\cdot}, p_{\cdot j}$, the problem is to find p_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ for the Problem P which is stated as follows:

Problem P :

$$
\begin{aligned}
\text{minimize} \quad & \sum_i \sum_j f(x_i - y_j) \, p_{ij} \\
\text{subject to} \quad & \\
& \sum_j p_{ij} = p_{i\cdot}, \; 1 \leq i \leq m \\
& \sum_i p_{ij} = p_{\cdot j}, \; 1 \leq j \leq n \\
& p_{ij} \geq 0, \; 1 \leq i \leq m, \; 1 \leq j \leq n.
\end{aligned}
$$

Let us define a set $B = \{(i, j) \mid p_{ij} \text{ is a basic variable}\}$. Note that if B is the basis set and if the basic solution to P corresponding to the set B is also feasible, it is optimal if and only if there exist α_i, $1 \leq i \leq m$, β_j, $1 \leq j \leq n$ such that

$$\alpha_i + \beta_j = c_{ij} \text{ if } (i, j) \in B \tag{3.1}$$

$$\alpha_i + \beta_j \leq c_{ij} \text{ if } (i, j) \notin B. \tag{3.2}$$

Now we prove the following theorem.

Theorem 3.2. *Consider the problem P. Let* $x_1 \leq x_2 \leq \cdots \leq x_n$ *and* $y_1 \leq y_2 \leq \cdots \leq y_n$ *be given numbers and* $c_{ij} = f(x_i - x_j)$ *where* $f: R \to R$ *is convex. Then the North West corner solution is optimal for Problem P.*

Proof. Suppose $i < r$ and $j < s$. Let $x_1 \leq x_2 \leq \cdots \leq x_i \leq \cdots \leq x_r \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_j \leq \cdots \leq y_s \leq \cdots \leq y_n$ be given numbers. It is shown that

$$x_i - y_s \leq x_i - y_j \leq x_r - y_j$$

$$x_i - y_s \leq x_r - y_s \leq x_r - y_j.$$

Then there exist $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 1$ such that

$$x_i - y_j = \lambda(x_i - y_s) + (1 - \lambda)(x_r - y_j)$$

$$x_r - y_s = \mu(x_i - y_s) + (1 - \mu)(x_r - y_j).$$

Table 1: North West corner rule solution

	$\cdots$	j	$\cdots$	k	$\cdots$	l	$\cdots$	s	$\cdots$
$\vdots$									
i		$*$						(i,s)	
$\vdots$		$\vdots$							
p		$*$	$\cdots$	$*$					
$\vdots$				$\vdots$					
q				$*$	$\cdots$	$*$			
$\vdots$						$\vdots$			
r						$*$	$\cdots$	$*$	
$\vdots$									

However since $x_i - y_j + x_r - y_s = (\lambda + \mu)(x_i - y_s) + (2 - \lambda - \mu)(x_r - y_j)$, it follows that $\lambda + \mu = 1$. Therefore, by convexity of f

$$f(x_i - y_j) \leq \lambda f(x_i - y_s) + (1 - \lambda) f(x_r - y_j)$$

$$f(x_r - y_s) \leq \mu f(x_i - y_s) + (1 - \mu) f(x_r - y_j)$$

$$f(x_i - y_j) + f(x_r - y_s) \leq (\lambda + \mu) f(x_i - y_s)$$
$$+ (2 - \lambda - \mu) f(x_r - y_j).$$

Using $\lambda + \mu = 1$ and $c_{ij} = f(x_i - x_j)$ it follows that $c_{ij} + c_{rs} \leq c_{is} + c_{rj}$ for all i, j, r, s such that $i < r$ and $j < s$. By Lemma 3.1, it follows that the North West corner rule produces an optimal solution. □

Corollary 3.3. *In problem P, suppose* $c_{ij} = (x_i - x_j)^2$ *or* $c_{ij} = |i - j|$. *Then North West corner solution is an optimal solution for Problem P.*

4. TRANSPORTATION PROBLEM AND A WEIGHTED VERSION OF KÖNIG-EGERVÁRY THEOREM

An assignment problem is a special case of a balanced transportation problem where $m = n$, $a_i = 1$, $\forall\ i \in A$ and $b_j = 1$, $\forall\ j \in B$. Various generalizations of transportation problem have been appeared in the literature. For details, see Goossens and Spieksma [26], Kasana and Kumar [29], Liu and Zhang [36] and the references cited therein. We consider the cardinality of a maximum matching and the size of a minimum vertex cover in a bipartite graph. Consider the entries of a matrix $A = (a_{ij}) \in R^{n\times n}$ as *points* and a row or a column as a *line.* A set of points is said to be *independent* if none of the lines of the matrix contains more than one point in the set. Suppose T is an independent set of points. Then an element of T is said to be an independent point. König-Egerváry theorem (Egerváry [17], König [31]) is stated as follows:

Theorem 4.1. *([17], [31]) Let S be a nonempty subset of points of a matrix $A = (a_{ij}) \in R^{n\times n}$, then the maximum number of independent points that can be selected in S is equal to the minimum number of lines covering all points in S.*

König-Egerváry theorem is used to obtain Hungarian method, and it is used to prove the finite convergence of the Hungarian method for linear assignment problem. Suppose that $\bar{C} = [\bar{c}_{ij}] \in R^{n\times n}$ is a cost matrix of the assignment problem. We obtain a reduced cost matrix $\bar{C}'$ of order n by subtracting the smallest element in each row and then subtracting the smallest element in each column. Note that all the elements of $\bar{C}'$ are non-negative and there is at least one zero in every row and every column. Recall that any two zero is said to be *independent* if they do not lie in the same line. Let t be the number of independent zeros in the reduced cost matrix $\bar{C}'$ and $t \leq n$. The König-Egerváry theorem states that maximum cardinality of an independent set of 0's is equal to minimum number of lines to cover all 0's.

In this section, we describe a weighted version of König-Egerváry theorem and use it to provide a weighted Hungarian method for solving a transportation problem. This states minium cut - maximum flow theorem of Ford and Fulkerson in a different way. Accordingly, a standard transportation problem can also be written as a linear assignment problem as follows:

Let $\mathcal{S}_1 = \{1, 2, \cdots, a_1\}$, $\mathcal{T}_1 = \{1, 2, \cdots, b_1\}$,

$$\mathcal{S}_r = \{\sum_{j=1}^{r-1} a_j + 1, \sum_{j=1}^{r-1} a_j + 2, \cdots, \sum_{j=1}^{r} a_j\},\ 2 \leq r \leq m \text{ and}$$

$$\mathcal{T}_s = \{\sum_{j=1}^{s-1} b_j + 1, \sum_{j=1}^{s-1} b_j + 2, \cdots, \sum_{j=1}^{s} b_j\}, 2 \leq s \leq n.$$

Let $\eta = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Consider a linear assignment problem in which total number of machines is equal to total number of jobs (η). Let $C = (c_{ij}) \in R^{m\times n}$ be the cost matrix of the transportation problem. We construct a cost matrix $\tilde{C} = (\tilde{c}_{ij})$ by copying $C_{i\cdot}$, a_i times for $i = 1, \cdots, m$ and $C_{\cdot j}$, b_j times for $j = 1, \cdots, n$. Thus $\tilde{C}_{p\cdot} = C_{l\cdot}\ \forall\ p \in \mathcal{S}_l$, $\tilde{C}_{\cdot p} = C_{\cdot l}\ \forall\ p \in \mathcal{T}_l$. The matrix constructed in this manner leads to a square cost matrix $\tilde{C}$ of order $\eta \times \eta$ for the linear assignment problem. So we arrive at an equivalent assignment problem of the transportation problem. Now looking at the equivalent linear assignment problem, we observe to see that there are mn blocks in $\tilde{C}$ where $(ij)^{th}$ block is of size $a_i \times b_j$ consisting of identical entries c_{ij}.

We explore the possibility of extending the Hungarian method for a transportation problem using the original cost matrix C of order $m \times n$. Note that in $\tilde{C}$, $(ij)^{th}$ block of size $a_i \times b_j$ consisting of identical entries c_{ij} can be treated as a single entry in C in the $(ij)^{th}$ position. We provide a weight a_i for the i^{th} row in C and a weight b_j for the j^{th} column in C. We now state a weighted version of König-Egerváry theorem.

In this theorem we use the following terminology. The entries of a matrix $C = (c_{ij}) \in R^{m\times n}$ are called *blocks.* The i^{th} row $C_{i\cdot}$ is known as a horizontal line with weight a_i and $C_{\cdot j,}$ the j^{th} column is a vertical line with weight b_j. A set of blocks is said to be *independent* if none of the lines of the matrix contains more than one block in the set. Suppose Λ is an independent set of blocks. Then an element of Λ is said to be an independent block.

Now we prove the following theorem.

Theorem 4.2. *If Ω is a nonempty subset of the blocks of a matrix C, then the maximum number of independent blocks that can be selected in Ω is equal to the lines with minimum total weight covering all the blocks in Ω.*

Proof. Note that in C, a horizontal line with weight a_i is equivalent to a_i rows and a vertical line with weight b_j is equivalent to b_j columns in $\tilde{C}$. Let $\tilde{\Omega}$ be a nonempty subset of points of a matrix $\tilde{C}$. Now by Theorem 4.1, the maximum number of independent points that can be selected in $\tilde{\Omega}$ is equal to the minimum number of lines covering all elements in $\tilde{\Omega}$. Now in $\tilde{C}$, drawing a_i horizontal lines is equivalent to drawing a horizontal line with weight a_i in C. Similarly in $\tilde{C}$, drawing b_j vertical lines is equivalent to drawing a vertical line with weight b_j in C. Since we do not distinguish between horizontal and vertical lines, the maximum number of independent blocks that can be selected in Ω is equal to the lines with minimum total weight covering all blocks of Ω. $\square$

The Hungarian method for the linear assignment problem was developed by Kuhn [33] which has computational complexity $O(n^4)$. Lawler [35] developed an order $O(n^3)$ version of the algorithm. Cechlárová [11] observed that in practical situations, it may be useful to get an overall picture about all the optima as well and obtains a generalization of the Berge's theorem.

We now apply weighted version of König-Egerváry theorem to get an weighted Hungarian method for solving transportation problems. Note that by weighted König-Egerváry theorem, maximum number of independent zero blocks that can be selected is equal to the lines with minimum total weight covering all the blocks. We describe the weighted Hungarian method based on weighted König-Egerváry theorem for solving the transportation problem.

The basic steps of the weighted Hungarian method are the same as in Hungarian method. The termination rule is as in Theorem 4.2, i.e., the weights of the lines drawn with minimum total weight is equal to η. Therefore, the proof of finite convergence also follows from Theorem 4.2.

We provide the basic steps of the weighted Hungarian method for solving the transportation problem:

Step 1: Get the reduced cost matrix by subtracting the smallest element in each row and then subtracting the smallest element in each column.

Step 2: Draw lines with minimum total weight to cover all zero blocks. Let the total weight of the lines drawn be ζ.

Step 3: If $\zeta = \eta$, optimal matrix has been reached. Get an optimal solution by assigning flows through blocks having zero entries. If $\zeta < \eta$, find the minimum of the entries not covered by any line. Let δ be the minimum of the uncovered entries. Subtract δ from all uncovered entries and add δ to all entries covered by two lines. With the new matrix go to step 2.

4.1. An illustration

Example 4.3. Consider the following transportation problem.

Table 1

Origin	Destination				a_i
	D_1	D_2	D_3	D_4	
O_1	10	7	3	6	3
O_2	1	6	8	3	5
O_3	7	4	5	3	7
b_j	3	2	6	4	

First we solve the above transportation problem using two phase method, namely North West corner rule, to find feasible solution and uv-method (primal-dual method) to find optimal solution. Note that we obtain $x_{11} = 3$, $x_{22} = 2$, $x_{23} = 3$, $x_{33} = 3$ and $x_{34} = 4$ by applying North West corner rule and finally we obtain $x_{13} = 3$, $x_{21} = 3$, $x_{24} = 2$, $x_{32} = 2$, $x_{33} = 3$ and $x_{34} = 2$ as optimal solution by applying uv-method.

Now we use the proposed weighted Hungarian method to solve the transportation problem.

(i) Following the step 1 of the proposed method, we subtract the smallest element in each row, and the smallest element in each column. We find the reduced cost matrix as in Table 2.

Table 2

7 – –	3 – –	0 – –	3
0	4	7	2
4	0	2	0

(ii) Now we consider the step 2 of the proposed method. In Table 2, row 1, column 1, column 2 and column 4 are crossed out. The lines to cover all zeros with minimum weight (12) is shown in Table 2.

(iii) Now we consider the step 3 of the proposed method. The lines drawn with minimum total weight is not equal to $\eta = 15$. The minimum of the not crossed out elements is subtracted from these elements and added to the elements which are on the intersection of two lines. Continuing step 3 for another two iterations we find the optimal solutions. See Table 3 and 4.

Table 3

9 – –	5 – –	0 – –	5
0	4	5	2
4 – –	0 – –	0 – –	0

Table 4

11 – –	5 – –	0 – –	5
0 – –	2 – –	3 – –	0
6 – –	0 – –	0 – –	0

We obtain the optimal solution based on the following approach. Inspecting the rows of the final reduced cost matrix we see that the row 1 contains only one zero which occurs in the 3^{rd} column. Since $\min(a_1, b_3) = a_1 = 3$, we have $x_{13} = 3$. We cross out row 1 (since no assignment will be made further) and update b_3 by $b_3 - a_1 = 3$. Now inspecting the columns we see that entry in the column 1 and row 2 contains 0. Since $\min(a_2, b_1) = b_1 = 3$, we have $x_{21} = 3$. We cross out column 1. Continuing in this manner we have $x_{32} = 2$, $x_{33} = 3$, $x_{24} = 2$ and $x_{34} = 2$.

Remark 4.4. *Note that it is quite likely to arise rows and columns with more than one zero. We call this phenomenon tie. As a tie breaking rule, we propose to select rows and columns arbitrarily with minimum number of zeros.*

5. CONCLUSION

At the outset, this paper considers some structured transportation problems which arise in sample surveys and other areas of statistics, and shows that optimal solution can be obtained by applying the North West corner rule. Subsequently, we consider a weighted version of König-Egerváry theorem and the corresponding version of Hungarian method. We propose the weighted Hungarian method to find

the solution of the transportation problem. The advantage of the proposed method is to find optimal solution of the transportation problem in a single phase. The limitation of weighted Hungarian method is mentioned in Remark 4.4. Finally, we propose a future scope of research on this study stating the following two questions:
(i) Can we identify the instances of transportation problems for which weighted Hungarian method can find rows and columns uniquely?
(ii) Can the weighted Hungarian method be used to solve nonlinear transportation problems?

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