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EFFICIENT MATRIX-FREE DIRECTION METHOD WITH LINE SEARCH FOR SOLVING LARGE-SCALE SYSTEM OF NONLINEAR EQUATIONS

Abubakar Sani HALILU
Department of Mathematics and Computer Science, Sule Lamido University, Kafin Hausa, Nigeria
abubakars.halilu@slu.edu.ng

Mohammed Yusuf WAZIRI
Department of Mathematical Sciences, Bayero University, Kano, Nigeria
mywaziri.mth@buk.edu.ng

Ibrahim YUSUF
Department of Mathematical Sciences, Bayero University, Kano, Nigeria
iyusuf.cs@buk.edu.ng

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Abstract: We proposed a matrix-free direction with an inexact line search technique to solve system of nonlinear equations by using double direction approach. In this article, we approximated the Jacobian matrix by appropriately constructed matrix-free method via acceleration parameter. The global convergence of our method is established under mild conditions. Numerical comparisons reported in this paper are based on a set of large-scale test problems and show that the proposed method is efficient for large-scale problems.

Keywords: Acceleration Parameter, Matrix-free, Inexact Line Search, Jacobian Matrix.

MSC: 65K05, 90C53, 65D32, 34G20.

1. INTRODUCTION

Systems of nonlinear equations form a family of problems that are equivalent to unconstrained optimization problems, and they often arise in the fields of science and technology. In recent years, researchers have considered various examples in this areas.

Matrix-free methods are very popular and widely used methods for solving the system of nonlinear equations. A typical system of nonlinear equations is represented by

$$F(x) = 0, \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear map.
Throughout the paper, the space $\mathbb{R}^n$ denote the $n-$dimensional real space equipped with the Euclidean norm $||\cdot||$. More applications of the problem (1) in economic equilibrium analysis, chemical equilibrium systems, compressive sensing, and control theory can be found in [14, 17, 21] and in the references therein. Some iterative methods for solving these problems include Newton and quasi-Newton methods [3, 12, 15, 18], the Gauss-Newton methods [7, 22], the Levenberg-Marquardt methods [16, 19, 23], the derivative-free methods [9, 13, 25, 29], the subspace methods [24], and the tensor methods [26].

The most popular schemes for solving (1) are based on successive linearization [3], where the search direction d_k is obtained by solving the following linear equation:

$$F(x_k) + F'(x_k)d_k = 0, \tag{2}$$

where $F'(x_k)$ is the Jacobian matrix of $F(x_k)$ at x_k or an approximation of it. The attractive features of Newtons method is easy implementation and rapid convergence [3]. However, this method requires the computation of Jacobian matrix, which invokes the first-order derivative of the system. It is well known that the computation of some function derivatives are costly in practice, sometimes they are not even available or could not be obtained exactly. In this case Newtons method cannot be directly applied [3, 11].

Based on this fact, the double direction method has been proposed in [2] and the iterative procedure is given as:

$$x_{k+1} = x_k + \alpha_k b_k + \alpha_k^2 c_k, \tag{3}$$

where x_{k+1} represents a new iterative point, x_k is the previous iteration, and $\alpha_k > 0$ denotes the step length, while b_k and c_k are search directions, respectively. We are interested in approximating the Jacobian with diagonal matrix via:

$$F'(x_k) \approx \gamma_k I,$$

where I is an identity matrix.
Furthermore, (1) can come from an unconstrained optimization problem, a saddle point, and equality constrained problem [7]. Let f be a norm function defined by

$$f(x) = \frac{1}{2}\|F(x)\|^2. \tag{4}$$

The nonlinear equations problem (1) is equivalent to the following global optimization problem

$$minf(x), \quad x \in \mathbb{R}^n. \tag{5}$$

The double direction method is proposed by Duranovic-Milicic [2], where using multi-step iterative information and curve search to generate new iterative points. However a double direction method for solving unconstrained optimization problem was presented by Petrovic and Stanimirovic [8]. In [9] Halilu and Waziri incorporated the work in [8] to solve the system of nonlinear equations, and approximated the Jacobian matrix with diagonal matrix via acceleration parameter. The global convergence of the scheme [9] is established under mild conditions. Furthermore, in order to improve the numerical performances and global convergence properties of double direction methods, transformation of double step length scheme is proposed in [5]. Recently, in [6], Halilu and Waziri proposed an enhanced matrix-free method via double step length approach for solving systems of nonlinear equations. The method was proven to be globally convergent by using the inexact line search proposed by Li and Fukushima [7]. Therefore, motivated by [8], we aimed at developing a matrix-free direction method with line search for solving systems of nonlinear equations, without computing the Jacobian matrix with less number of iterations and CPU time, that is globally convergent.

There are some known procedures for driving the search directions [1, 11, 12, 28]. The step length α_k can also be computed either exact or inexact. It is very expensive to find exact step length in practical computation. Therefore, the most frequently used line search in practice is inexact line search [9, 10, 15, 20]. A basic requirement of the line search is to sufficiently decrease the function values, i.e., to establish

$$\|F(x_{k+1})\| \leq \|F(x_k)\|.$$

We organized the paper as follows; In the next section, we present the proposed method, and th convergence results are presented in section 3. Some numerical experiment results are reported in section 4. Finally, we present concluding remarks in section 5.

2. DETAILS OF THE METHOD

In this section, we propose to reduce the two directions vectors (3) into a single one. This is made possible by making the two directions to be equal, i.e $b_k = c_k$. We suggest, that the search directions b_k and c_k in (3) be defined as:

$$b_k = c_k = -\gamma_k^{-1} F(x_k), \tag{6}$$

By putting (6) into (3), we obtained

$$x_{k+1} = x_k - \alpha_k(\gamma_k^{-1}(1 + \alpha_k))F(x_k). \tag{7}$$

From (7) we can easily show that our direction is

$$d_k = -(1+\alpha_k)\gamma_k^{-1}F(x_k). \tag{8}$$

We adopt the acceleration parameter used in [5] in order to improve good direction towards the solution. The technique in [5] generates a sequence of iterates $\{x_k\}$ such that $x_{k+1} = x_k + (\alpha_k + \frac{1}{2}\alpha_k\gamma_k)d_k$ and the acceleration parameter γ_k is obtained by using Taylor's expansion of the first order as:

$$\gamma_{k+1} = \frac{||y_k||^2}{y_k^T s_k}, \tag{9}$$

with $y_k = F(x_{k+1}) - F(x_k)$ and $s_k = x_{k+1} = x_k$.
So, from (7) and (8), we have the general scheme as:

$$x_{k+1} = x_k + \alpha_k d_k. \tag{10}$$

We then used the derivative-free line search proposed by Li and Fukushima [7] in order to compute our step length α_k.
Let $\omega_1 > 0$, $\omega_2 > 0$ and $r \in (0,1)$ be constants and let η_k be a given positive sequence such that

$$\sum_{k=0}^{\infty} \eta_k < \eta < \infty, \tag{11}$$

and

$$f(x_k + \alpha_k d_k) - f(x_k) \leq -\omega_1\|\alpha_k F(x_k)\|^2 - \omega_2\|\alpha_k d_k\|^2 + \eta_k f(x_k). \tag{12}$$

Let i_k be the smallest non negative integer i such that (12) holds for $\alpha = r^i$. Let $\alpha_k = r^{i_k}$.

Now, we describe the algorithm of the proposed method as follows:
Algorithm 1(EMD)
STEP 1: Given x_0, $\gamma_0 = 0.01$, $\alpha > 0$, $\epsilon = 10^{-4}$, set $k = 0$.
STEP 2: Compute $F(x_k)$.
STEP 3: Test the stopping criterion. If yes, then stop; otherwise, continue with Step 4.
STEP 4: Compute search direction d_k (using (8)).
STEP 5: Compute step length α_k(using (12)).
STEP 6: Set $x_{k+1} = x_k + \alpha_k d_k$.
STEP 7: Compute $F(x_{k+1})$.
STEP 8: Determine γ_{k+1}(using (9)).
STEP 9: Set k=k+1, and go to STEP 3.

3. CONVERGENCE RESULT

In this section, we present the global convergence of our method (EMD). To begin with, let us defined the level set

$$\Omega = \{x | \|F(x)\| \leq \|F(x_0)\|\}. \tag{13}$$

In order to analyze the convergence of algorithm 1, we need the following assumption:

Assumption 1

(1) There exists $x^* \in \mathbb{R}^n$ such that $F(x^*) = 0$.

(2) F is continuously differentiable in some neighborhood, say N of x^* containing Ω.

(3) The Jacobian of F is bounded and positive definite on N, i.e., there exists a positive constants $M > m > 0$ such that

$$\|F'(x)\| \leq M \quad \forall x \in N, \tag{14}$$

and

$$m\|d\|^2 \leq d^T F'(x)d \quad \forall x \in N, d \in \mathbb{R}^n. \tag{15}$$

Remarks:

Assumption 1 implies that there exist constants $M > m > 0$ such that

$$m\|d\| \leq \|F'(x)d\| \leq M\|d\| \quad \forall x \in N, d \in \mathbb{R}^n. \tag{16}$$

$$m\|x - y\| \leq \|F(x) - F(y)\| \leq M\|x - y\| \quad \forall x, y \in N. \tag{17}$$

In particular $\forall x \in N$ we have

$$m\|x - x^*\| \leq \|F(x)\| \leq \|F(x) - F(x^*)\| \leq M\|x - x^*\|,$$

where x^* stands for the unique solution of (1) in N. Since $\gamma_k I$ approximates $F'(x_k)$ along direction s_k, we can state another assumption.

Assumption 2

$\gamma_k I$ is a good approximation to $F'(x_k)$, i.e

$$\|(F'(x_k) - \gamma_k I)d_k\| \leq \epsilon\|F(x_k)\|. \tag{18}$$

where $\epsilon \in (0, 1)$ is a small quantity [18].

Lemma 3.1. Suppose that assumption 2 holds and $\{x_k\}$ be generated by algorithm 1. Then, d_k is a descent direction for $f(x_k)$ at x_k i.e,

$$\nabla f(x_k)^T d_k < 0. \tag{19}$$

proof. From (6), we have

$$\begin{aligned} \nabla f(x_k)^T d_k &= F(x_k)^T F'(x_k)d_k, \\ &= F(x_k)^T [(F'(x_k) - \gamma_k I)d_k - (1 + \alpha_k)F(x_k)], \\ &= F(x_k)^T (F'(x_k) - \gamma_k I)d_k - (1 + \alpha_k)\|F(x_k)\|^2, \end{aligned} \tag{20}$$

by Chauchy-Schwarz we have,

$$\begin{aligned}\nabla f(x_k)^T d_k &\leq \|F(x_k)\|\|(F'(x_k)-\gamma_k I)d_k\| - (1+\alpha_k)\|F(x_k)\|^2,\\ &\leq -(1-\epsilon)\|F(x_k)\|^2 - \|\sqrt{\alpha_k}F(x_k)\|^2,\\ &\leq -(1-\epsilon)\|F(x_k)\|^2.\end{aligned} \tag{21}$$

Hence for $\epsilon \in (0,1)$ this lemma is true.
By the above lemma, we can deduce that the norm function $f(x_k)$ is a descent along d_k which means that $\|F(x_{k+1})\| \leq \|F(x_k)\|$ is true.
Lemma 3.2. Suppose that assumption 2 holds and $\{x_k\}$ be generated by algorithm 1. Then $\{x_k\} \subset \Omega$.
Proof. By lemma 3.1, we have $\|F(x_{k+1})\| \leq \|F(x_k)\|$. Moreover, we have for all k

$$\|F(x_{k+1})\| \leq \|F(x_k)\| \leq \|F(x_{k-1})\| \leq \ldots \leq \|F(x_0)\|.$$

This implies that $\{x_k\} \subset \Omega$.
Lemma 3.3. Suppose that assumption 1 holds and $\{x_k\}$ is generated by algorithm 1. Then there exists a constant $m > 0$ such that for all k

$$y_k^T s_k \geq m\|s_k\|^2. \tag{22}$$

Proof. By mean-value theorem, we have, $y_k^T s_k = s_k^T(F(x_{k+1})-F(x_k)) = s_k^T F'(\xi)s_k \geq m\|s_k\|^2$.
Where $\xi = x_k + \zeta(x_{k+1} - x_k)$, $\zeta \in (0,1)$; the last inequality follows from (15). The proof is completed.
Using $y_k^T s_k \geq m\|s_k\|^2 > 0$, γ_{k+1} is always generated by the update of formula (9), and we can deduce that $\gamma_{k+1}I$ inherits the positive definiteness of $\gamma_k I$. By the above lemma and (17), we obtained

$$\frac{y_k^T s_k}{\|s_k\|} \geq m, \qquad \frac{\|y_k\|^2}{y_k^T s_k} \leq \frac{M^2}{m}. \tag{23}$$

Lemma 3.4. Suppose that assumption 2 holds and $\{x_k\}$ is generated by algorithm 1. Then we have

$$\lim_{k\to\infty} \|\alpha_k d_k\| = \lim_{k\to\infty} \|s_k\| = 0, \tag{24}$$

and

$$\lim_{k\to\infty} \|\alpha_k F(x_k)\| = 0. \tag{25}$$

Proof. By (12) we have for all $k > 0$

$$\begin{aligned}\omega_2\|\alpha_k d_k\|^2 &\leq \omega_1\|\alpha_k F(x_k)\|^2 + \omega_2\|\alpha_k d_k\|^2,\\ &\leq \|F(x_k)\|^2 - \|F(x_{k+1})\|^2 + \eta_k\|F(x_k)\|^2,\end{aligned} \tag{26}$$

by summing the above inequality, we have

$$\begin{aligned}\omega_2 \sum_{i=0}^{k} \|\alpha_i d_i\|^2 &\leq \sum_{i=0}^{k} \left(\|F(x_i)\|^2 - \|F(x_{i+1})\|^2\right) + \sum_{i=0}^{k} \eta_i \|F(x_i)\|^2, \\ &= \|F(x_0)\|^2 - \|F(x_{k+1})\|^2 + \sum_{i=0}^{k} \eta_i \|F(x_i)\|^2, \\ &\leq \|F(x_0)\|^2 + \|F(x_0)\|^2 \sum_{i=0}^{k} \eta_i, \\ &\leq \|F(x_0)\|^2 + \|F(x_0)\|^2 \sum_{i=0}^{\infty} \eta_i.\end{aligned} \tag{27}$$

So from the level set and fact that $\{\eta_k\}$ satisfies (11) then the series $\sum_{i=0}^{\infty} \|\alpha_i d_i\|^2$ converged. This implies (24). By similar arguments as the above but with $\omega_1 \|\alpha_k F(x_k)\|^2$ on the left hand side, we obtain (25).

Lemma 3.5. Suppose that assumption 2 holds and $\{x_k\}$ is generated by algorithm 1. Then there exist a constant $m_3 > 0$ such that for all $k > 0$,

$$\|d_k\| \leq m_3. \tag{28}$$

Proof. From (8) and (17) we have

$$\begin{aligned}\|d_k\| &= \left\| -\frac{(1+\alpha_k)F(x_k) y_k^T s_k}{\|y_k\|^2} \right\|, \\ &\leq \frac{(1+\alpha_k)\|F(x_k)\| \|s_k\| \|y_k\|}{m^2 \|s_k\|^2}, \\ &\leq \frac{(1+\alpha_k)\|F(x_k)\| M \|s_k\|}{m^2 \|s_k\|}, \\ &\leq \frac{(1+\alpha_k)\|F(x_k)\| M}{m^2}, \\ &= \frac{\|F(x_k)\| M + \|\alpha_k F(x_k)\| M}{m^2}, \\ &\leq \frac{(\|F(x_0)\| + P) M}{m^2},\end{aligned} \tag{29}$$

where P is some positive constant. Taking $m_3 = \frac{(\|F(x_0)\|+P)M}{m^2}$, we have (28). We can deduce that for all k, (28) holds.

Now we are going to establish a global convergence theorem to show that under some suitable conditions, there exist an accumulation point of $\{x_k\}$ which is a solution to problem (1).

Theorem 3.1. Suppose that assumption 2 holds,$\{x_k\}$ is generated by algorithm

1. Assume further for all $k > 0$,

$$\alpha_k \geq c\frac{|F(x_k)d_k|}{\|d_k\|^2}, \tag{30}$$

where c is some positive constant. Then

$$\lim_{k\to\infty} \|F(x_k)\| = 0. \tag{31}$$

Proof. From lemma 3.5 we have (28). Therefore by (24) and the boundedness of $\{\|d_k\|\}$, we have

$$\lim_{k\to\infty} \alpha_k\|d_k\|^2 = 0. \tag{32}$$

From (30) and (32) it follows that

$$\lim_{k\to\infty} |F(x_k)^T d_k| = 0. \tag{33}$$

On the other hand, (8) leads to,

$$\begin{aligned} -\gamma_k F(x_k)^T d_k &= (1+\alpha_k)\|F(x_k)\|^2, \\ &= \|F(x_k)\|^2 + \|\alpha_k F(x_k)\|^2, \end{aligned} \tag{34}$$

and

$$\begin{aligned} \|F(x_k)\|^2 &= \| -\gamma_k F(x_k)^T d_k\| - \|\alpha_k F(x_k)\|^2 \\ &\leq |\gamma_k||F(x_k)^T d_k|, \end{aligned} \tag{35}$$

but

$$\gamma_k^{-1} = \frac{y_{k-1}^T s_{k-1}}{\|y_{k-1}\|^2} \geq \frac{m\|s_{k-1}\|^2}{\|y_{k-1}\|^2} \geq \frac{m\|s_{k-1}\|^2}{M^2\|s_{k-1}\|^2} = \frac{m}{M^2}.$$

Then

$$|\gamma_k^{-1}| \geq \frac{m}{M^2},$$

so from (35),

$$\|F(x_k)\|^2 \leq |F(x_k)^T d_k| \left(\frac{M^2}{m}\right). \tag{36}$$

Thus

$$0 \leq \|F(x_k)\|^2 \leq |F(x_k)^T d_k| \left(\frac{M^2}{m}\right) \longrightarrow 0. \tag{37}$$

Therefore

$$\lim_{k\to\infty} \|F(x_k)\| = 0. \tag{38}$$

The proof is completed.

4. NUMERICAL RESULTS

In this section, the performance of the proposed method is compared with a derivative-free CG method and its global convergence for solving symmetric nonlinear equations [10]. For both methods the following parameters are set, $\omega_1 = \omega_2 = 10^{-4}$, $\alpha_0 = 0.01$, $r = 0.2$ and $\eta_k = \dfrac{1}{(k+1)^2}$.

The employed computational codes ware written in Matlab 7.9.0 (R2009b) and run on a personal computer 2.00 GHz CPU processor and 3 GB RAM memory. We stop the iteration if the total number of iterations exceeds 1000 or $\|F(x_k)\| \leq 10^{-4}$. We claim that the method fails, and use the symbol "-" to represent failure due to: (i) Memory requirement, (ii) Number of iterations exceed 1000, (iii) If $\|F(x_k)\|$ is not a number. The methods were tested on some Benchmark test problems with different initial points. Problems 1-7 below are from [10] and problems 9 and 10 are from [27], while problem 8 is an artificial problem.

Problem 1:

$$F(x) = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix} x + (e_1^x - 1, ..., e_n^x - 1)^T. \tag{39}$$

Problem 2:

$$F(x) = \begin{pmatrix} 2 & -1 & & & \\ 0 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix} x + (sinx_1 - 1, ..., sinx_n - 1)^T. \tag{40}$$

Problem 3:

$$\begin{aligned} F_1(x) &= x_1(x_1^2 + x_2^2) - 1, \\ F_i(x) &= x_i(x_{i-1}^2 + 2x_i^2 + x_{i+1}^2), \\ F_n(x) &= x_n(x_{n-1}^2 + x_n^2). \\ i &= 2, 3, ..., n-1. \end{aligned} \tag{41}$$

Problem 4:

$$
\begin{aligned}
F_{3i-2}(x) &= x_{3i} - 2x_{3i-1} - x_{3i}^2 - 1, \\
F_{3i-1}(x) &= x_{3i-2}x_{3i-2}x_{3i} - x_{3i-2}^2 + x_{3i-1}^2 - 2, \\
F_{3i}(x) &= e^{-x_{3i-2}} - e^{-x_{3i-1}}. \\
i &= 1, ..., \frac{n}{3}.
\end{aligned} \tag{42}
$$

Problem 5:

$$
\begin{aligned}
F_i(x) &= (1 - x_i^2) + x_i(1 + x_i x_{n-2} x_{n-1} x_n) - 2. \\
i &= 1, 2, ..., n.
\end{aligned} \tag{43}
$$

Problem 6:

$$
\begin{aligned}
F_1(x) &= x_1^2 - 3x_1 + 1 + cos(x_1 - x_2), \\
F_i(x) &= x_1^2 - 3x_i + 1 + cos(x_i - x_{i-1}). \\
i &= 1, 2, ..., n.
\end{aligned} \tag{44}
$$

Problem 7:

$$
\begin{aligned}
F_i(x) &= x_i - 0.1x_{i+1}^2, \\
F_n(x) &= x_n - 0.1x_1^2. \\
i &= 1, 2, ..., n - 1.
\end{aligned} \tag{45}
$$

Problem 8:

$$
\begin{aligned}
F_i(x) &= 0.i(1 - x_i)^2 - e^{-x_i^2}, \\
F_n(x) &= \frac{n}{10}(1 - e^{-x_n^2}). \\
i &= 1, 2, ..., n - 1.
\end{aligned} \tag{46}
$$

Problem 9. The discretized Chandrasehars H-equation:

$$
\begin{aligned}
F_i(x) &= x_i - \left(1 - \frac{c}{2n}\sum_{j=1}^{n}\frac{\mu_i x_j}{\mu_i + \mu_j}\right)^{-1} \\
i &= 1, 2, ..., n, \quad j = 1, 2, ..., n.
\end{aligned} \tag{47}
$$

with $c \in [0, 1)$ and $\mu = \frac{i-0.5}{n}$, for $1 \leq i \leq n$. (In our experiment we take c =0.1).

Problem 10.

$$
\begin{aligned}
F_i(x) = 2\left(n + i(1 - cosx_i) - sinx_i - \sum_{j=1}^{n} cosx_j\right)(2sinx_i - cosx_i) \\
i = 1, 2, ..., n.
\end{aligned} \tag{48}
$$

Table 1: Test results for the two methods, where e=Ones(n,1)

			EMD			*DFCG*		
Problems	x_0	n	Iter	Time(s)	$\|F(x_k)\|$	Iter	Time(s)	$\|F(x_k)\|$
1	0.5*e	10	17	0.046537	6.53E-05	33	0.137884	9.74E-05
		100	20	0.097375	8.55E-05	38	0.182246	9.55E-05
		1000	19	0.529183	7.73E-05	53	2.285821	8.72E-05
		2000	24	2.108369	8.97E-05	54	7.791001	8.10E-05
2	e	10	14	0.052196	7.09E-05	49	0.18529	4.08E-05
		100	15	0.050377	9.85E-05	60	0.291577	8.65E-05
		1000	17	0.492775	3.77E-05	63	2.874518	9.31E-05
		2000	17	1.575679	4.12E-05	61	9.321487	9.30E-05
3	0.01*e	10	18	0.005688	5.46E-05	52	0.021726	9.57E-05
		100	25	0.010826	9.24E-05	52	0.021726	9.57E-05
		1000	24	0.037681	6.58E-05	54	0.105493	8.83E-05
		2000	27	0.076132	9.43E-05	54	0.176152	8.43E-05
		3000	26	0.088922	8.47E-05	62	0.237935	9.52E-05
		50000	26	1.229412	5.90E-05	55	3.550896	7.53E-05
4	0.1*e	10	15	0.029809	5.75E-05	47	0.018898	8.07E-05
		100	17	0.012018	2.88E-05	66	0.034618	9.72E-05
		1000	17	0.016923	9.14E-05	60	0.072484	8.25E-05
		5000	19	0.086431	7.42E-05	57	0.308569	9.39E-05
		10000	20	0.163747	5.69E-05	58	0.637811	6.51E-05
5	0.7*e	10	15	0.004618	3.52E-05	431	0.176174	9.54E-07
		100	16	0.008113	6.67E-05	431	0.31315	3.02E-06
		1000	17	0.027731	4.22E-05	431	0.996263	9.54E-06
		5000	17	0.086502	9.44E-05	431	4.354077	2.13E-05
		10000	18	0.148485	8.01E-05	431	8.684382	3.02E-05
6	0.4*e	10	14	0.027285	7.76E-05	-	-	-
		100	15	0.002827	4.91E-05	-	-	-
		1000	16	0.0171	9.31E-05	-	-	-
		5000	17	0.071727	4.16E-05	-	-	-
		10000	17	0.125057	5.89E-05	-	-	-
7	e	10	10	0.00801	5.51E-05	5	0.036751	5.23E-06
		100	12	0.016974	2.18E-05	5	0.017619	2.35E-05
		1000	12	0.349293	6.91E-05	5	0.206744	7.52E-05
		5000	13	2.454612	9.27E-05	6	1.910818	3.28E-08
		10000	14	6.537241	2.62E-05	6	5.167084	4.64E-08
8	0.5*e	10	4	0.069137	7.61E-05	14	0.005212	5.80E-05
		100	4	0.001451	5.12E-05	13	0.008095	6.11E-05
		1000	9	0.008449	6.57E-05	27	0.057786	6.03E-05
		5000	10	0.046607	3.54E-05	23	0.138946	6.18E-05
		10000	7	0.072395	2.36E-05	36	0.466298	1.29E-06
9	-10*e	10	16	0.026132	3.67E-05	39	0.033934	9.61E-05
		100	16	0.014753	6.26E-05	48	0.056708	6.08E-05
		1000	18	0.027309	5.27E-05	63	0.115058	6.53E-05
		5000	16	0.084268	3.11E-05	69	0.442739	9.04E-05
		10000	22	0.166954	7.40E-05	-	-	-
10	-20*e	10	12	0.015719	7.57E-05	33	0.025701	5.81E-06
		100	14	0.013666	2.81E-05	-	-	-
		1000	14	0.063125	7.38E-06	-	-	-
		5000	19	0.582888	8.56E-05	-	-	-
		10000	18	0.997046	2.73E-05	-	-	-

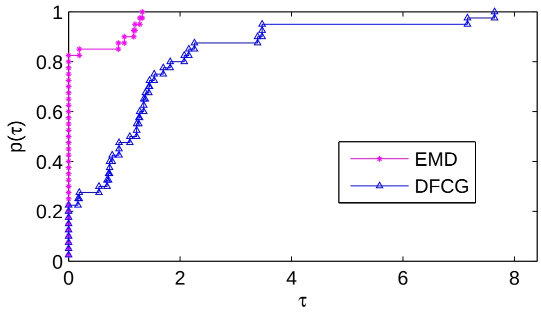

Figure 1: Performance profile with respect to the number of iterations

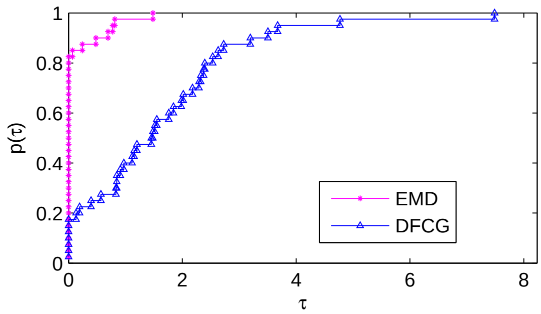

Figure 2: Performance profile with respect to the CPU time (in second)

The numerical results of the two methods are reported in Table 1, where "Iter" and "Time" stand for the total number of all iterations and the CPU time in seconds, respectively, while $\|F(x_k)\|$ is the norm of the residual at the stopping point. From Table 1, we can easily observe that both methods attempt to solve the systems of nonlinear equations (1), but the better efficiency and effectiveness of our algorithm is clear for it solves where DFCG fails. This is quite evident for instance with problems 6, 9 and 10. In particular, the EMD method considerably outperforms the DFCG for almost all the tested problems, as it has the smallest number of iterations and shorter CPU time, which is even smaller than the CPU time for the DFCG method.

Figures (1-2) show the performance of our method relative to the number of iterations and CPU time, which were evaluated using the profiles of Dolan and Moré [4]. That is, for each method, we plot the fraction $P(\tau)$ of the problems for which the method is within a factor τ of the best time. The top curve is the method that solved the most problems in a time within a factor τ of the best time.

5. CONCLUSION

In this paper, an efficient matrix-free direction method with line search for solving large scale systems of nonlinear equations is derived. It is a fully matrix-free iterative method which possesses global convergence under some reasonable conditions. Numerical comparisons using a set of large-scale test problems show that the proposed method is practically quite effective.

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