FINDING EFFICIENT SOLUTIONS IN THE INTERVAL MULTI-OBJECTIVE LINEAR PROGRAMMING MODELS

Aida BATAMIZ
Mathematics Faculty, University of Sistan and Baluchestan
Zahedan, IRAN
aida.batamiz@yahoo.com

Mehdi ALLAHDADI
Mathematics Faculty, University of Sistan and Baluchestan
Zahedan, IRAN
m_allahdadi@math.usb.ac.ir

Abstract: The aim of our paper is to obtain efficient solutions to the interval multi-objective linear programming (IMOLP) models. In this paper, we propose a new method to determine the efficient solutions in the IMOLP models by using the expected value and variance operators (EVV operators). First, we define concepts of the expected value, variance, and uncertainty distributions, and present some properties of the EVV operators. Then, we introduce the IMOLP model under these operators. An IMOLP model consist of separate ILPs, but using the EVV operators and the uncertainty distributions, it can be converted into the interval linear programming (ILP) models under the EVV operators (EVV-ILP model). We show that optimal solutions of the EEV-ILP model are the efficient solutions of IMOLP models with uncertainty variables. The proposed method, which is called EVV, is not hard to solve. Finally, Monte Carlo simulation is used to show its performance assessment.

1. INTRODUCTION

Multi-objective models play a very important role in real-world decision making problems. Hence, these models are interesting for many researchers who work in statistical fields, engineering sciences, mathematics, and seek to calculate and find optimal and efficient solutions to such models. One method for modelling problems under uncertainty is the interval linear programming (ILP). Generally, for solving ILP models, two sub-models are proposed. Solution region of the ILP is determined by solving these two sub-models. One of these sub-models obtains the best value (optimistic model), and the other gives the worst value (pessimistic model) for the objective functions.

All methods used in multi-objective linear programming (MOLP) and ILP can be used for solving the interval multi-objective linear programming (IMOLP) models, such as interactive approaches, optimization methods, goal programming, and fuzzy method, which have been used for determining efficient solutions and optimal solutions of these models [18, 20, 31, 33]. There are some methods for obtaining possible and necessary efficient solutions, efficient solutions and weak efficient solution in the IMOLP models [12, 13, 17, 20]. Some methods have been proposed which use the concept of regret and the weighted sum of maximum regrets for obtaining efficient solutions. These are the criteria for finding possible and necessary efficient solutions in the IMOLP. Thereafter, some modified maximum regret approaches were presented to find a necessarily efficient solution in the IMOLP model [14, 21, 34, 35, 36].

A methodology for solving the IMOLP models is extended through a fuzzy set based approach by Razavi Hajiagha [33]. The other methods to solve the IMOLP models include the ε-constraint, Lexicographic and WILP methods [2]. Dechao et al. [9] employed an admissible order and interval ordered weighted aggregation operator to transform a IMOLP problem into an interval weighted sum scalarization multiobjective optimization problem whose solution can be derived by solving several related real-valued programming problems, and the Pareto optimal solution of this IMOLP problem can likewise be obtained. Bharati and Singh developed a new method for obtaining the solution of the MOLP models based on interval-valued intuitionistic fuzzy sets [7].

Uncertainty theory was founded by Liu in 2007, and a branch of mathematics for modelling under uncertainty was introduced [24]. Also, using the definitions in the field of expected value, he established a new concept of this operator to uncertainties and modelling. The applications of uncertainty theory were investigated by Jiao and Yao [22], Wang et al. [39, 40] and [41], Guo et al. [11], Liu and Yao [30], and Li et al [29] and Zheng et al. [44].

In this paper, we discuss the IMOLP models under the expected value and variance operators (EVV operators) using the uncertainty theory [24, 26, 27, 28]. Then, we obtain the efficient solutions using EVV operators and the uncertainty distributions that can convert the IMOLP models into the interval linear programming (ILP) models (that we call EVV-ILP model). The efficient solutions are found by solving the EVV-ILP models. This method, which is called EVV method, is
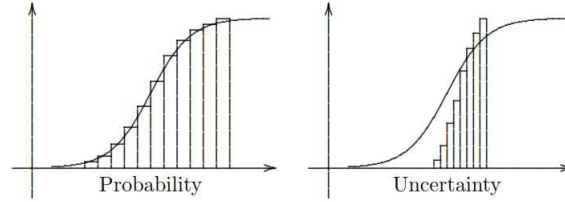
Figure 1: Difference of uncertainty and probability

considered as a suitable tool in decision making problems.

Our paper is organized as follows. Section 2 reviews some basic results of the uncertainty theory. Section 3 presents the IMOLP model and defines different efficient solution concepts. Section 4 establishes efficient solutions in the IMOLP under the EVV operators by uncertainty theory. In section 5, Monte Carlo simulation is reviewed to demonstrate the performance assessement of the proposed method, where it is supposed that all coefficients are random samples with normal contributions. Section 6 presents two examples solved by using EVV method and Monte Carlo simulation.

2. UNCERTAINTY

Real-world decisions are usually made in a state of uncertainty. To rationally deal with uncertainty, there exist two mathematical systems: probability theory [23], and uncertainty theory [24]. Uncertainty theory is a branch of mathematics demonstrating belief degrees. A belief degree presents the degree which at we believe the event will happen. If we believe that the estimated uncertainty distribution is close enough to the belief degrees hidden in the mind of the domain experts, then we may use uncertainty theory to deal with our own models on the basis of the estimated uncertainty distributions. This section provides axioms of uncertainty theory and fundamental concepts and uncertainty distribution in uncertainty theory (here, we just study linear distribution). As shown in Fig. 1, probability theory is only applicable to modelling frequencies, and uncertainty theory to modelling belief degrees [27]. From the strictly mathematical viewpoint, uncertainty theory is an alternative theory of measure. Thus uncertainty theory starts with a measurable space. So, we will have the following definitions.

Definition 1. *[44] Let* $(\Gamma, \mathcal{L})$ *is a measurable space,* Γ *a nonempty set, and* $\mathcal{L}$ *a* σ-*algebra over* Γ. *Each element* Λ *in* $\mathcal{L}$ *is called an event. A set function* $\mathcal{M}$ *from* $\mathcal{L}$ *to* [0, 1] *is called an uncertain measure if it satisfies the following axioms:*

Axiom 1 (Normality axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (Duality axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (Sub-additively axiom) For every countable sequence of events Λ_1, Λ_2, ..., we have $\mathcal{M}\left\{\bigcup\limits_{i=1}^{\infty} \Lambda_i\right\} \leq \sum\limits_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called uncertainty

space.

Axiom 4 (Product axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty space for $k = 1, 2, ...$ A product uncertain measure is defined as follows:

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_i\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\left\{\Lambda_k\right\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, ...$ Indeed, a product uncertain measure is defined for rectangles, and the notation $\bigwedge$ is the minimum operator.

The set function $\mathcal{M}$ is called an uncertain measure if it satisfies normality, duality, and subadditivity axioms. An uncertain measure is interpreted as the personal belief degree of an uncertain event that may happen.

Definition 2. *[27] The smallest σ-algebra B containing all open intervals is called the Borel algebra over the set of real numbers, and any element in B is called a Borel set. An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers i.e., for any Borel set B of real numbers, the set following is an event.*

$$\{\xi \in B\} = \{\gamma \in \Gamma \,|\, \xi(\gamma) \in B\}.$$

Note that an uncertainty distribution presents incomplete information of uncertain variable. Therefore, it is very important to know the uncertainty distribution. So, we will have the following theorems and definitions.

Definition 3. *[28] For any real number x, the uncertainty distribution Φ of an uncertain variable ξ is defined as follows:*

$$\Phi(x) = M\{\xi \leq x\}.$$

Theorem 4. *[32] A function $\Phi(x) : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.*

Definition 5. *[25] An uncertain variable ξ is regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $0 < \alpha < 1$.*

Definition 6. *[25] Assume that ξ is an uncertain variable with regular uncertainty distribution Φ. Then the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ.*

Theorem 7. *[25] Let $\xi_1, \xi_2, ..., \xi_n$ be uncertain variables and f is a real-valued measurable function then, $f(\xi_1, \xi_2, ..., \xi_n)$ is an uncertain variable.*

Definition 8. *[24] Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E\left[\xi\right] = \int_0^\infty M\left\{\xi \geq x\right\}dx - \int_{-\infty}^0 M\left\{\xi \leq x\right\}dx.$$

Theorem 9. *[25] Let us have $\xi_1, \xi_2, ..., \xi_n$ and f be a real-valued measurable function then, $f(\xi_1, \xi_2, ..., \xi_n)$ is an uncertain variable. Let ϕ be a regular uncertainty distribution. If the expected value exists, then, $E\left[\xi\right] = \int_0^1 \phi^{-1}\left(\alpha\right)d\alpha$.*

If ξ and η are independent with finite expected values, then for any real numbers a and b, we have $E\left[a\xi + b\eta\right] = aE\left[\xi\right] + bE\left[\eta\right]$.

Theorem 10. *[42] Let f and g be comonotonic functions. Then for any ξ, we have $E\left[f(\xi) + g(\xi)\right] = E\left[f(\xi)\right] + E\left[g(\xi)\right]$. If e is a finite expected value, then the variance of ξ is $Var\left[\xi\right] = E\left[(\xi - e)^2\right]$.*

Theorem 11. *[41] If e is a finite expected value, a and b are real numbers, then $Var\left[a\xi + b\right] = a^2 Var\left[\xi\right]$.*

Theorem 12. *[43] Let ϕ be a regular uncertainty distribution and finite expected value e. Then $Var\left[\xi\right] = \int_0^1 \left(\phi^{-1}\left(\alpha\right) - e\right)^2 d\alpha$.*

Now, we define linear uncertainty distribution as follows.

Definition 13. *[24] ξ is called linear if it has a linear uncertainty distribution*

$$\phi\left(x\right) = \begin{cases} 0 & if \ x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & if \ x \geq b \end{cases}$$

denoted by $\mathcal{L}\left(a, b\right)$, where a and b are real numbers with $a < b$.

Definition 14. *The linear uncertain variable $\xi \in \mathcal{L}\left(a, b\right)$ has an expected value $E\left[\xi\right] = \frac{a+b}{2}$ and a variance $Var\left[\xi\right] = \frac{(b-a)^2}{12}$.*

3. INTERVAL MULTI-OBJECTIVE LINEAR PROGRAMMING

In this section, we define fundamental concepts about the ILP and IMOLP models. An interval number $[X^-, X^+]$ is shown as $X^\pm$ where $X^- \leq X^+$. If $X^- = X^+$, then $X^\pm$ is degenerate. If A^- and A^+ are two matrices in $\mathbb{R}^{m \times n}$ such that $A^- \leq A^+$, then the set of matrices $\mathbf{A}^\pm = [A^-, A^+] = \{A|\ A^- \leq A \leq A^+\}$ is called an interval matrix and the matrices A^- and A^+ are called its bounds. Centre and radius matrices are defined as: $\Delta_{A^\pm} = \frac{1}{2}\left(A^+ - A^-\right)$ and $A^c = \frac{1}{2}(A^- + A^+)$. A special case of an interval matrix is an interval vector $\mathbf{x}^\pm = \{x|x^- \leq x \leq x^+\}$

where $x^-, x^+ \in \mathbb{R}^n$ [1].

Consider the following IMOLP where k is the number of objective functions:

$$\max \quad z^\pm = C^\pm \mathbf{x}^\pm = (c_1^\pm \mathbf{x}^\pm, \, c_2^\pm \mathbf{x}^\pm, \, ..., \, c_k^\pm \mathbf{x}^\pm)^T$$
$$s.t.$$
$$A^\pm \mathbf{x}^\pm \leq b^\pm$$
$$\mathbf{x}^\pm \geq 0. \tag{1}$$

The characteristic model of model (1) is:

$$\max \quad z = C\mathbf{x} = (c_1\mathbf{x}, \, ..., \, c_k\mathbf{x})^T$$
$$s.t.$$
$$A\mathbf{x} \leq b$$
$$\mathbf{x} \geq 0, \tag{2}$$

where $C \in C^\pm$, $A \in A^\pm$, $b \in b^\pm$, and $\mathbf{x} \in \mathbf{x}^\pm$. Model (2) is referred to as a MOLP.

Definition 15. *Let X be the feasible space of MOLP (2). $\overline{\mathbf{x}} \in X$ is efficient if and only if there is no $\mathbf{x} \in X$ such that $C\mathbf{x} \geq C\overline{\mathbf{x}}$, $C\mathbf{x} \neq C\overline{\mathbf{x}}$.*

In each MOLP, there is rarely a point that can simultaneously maximize all objective functions. Also, we can define a given point as efficient for the IMOLP model if it is efficient for at least one characteristic MOLP. Consider the following ILP model:

$$\max \quad z^\pm = \mathbf{c}^\pm \mathbf{x}^\pm$$
$$s.t. \quad A^\pm \mathbf{x}^\pm \leq b^\pm$$
$$\mathbf{x}^\pm \geq 0. \tag{3}$$

The characteristic model of ILP model (3) is:

$$\max \quad z = \mathbf{c}\mathbf{x}$$
$$s.t. \quad A\mathbf{x} \leq b$$
$$\mathbf{x} \geq 0, \tag{4}$$

where $c \in c^\pm$, $A \in A^\pm$, $b \in b^\pm$, and $\mathbf{x} \in \mathbf{x}^\pm$.

To solve the ILP model, the point $\mathbf{x}$ is feasible if $\mathbf{x} \in S = \{\mathbf{x} : A^- \mathbf{x} \leq b^+, \, \mathbf{x} \geq 0\}$ and it is optimal if there exists at least one characteristic model (4) such that $\mathbf{x}$ is the optimal solution. There are many methods for solving the ILP models. One of the methods is the BWC method which determines optimistic and pessimistic values for the objective function, proposed by Tong [37]. In this method, the ILP model is converted into two sub-models. One sub-model has the largest feasible space (the best model), and the other has the smallest feasible space (the worst model) [37]. The sub-models are respectively as follows:

$$\max \quad z^+ = c^+ \mathbf{x}$$
$$s.t. \quad A^- \mathbf{x} \leq b^+$$
$$\mathbf{x} \geq 0, \tag{5}$$

$$\begin{aligned}
\max \quad & \mathrm{z}^- = c^- \mathbf{x} \\
s.t. \quad & \mathrm{A}^+ \mathbf{x} \leq \mathrm{b}^- \\
& \mathbf{x} \geq 0.
\end{aligned} \tag{6}$$

The BWC method was extended by Chinneck and Ramadan for ILP models with equality constraints [8]. A novel ILP method was proposed by Huang and Moore [16], and Huang and Cao [15] who analyzed the principals of the two-step method (TSM). Some solutions obtained through the BWC, ILP, and TSM may be infeasible. For solving ILP models, new solving methods named three-step method (ThSM) and robust two-step method (RTSM) were developed [10],[15]. To guarantee that the given solutions of ILP method are completely feasible, Zhou et al. [45] proposed the modified ILP method (MILP). Since the obtained solutions from the MILP may be non-optimal, then two improved ILP and MILP (IILP and IMILP) methods were proposed [5],[6]. The solutions to these methods are completely feasible and optimal [4].

4. EFFICIENT SOLUTIONS IN THE IMOLP WITH THE EXPECTED VALUE AND VARIANCE

Uncertainty programming is a type of programming which involves uncertainty variables. Assume that $\mathbf{x}$ and ξ are decision variable and uncertainty vector respectively. Using the described concepts, definitions, and theorems, we examine the IMOLP model under EVV operators. Then we convert the IMOLP into the EVV-ILP model and obtain efficient solutions of the IMOLP model.
In [44], the authors have defined the expected value and variance efficient solutions for the MOLP models. Since a characteristic model of the IMOLP is a concrete realization of interval values, we can generalize these definitions for the IMOLP, which is a family of the MOLPs.
Model (7) displays the MOLP with interval coefficients and uncertainty variables.

$$\begin{aligned}
\max \quad & C^{\pm}(\mathbf{x}^{\pm}, \xi) = (c_1^{\pm}(\mathbf{x}^{\pm}, \xi_1), c_2^{\pm}(\mathbf{x}^{\pm}, \xi_2), ..., c_k^{\pm}(\mathbf{x}^{\pm}, \xi_k))^T \\
s.t. \quad & \\
& A^{\pm} \mathbf{x}^{\pm} \leq b^{\pm} \\
& \mathbf{x}^{\pm} \geq 0,
\end{aligned} \tag{7}$$

where $\mathbf{x}^{\pm}$ is a decision vector and ξ is a known uncertain vector which is continuous and defined on the uncertainty space $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$. Since an uncertain objective function $C^{\pm}(\mathbf{x}, \xi)$ cannot be directly maximized, we maximize the expected value and minimize the variance value[24], i.e.,

$$\begin{aligned}
\max \quad & E\left[C^{\pm}(\mathbf{x}^{\pm}, \xi)\right] \\
s.t. \quad & \\
& A^{\pm} \mathbf{x}^{\pm} \leq b^{\pm} \\
& \mathbf{x}^{\pm} \geq 0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\min \quad & Var\left[C^{\pm}\left(\mathbf{x}^{\pm},\xi\right)\right] \\
s.t. \quad & \\
& A^{\pm}\mathbf{x}^{\pm} \le b^{\pm} \\
& \mathbf{x}^{\pm} \ge 0.
\end{aligned} \tag{9}$$

Note that if model (7) is a minimization model, then we minimize both expected value and variance models.

Also, in section 4.1, we apply the expected value operator on the MOLP with interval coefficients and uncertainty variables using the uncertainty theory and introduce EV-ILP model. Then in section 4.2, we apply the variance operator on the MOLP with interval coefficients and uncertainty variables, and introduce V-ILP model. Finally, we introduce EVV-ILP model.

4.1. Expected value operator in the IMOLP

The expected value is the average value of an uncertain variable and represents the size of an uncertain variable [24]. Also, we apply the expected value operator on the IMOLP under uncertainty variables as follows. Indeed, the objective functions are the expected value of the interval objective functions of model (7). Also, we have:

$$\begin{aligned}
\max \quad & E(C^{\pm}(\mathbf{x}^{\pm},\xi)) = (E(c^{\pm T}_1(\mathbf{x}^{\pm},\xi_1)),\ E(c^{\pm T}_2(\mathbf{x}^{\pm},\xi_2)),...,\ E(c^{\pm T}_k(\mathbf{x}^{\pm},\xi_k)))^T \\
s.t. \quad & \\
& A^{\pm}\mathbf{x}^{\pm} \le b^{\pm} \\
& \mathbf{x}^{\pm} \ge 0,
\end{aligned} \tag{10}$$

where $C^{\pm}_l = (c^{\pm}_{l1},...,c^{\pm}_{ln})^T$, $l = 1,...,k$ are objective functions and $\mathbf{x} \in R^n$. $\xi_j = (\xi^j_1,...,\xi^j_n)$, $j = 1,...,n$ is a continuous uncertain vector such that its components are defined uncertain variables on the space $(\Gamma, \mathcal{L}, \mathcal{M})$. The uncertain distribution of variables ξ_i is known. Also, the feasible space for the model is nonempty, convex, and compact. Model (10) is called EV-ILP model. Indeed, in this model, we have:

$$C^{\pm}\mathbf{x}^{\pm} = \begin{pmatrix}
c_{1,1}{}^{\pm}x^{\pm}_1 & c_{1,2}{}^{\pm}x^{\pm}_2 & ... & c_{1,n}{}^{\pm}x^{\pm}_n \\
c_{2,1}{}^{\pm}x^{\pm}_1 & c_{2,2}{}^{\pm}x^{\pm}_2 & ... & c_{2,n}{}^{\pm}x^{\pm}_n \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
c_{k,1}{}^{\pm}x^{\pm}_1 & c_{k,2}{}^{\pm}x^{\pm}_2 & ... & c_{k,n}{}^{\pm}x^{\pm}_n
\end{pmatrix}.$$

Also, $E(C^{\pm}\mathbf{x}^{\pm}, \xi)$ can be written as follows:

$$E(C^{\pm}\mathbf{x}^{\pm}, \xi) = \begin{pmatrix} E(c_{1,1}{}^{\pm}x_1^{\pm}, \xi_1) & E(c_{1,2}{}^{\pm}x_2^{\pm}, \xi_1) & ... & E(c_{1,n}{}^{\pm}x_n^{\pm}, \xi_1) \\ E(c_{2,1}{}^{\pm}x_1^{\pm}, \xi_2) & E(c_{2,2}{}^{\pm}x_2^{\pm}, \xi_2) & ... & E(c_{2,2}{}^{\pm}x_n^{\pm}, \xi_2) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ E(c_{k,1}{}^{\pm}x_1^{\pm}, \xi_k) & E(c_{k,2}{}^{\pm}x_2^{\pm}, \xi_k) & ... & E(c_{k,n}{}^{\pm}x_n^{\pm}, \xi_k) \end{pmatrix}.$$

In [44], the authors have defined the expected value efficient solutions for the MOLP models. Since an IMOLP is a family of the MOLPs, we can generalize this definition for the IMOLPs.

Definition 16. *The solution* $\mathbf{x}^* \in S$ *is an expected value efficient solution (EV efficient solution) for model (7) if it is an optimal solution of EV-ILP model (10), in this case, there is no* $\overline{\mathbf{x}} \in S$ *such that* $E\left[C^{\pm}\left(\overline{\mathbf{x}}, \xi\right)\right] \leq E\left[C^{\pm}\left(\mathbf{x}^*, \xi\right)\right].$

So we will have $E\left[c_l^{\pm}\left(\overline{\mathbf{x}}, \xi_l\right)\right] \leq E\left[c_l^{\pm}\left(\mathbf{x}^*, \xi_l\right)\right]$ *for* $l = 1, ..., k$ *and for at least one* $1 \leq l_0 \leq p$, $E\left[c_{l_o}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l_0}\right)\right] < E\left[c_{l_o}^{\pm}\left(\mathbf{x}^*, \xi_{l_0}\right)\right].$

4.2. Variance operator in the IMOLP

The variance of an uncertain variable is a degree of the spread of the distribution around its expected value. A small value of variance shows the uncertain variable is firmly concentrated around its expected value; and a large value of variance indicates the uncertain variable has a widespread around its expected value. Variance is an important tool in the sciences, where some concepts use it including statistics and Monte Carlo simulation. By considering a variance operator on the IMOLP under uncertainty variables, we have:

$$\begin{aligned} \min \quad & Var(C^{\pm}\mathbf{x}^{\pm}, \xi) = (Var(c^{\pm}{}_1^T\mathbf{x}^{\pm}, \xi_1), Var(c^{\pm}{}_2^T\mathbf{x}^{\pm}, \xi_2), ..., Var(c^{\pm}{}_k^T\mathbf{x}^{\pm}, \xi_k))^T \\ s.t. \quad & \\ & A^{\pm}\mathbf{x}^{\pm} \leq b^{\pm} \\ & \mathbf{x}^{\pm} \geq 0, \end{aligned}$$

(11)

such that the objective functions are the variance of the interval objective functions of model (7). Model (11) is called V-ILP model. Also, we have:

$$Var(C^{\pm}\mathbf{x}^{\pm}, \xi) = \begin{pmatrix} Var(c_{1,1}{}^{\pm}x_1^{\pm}, \xi_1) & Var(c_{1,2}{}^{\pm}x_2^{\pm}, \xi_1) & ... & Var(c_{1,n}{}^{\pm}x_n^{\pm}, \xi_1) \\ Var(c_{2,1}{}^{\pm}x_1^{\pm}, \xi_2) & Var(c_{2,2}{}^{\pm}x_2^{\pm}, \xi_2) & ... & Var(c_{2,n}{}^{\pm}x_n^{\pm}, \xi_2) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ Var(c_{k,1}{}^{\pm}x_1^{\pm}, \xi_k) & Var(c_{k,2}{}^{\pm}x_2^{\pm}, \xi_k) & ... & Var(c_{k,n}{}^{\pm}x_n^{\pm}, \xi_k) \end{pmatrix}.$$

In [44], the authors have defined the variance efficient solutions for the MOLP models. Since an IMOLP is a family of the MOLPs, we can generalize this definition for the IMOLPs.

Definition 17. *The solution $\mathbf{x}^* \in S$ is a variance efficient solution (V efficient solution) to model (7) if it is an optimal solution for V-ILP model (11), and there is no $\overline{\mathbf{x}} \in S$ such that $Var\left[c_l^{\pm}\left(\overline{\mathbf{x}}, \xi_l\right)\right] \leq Var\left[c_l^{\pm}\left(\mathbf{x}^*, \xi_l\right)\right]$, $l = 1, ..., k$ and for at least one, $1 \leq l_0 \leq k$, $Var\left[c_{l_o}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l_0}\right)\right] < Var\left[c_{l_o}^{\pm}\left(\mathbf{x}^*, \xi_{l_0}\right)\right].$*

Consider the following model in which the objective function of model (7) is minimization. Therefore, we minimize the expected value and variance.

$$
\begin{aligned}
&\min \quad \{E\left[C^{\pm}\left(\mathbf{x}^{\pm}, \xi\right)\right], Var\left[C^{\pm}\left(\mathbf{x}^{\pm}, \xi\right)\right]\} \\
&s.t. \\
&\qquad A^{\pm}\mathbf{x}^{\pm} \leq b^{\pm} \\
&\qquad \mathbf{x}^{\pm} \geq 0
\end{aligned}
\tag{12}
$$

In order to establish the relation between the expected value efficient solutions set and variance efficient solutions set, we use the following definitions and theorems.

Definition 18. *The solution $\mathbf{x}^* \in S$ is an EVV efficient solution for model (7) if it is an optimal solution to EVV-ILP model (12). In this case, there is no $\overline{\mathbf{x}} \in S$ such that*

$$
\begin{cases}
E\left[c_l^{\pm}\left(\overline{\mathbf{x}}, \xi_l\right)\right] \leq E\left[c_l^{\pm}\left(\mathbf{x}^*, \xi_l\right)\right], & l = 1, ..., k, \\
or \\
Var\left[c_l^{\pm}\left(\overline{\mathbf{x}}, \xi_l\right)\right] \leq Var\left[c_l^{\pm}\left(\mathbf{x}^*, \xi_l\right)\right], & l = 1, ..., k,
\end{cases}
$$

and for at least one, $1 \leq l_0 \leq k$,

$$
\begin{cases}
E\left[c_{l_o}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l_0}\right)\right] < E\left[c_{l_o}^{\pm}\left(\mathbf{x}^*, \xi_{l_0}\right)\right] \\
or \\
Var\left[c_{l_o}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l_0}\right)\right] < Var\left[c_{l_o}^{\pm}\left(\mathbf{x}^*, \xi_{l_0}\right)\right].
\end{cases}
$$

Theorem 19. *If S_{EV} is a set of expected value efficient solutions, S_V is the set of variance efficient solutions and S_{EVV} is the set of expected value-variance efficient solutions, then*

$$
S_{EV} \cup S_V = S_{EVV}.
$$

Proof. First, suppose that $\mathbf{x}^* \in S_{EV} \cup S_V$ and $\mathbf{x}^* \notin S_{EVV}$. So, by the definition of EVV efficient solution, there exists $\overline{\mathbf{x}} \in S$ such that

$$
E\left[c_l^{\pm}\left(\overline{\mathbf{x}}, \xi_l\right)\right] \leq E\left[c_l^{\pm}\left(\mathbf{x}^*, \xi_l\right)\right] \quad or \quad Var\left[c_l^{\pm}\left(\overline{\mathbf{x}}, \xi_l\right)\right] \leq Var\left[c_l^{\pm}\left(\mathbf{x}^*, \xi_l\right)\right], \quad l = 1, ..., k,
$$

and there exists at least one, $1 \leq l_0 \leq k$ such that

$$
E\left[c_{l0}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l0}\right)\right] < E\left[c_{l0}^{\pm}\left(\mathbf{x}^*, \xi_{l0}\right)\right] \quad or \quad Var\left[c_{l0}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l0}\right)\right] < Var\left[c_{l0}^{\pm}\left(\mathbf{x}^*, \xi_{l0}\right)\right].
$$

If $E\left[c_{l0}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l0}\right)\right] < E\left[c_{l0}^{\pm}\left(\mathbf{x}^*, \xi_{l0}\right)\right]$, then $x^* \notin S_{EV}$, and if $Var\left[c_{l0}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l0}\right)\right] < Var\left[c_{l0}^{\pm}\left(\mathbf{x}^*, \xi_{l0}\right)\right]$, then $\mathbf{x}^* \notin S_V$. In both cases we have $\mathbf{x}^* \notin S_{EV} \cup S_V$, which contradicts the assumption.

To show the converse, suppose $\mathbf{x}^* \in S_{EVV}$. Therefore, there is no $\overline{\mathbf{x}} \in S$ such that

$$\begin{cases} E\left[c_l^{\pm}\left(\overline{\mathbf{x}}, \xi_l\right)\right] \leq E\left[c_l^{\pm}\left(\mathbf{x}^*, \xi_l\right)\right], & l = 1, ..., k, \\ or \\ Var\left[c_l^{\pm}\left(\overline{\mathbf{x}}, \xi_l\right)\right] \leq Var\left[c_l^{\pm}\left(\mathbf{x}^*, \xi_l\right)\right], & l = 1, ..., k, \end{cases}$$

and for at least one $1 \leq l_0 \leq k$,

$$\begin{cases} E\left[c_{l_o}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l_0}\right)\right] < E\left[c_{l_o}^{\pm}\left(\mathbf{x}^*, \xi_{l_0}\right)\right] \\ or \\ Var\left[c_{l_o}^{\pm}\left(\overline{\mathbf{x}}, \xi_{l_0}\right)\right] < Var\left[c_{l_o}^{\pm}\left(\mathbf{x}^*, \xi_{l_0}\right)\right], \end{cases}$$

which results $\mathbf{x}^* \in S_{EV}$ or $\mathbf{x}^* \in S_V$, and hence, $\mathbf{x}^* \in S_{EV} \cup S_V$. $\square$

Now, according to the above definitions and theorems, the interval multi-objective linear programming model under uncertainty variables and EVV operators can be converted into the following EVV-ILP model:

$$\begin{aligned} \min \quad & \left\{E\left(c_l^-\right), E(c_l^+), Var(c_l^-),\ Var(c_l^+)\right\}, l = 1, ..., k \\ s.t. \quad & \\ & A^{\pm}\mathbf{x}^{\pm} \leq b^{\pm} \\ & \mathbf{x}^{\pm} \geq 0 \end{aligned} \qquad (13)$$

This model arises from the construction of a model with $4k$ objective functions involving the expected value and the variance of uncertain objective functions.

5. MONTE CARLO SIMULATION

Monte Carlo methods are an expansive class of computational algorithms that depends on repeated random sampling to get numerical results. The essential idea in Monte Carlo methods is randomness to solve problems that might be deterministic on a fundamental level. A Monte Carlo method is applied for assessment solving methods [38]. A Monte Carlo simulation can be used to describe any technique that approximates solutions to quantitative models by using statistical sampling, or describe a method with uncertainty in model parameters. Also, it is a type of simulation that represents uncertainty in problems.
In Monte Carlo simulation, the model is simulated a large number (e.g., 1000) of times. In each simulation, all of the uncertain parameters are sampled (i.e., a single random value is selected from the specified uncertainty distribution describing each parameter). Then, the model is simulated through time (with respect to given parameters) such that the performance of the system can be computed. This results in large number of separate and independent results.
Different examples for using Monte Carlo method include modelling data sources with uncertainty in information sources, in math, in physics, and, in assessment of multi-dimensional definite integrals with hard boundary condition. Also, Monte Carlo simulation has been used to explore the values of the objective function in

the IMOLP with equality constraints by considering some distribution functions including normal, uniform, and beta [3].

In this paper, we use Monte Carlo simulation for the examples based on the normal distribution function and compare the solutions obtained with the solution obtained by the EVV method to show the performance assessment of our method. Indeed, by using Monte Carlo simulation, we transform the IMOLP to the MOLP such that we generate 1000 random samples from intervals by using the normal distribution, in which the IMOLP model is converted into the MOLP. In this case, by using the weighted sum method, we transform the MOLP to LP model, and solved them by using matlab. Actually, optimal solutions of the LP model and the efficient solutions in the MOLP, and hence the IMOLP are equivalent.

6. NUMERICAL EXAMPLES

In this section, we solve two examples by using the EVV method.

Example 6.1. Consider an IMOLP as follows:

$$
\begin{aligned}
\min \quad & z_1^{\pm} = [-3, -2]\, x_1 + [-2.5, -1.5]\, x_2 \\
\min \quad & z_2^{\pm} = [-4, -3]\, x_1 + [-0.8, -0.5]\, x_2 \\
s.t. \quad & \\
& 3x_1 + 4x_2 \leq 42 \\
& 3x_1 + x_2 \leq 24 \\
& x_1 \geq 0, \ \ 0 \leq x_2 \leq 9.
\end{aligned}
\tag{14}
$$

The feasible region of model (14) is given in Fig. 2.
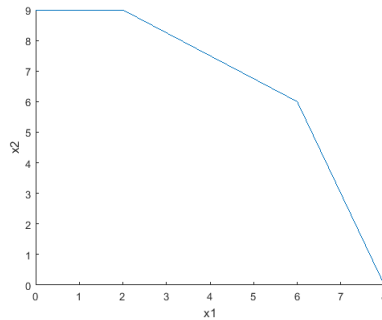
Figure 2: The feasible region of model (14)

Solving by using the EVV method

First, we apply the expected value operator for obtaining efficient points:

$$E\left(\begin{array}{cc} [-3, -2] & [-2.5, -1.5] \\ [-4, -3] & [-0.8, -0.5] \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) =$$

$$\left(\begin{array}{cc} E([-3, -2]\,x_1) & E([-2.5, -1.5]\,x_2) \\ E([-4, -3]\,x_1) & E([-0.8, -0.5]\,x_2) \end{array}\right) =$$

$$\left(\begin{array}{c} E([-3x_1 - 2.5x_2, -2x_1 - 1.5x_2]) \\ E([-4x_1 - 0.8x_2, -3x_1 - 0.5x_2]) \end{array}\right)$$

Therefore, we have:

$$E(z_1^{\pm}) = \left[E(z_1^{-}), E(z_1^{+})\right] = E([-3x_1\xi_1 - 2.5x_2\xi_1, -2x_1\xi_1 - 1.5x_2\xi_1]),$$
$$E(z_2^{\pm}) = \left[E(z_2^{-}), E(z_2^{+})\right] = E([-4x_1\xi_2 - 0.8x_2\xi_2, -3x_1\xi_2 - 0.5x_2\xi_2]),$$

and hence:

$$E(z_1^{-}) = E(-3x_1\xi_1 - 2.5x_2\xi_1) = -3x_1E(\xi_1) - 2.5x_2E(\xi_1)$$

$$E(z_1^{+}) = E(-2x_1\xi_1 - 1.5x_2\xi_1) = -2x_1E(\xi_1) - 1.5x_2E(\xi_1)$$

$$E(z_2^{-}) = E(-4x_1\xi_2 - 0.8x_2\xi_2) = -4x_1E(\xi_2) - 0.8x_2E(\xi_2)$$

$$E(z_2^{+}) = E(-3x_1\xi_2 - 0.5x_2\xi_2) = -3x_1E(\xi_2) - 0.5x_2E(\xi_2)$$

The expected value and variance are calculated with respect to the uncertainty distributions. For example, suppose that ξ_1 and ξ_2 have the following distributions (we can use another uncertainty distributions).

$$\xi_1 \sim \mathcal{L}(1,\, 3),\ \xi_2 \sim \mathcal{L}(2,\, 4)$$

Also:

$$\phi_1(x) = \begin{cases} 0 & x \leq 1 \\ \frac{x-1}{2} & 1 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases},\ \phi_2(x) = \begin{cases} 0 & x \leq 2 \\ \frac{x-2}{2} & 2 \leq x \leq 4 \\ 1 & x \geq 4 \end{cases}$$

So, $\phi_1^{-1}(\alpha) = 1 + 2\alpha$, $\phi_2^{-1}(\alpha) = 2 + 2\alpha$. Now, by using Theorems 9 and 12 or definition 14, we have:

$E(\xi_1) = \int_0^1 \phi_1^{-1}(\alpha)d\alpha = \int_0^1 (1 + 2\alpha)d\alpha = 2,$
$E(\xi_2) = 3,$
$Var[\xi_1] = \int_0^1 \left(\phi_1^{-1}(\alpha) - E[\xi_1]\right)^2 d\alpha = \int_0^1 (1 + 2\alpha - 2)^2 d\alpha = \frac{1}{3},$
$Var[\xi_2] = \frac{1}{3},$

In this case, we substitute $E(\xi_1)$ and $E(\xi_2)$; therefore, we have:

$$E(z_1^{-}) = -3x_1E(\xi_1) - 2.5x_2E(\xi_1) = -6x_1 - 5x_2$$
$$E(z_1^{+}) = -2x_1E(\xi_1) - 1.5x_2E(\xi_1) = -4x_1 - 3x_2$$
$$E(z_2^{-}) = -4x_1E(\xi_2) - 0.8x_2E(\xi_2) = -12x_1 - 2.4x_2$$
$$E(z_2^{+}) = -3x_1E(\xi_2) - 0.5x_2E(\xi_2) = -9x_1 - 1.5x_2$$

Also, the IMOLP model is converted into the following model:

$$\min \quad \left\{E(z_1^-), E(z_1^+), E(z_2^-), E(z_2^+)\right\}$$
$$s.t.$$
$$3x_1 + 4x_2 \leq 42$$
$$3x_1 + x_2 \leq 24$$
$$x_1 \geq 0, \quad 0 \leq x_2 \leq 9$$

In other words:

$$\min \quad \left\{-6\,x_1 - 5x_2, \ -4\,x_1 - 3x_2, \ -12\,x_1 - 2.4x_2, \ -9\,x_1 - 1.5x_2\right\}$$
$$s.t.$$
$$3x_1 + 4x_2 \leq 42$$
$$3x_1 + x_2 \leq 24$$
$$x_1 \geq 0, \quad 0 \leq x_2 \leq 9.$$

Now, by using the variance operator:

$$Var(z_1^\pm) = \left[Var(z_1^-), Var(z_1^+)\right] = Var([-3x_1\xi_1 - 2.5x_2\xi_1, \ -2x_1\xi_1 - 1.5x_2\xi_1]),$$
$$Var(z_2^\pm) = \left[Var(z_2^-), Var(z_2^+)\right] = Var([-4x_1\xi_2 - 0.8x_2\xi_2, \ -3x_1\xi_2 - 0.5x_2\xi_2]),$$

and so:

$$Var(z_1^-) = Var(-3x_1\xi_1 - 2.5x_2\xi_1) = 9x_1{}^2 Var(\xi_1) - 6.25x_2{}^2 Var(\xi_1) = 3\,x_1{}^2 - 6.25/3x_2{}^2,$$

$$Var(z_1^+) = Var(-2x_1\xi_1 - 1.5x_2\xi_1) = 4x_1{}^2 Var(\xi_1) - 2.25x_2{}^2 Var(\xi_1) = 4/3x_1{}^2 - 2.25/3x_2{}^2,$$

$$Var(z_2^-) = Var(-4x_1\xi_2 - 0.8x_2\xi_2) = 16x_1{}^2 Var(\xi_2) - 0.64x_2{}^2 Var(\xi_2) = 16/3\,x_1{}^2 - 0.64/3x_2{}^2,$$

$$Var(z_2^+) = Var(-3x_1\xi_2 - 0.5x_2\xi_2) = 9x_1{}^2 Var(\xi_2) - 0.25x_2{}^2 Var(\xi_2) = 3\,x_1{}^2 - 0.25/3x_2{}^2.$$

Also, we have:

$$\min \quad \left\{3\,x_1{}^2 - 6.25/3x_2{}^2, \ 4/3x_1{}^2 - 2.25/3x_2{}^2, \ 16/3\,x_1{}^2 - 0.64/3x_2{}^2, 3\,x_1{}^2 - 0.25/3x_2{}^2\right\}$$
$$s.t.$$
$$3x_1 + 4x_2 \leq 42$$
$$3x_1 + x_2 \leq 24$$
$$x_1 \geq 0, \quad 0 \leq x_2 \leq 9,$$

finally, by using model (13), we have:

$$\min \quad \begin{cases} -6\,x_1 - 5x_2,\ -4\,x_1 - 3x_2,\ -12\,x_1 - 2.4x_2,\ -9\,x_1 - 1.5x_2, \\ 3\,x_1{}^2 - 6.25/3x_2{}^2,\ 4/3x_1{}^2 - 2.25/3x_2{}^2, \\ 16/3\,x_1{}^2 - 0.64/3x_2{}^2,\ 3x_1{}^2 - 0.25/3x_2{}^2 \end{cases}$$

$s.t.$

$$3x_1 + 4x_2 \le 42$$
$$3x_1 + x_2 \le 24$$
$$x_1 \ge 0, \quad 0 \le x_2 \le 9$$

Therefore, it can be said, the two-objective models with interval coefficients are converted into eight models, by solving them separately, the efficient solutions are obtained as: $(6,6)\,,(8,0),\ (0,9)$.

Solving by using Monte Carlo simulation

In this case, we use Monte Carlo simulation when the interval coefficients of the ILP can be replaced by a random variable with uncertainty distributions. Also, for example, we only determine the optimal solutions under the normal distribution, and then we compare the solutions obtained through the Monte Carlo simulation and the solutions obtained by our method.

First, we generate 1000 random samples with normal distribution for coefficients c_{11}, c_{12}, c_{21}, c_{22} in IMOLP model (14), then the IMOLP is converted into the MOLP. We solve the MOLP through the weighted sum method. This process is repeated 1000 times and solved by using matlab. Fig. 3 shows generated random samples and the range of the optimal values of the objective function that is $[-28, -18]$. Also, for example, we show generated samples in the interval $c_{11} = [-3, -2]$ with respect to $w_1 = 0.1$, $w_2 = 0.9$. The results are shown in Fig. 3 and Fig. 4. It is noteworthy that the obtained efficient solution with respect to $w_1 = 0.1$, $w_2 = 0.9$ is the point $(0, 9)$.

The results are shown in Fig. 5 and Fig. 6 for $c_{11} = [-3, -2]$ with respect to $w_1 = 0.6$, $w_2 = 0.4$, in this case, the obtained efficient solution is point $(8, 0)$. The other solution is $(6, 6)$.

The solutions obtained through Monte Carlo simulation are the same as our method. The results are given in Table 1.

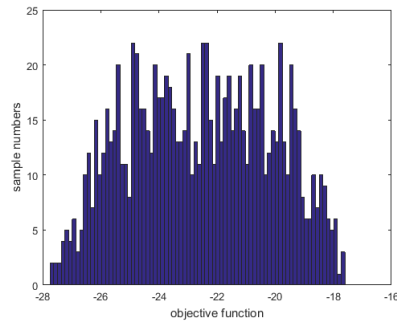Note that the results obtained by Monte Carlo simulation are got by using matlab.

Figure 3: Values of the objective function resulted from Monte Carlo simulation for normal distribution of IMOLP model (14)
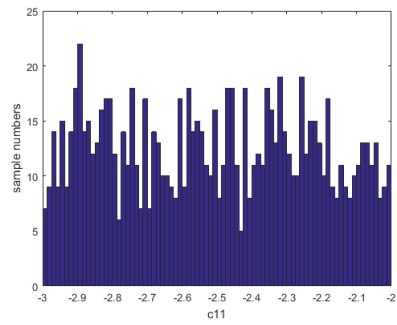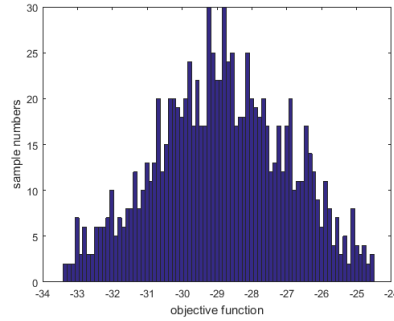

Figure 4: Values of $c11$ resulted from Monte Carlo simulation for normal distribution of IMOLP model (14)

Example 6.1 has been discussed in [2, 19, 31, 35]. Note that the results obtained are stated briefly in Table 2. The solutions obtained in our method are $(6,6)^T$, $(8,0)^T$ and $(0,9)^T$ and generalized ε-constraint are $(6,6)^T$, $(8,0)^T$. Solutions in the WILP method obtained $(6,6)^T$, $(2,9)^T$, and the solution in the admissible order, modified maximum regret and weighted sum of maximum regrets methods is $(6,6)^T$.

Figure 5: Values of the objective function resulted from Monte Carlo simulation for normal distribution of IMOLP model (14)
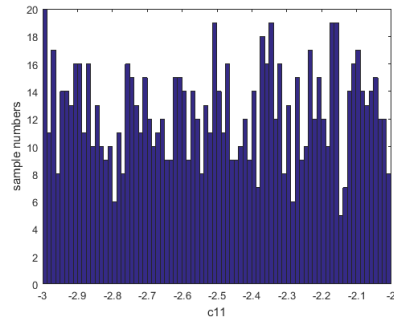

Figure 6: Values of $c11$ resulted from Monte Carlo simulation for normal distribution of IMOLP model (14)

Example 6.2. Consider the IMOLP as follows:

$$
\begin{aligned}
\min \quad & z_1^{\pm} = [1, 3]\, x_1 + [-1, 1.5]\, x_2 \\
\min \quad & z_2^{\pm} = [0.5, 2]\, x_1 + [-1.5, -1]\, x_2 \\
\min \quad & z_3^{\pm} = [1, 2]\, x_1 + [-2, -1.5]\, x_2 \\
\min \quad & z_4^{\pm} = [2, 4]\, x_1 + [-2.5, -2]\, x_2 \\
s.t. \quad & \\
& [1, 2]\, x_1 + [1.5, 3]\, x_2 \leq [4, 6] \\
& [1, 3]\, x_1 + [2.5, 3.5]\, x_2 \leq [12, 12] \\
& x_1, x_2 \geq 0.
\end{aligned}
\tag{15}
$$

Solving by using the EVV method

First, we apply the expected value operator for obtaining the efficient solutions. Therefore, we have:

Table 1: Efficient solutions obtained by Monte Carlo with respect to different weights for example 6.1

Weights	Efficient solutions
$w_1 = 0.1$, $w_2 = 0.9$	$(0, 9)$
$w_1 = 0.6$, $w_2 = 0.4$	$(8, 0)$
$w_1 = 0.5$, $w_2 = 0.5$	$(6, 6)$

Table 2: Numerical results obtained by using different methods for example 6.1.

Methods	Efficient solutions
Our method	$(6,6)^T$, $(8,0)^T$, $(0,9)^T$
WILP	$(6,6)^T$, $(2,9)^T$
Generalized ε -constraint	$(6,6)^T$, $(8,0)^T$
Generalized lexicographic	$(6,6)^T$
Admissible order	$(6,6)^T$
Modified maximum regret	$(6,6)^T$
Weighted sum of maximum regrets	$(6,6)^T$

$$E(z_1^\pm) = \left[E(z_1^-), E(z_1^+) \right] = E([x_1\xi_1 - x_2\xi_1, \ 3x_1\xi_1 + 1.5x_2\xi_1]),$$
$$E(z_2^\pm) = \left[E(z_2^-), E(z_2^+) \right] = E([0.5x_1\xi_2 - 1.5x_2\xi_2, \ 2x_1\xi_2 - x_2\xi_2]),$$
$$E(z_3^\pm) = \left[E(z_3^-), E(z_3^+) \right] = E([x_1\xi_3 + 2x_2\xi_3, \ 2x_1\xi_3 - 1.5x_2\xi_3]),$$
$$E(z_4^\pm) = \left[E(z_4^-), E(z_4^+) \right] = E([2x_1\xi_4 - 2.5x_2\xi_4, \ 4x_1\xi_4 - 2x_2\xi_4]),$$

and hence:

$$E(z_1^-) = x_1 E(\xi_1) - x_2 E(\xi_1), E(z_1^+) = 3x_1 E(\xi_1) + 1.5x_2 E(\xi_1),$$

$$E(z_2^-) = 0.5x_1 E(\xi_2) - 1.5x_2 E(\xi_2), E(z_2^+) = 2x_1 E(\xi_2) - x_2 E(\xi_2),$$

$$E(z_3^-) = x_1 E(\xi_3) - 2x_2 E(\xi_3), E(z_3^+) = 2x_1 E(\xi_3) - 1.5x_2 E(\xi_3),$$

$$E(z_4^-) = 2x_1 E(\xi_4) - 2.5x_2 E(\xi_4), E(z_4^+) = 4x_1 E(\xi_4) - 2x_2 E(\xi_4).$$

The expected value and variance are calculated with respect to the uncertainty distributions. For example, suppose that ξ_1, ξ_2, ξ_3 and ξ_4 have the following distributions (we can use the other uncertainty distributions).

$\xi_1 \sim \mathcal{L}(1, 3)$, $\xi_2 \sim \mathcal{L}(2, 4)$, $\xi_3 \sim \mathcal{L}(3, 5)$, $\xi_4 \sim \mathcal{L}(4, 6)$

So, $\phi_1^{-1}(\alpha) = 1 + 2\alpha$, $\phi_2^{-1}(\alpha) = 2 + 2\alpha$, $\phi_3^{-1}(\alpha) = 3 + 2\alpha$, $\phi_4^{-1}(\alpha) = 4 + 2\alpha$.

Also, we have:

$E(\xi_1) = \int_0^1 \phi_1^{-1}(\alpha)d\alpha = \int_0^1 (1 + 2\alpha)d\alpha = 2$, $E(\xi_2) = 3$, $E(\xi_3) = 4$, $E(\xi_4) = 5$,

$Var\left[\xi_1\right] = \int_0^1 \left(\phi_1^{-1}(\alpha) - E\left[\xi_1\right]\right)^2 d\alpha = \int_0^1 \left(1 + 2\alpha - 2\right)^2 d\alpha = \frac{1}{3}$,
$Var\left[\xi_2\right] = \frac{1}{3}, Var\left[\xi_3\right] = \frac{1}{3}, Var\left[\xi_4\right] = \frac{1}{3}$.

Therefore, by using $E\left(\xi_1\right)$, $E\left(\xi_2\right)$, $E\left(\xi_3\right)$, and $E\left(\xi_4\right)$, we have:

$$\min \quad \begin{cases} 2\,x_1 - 2x_2,\, 6x_1 + 3x_2,\, 1.5\,x_1 - 4.5x_2, \\ 6x_1 - 3x_2,\, 4\,x_1 - 8x_2,\, 8\,x_1 - 6x_2, \\ 10\,x_1 - 12.5x_2,\, 20\,x_1 - 10x_2 \end{cases}$$

s.t.

$$[1,2]\,x_1 + [1.5,3]\,x_2 \le [4,6]$$
$$[1,3]\,x_1 + [2.5,3.5]\,x_2 \le [12,12]$$
$$x_1, x_2 \ge 0$$

Now, by using the variance operator:

$Var(z_1^-) = Var(x_1\xi_1 - x_2\xi_1) = x_1{}^2 Var(\xi_1) - x_2{}^2 Var(\xi_1) = 1/3\,x_1{}^2 - 1/3x_2{}^2$,
$Var(z_1^+) = 3x_1{}^2 + 2.25/3x_2{}^2$,
$Var(z_2^-) = 0.25/3\,x_1{}^2 - 2.25/3x_2{}^2$,
$Var(z_2^+) = 4/3\,x_1{}^2 - 1/3x_2{}^2$,
$Var(z_3^-) = 1/3\,x_1{}^2 - 4/3x_2{}^2$,
$Var(z_3^+) = 4/3\,x_1{}^2 - 2.25/3x_2{}^2$,
$Var(z_4^-) = 4/3\,x_1{}^2 - 6.25/3x_2{}^2$,
$Var(z_4^+) = 16/3\,x_1{}^2 - 4/3x_2{}^2$.

Also, we have:

$$\min \quad \begin{cases} 1/3\,x_1{}^2 - 1/3x_2{}^2,\, 3x_1{}^2 + 2.25/3x_2{}^2, \\ 0.25/3\,x_1{}^2 - 2.25/3x_2{}^2,\, 4/3\,x_1{}^2 - 1/3x_2{}^2, \\ 1/3\,x_1{}^2 - 4/3x_2{}^2,\, 4/3\,x_1{}^2 - 2.25/3x_2{}^2, \\ 4/3\,x_1{}^2 - 6.25/3x_2{}^2,\, 16/3\,x_1{}^2 - 4/3x_2{}^2 \end{cases}$$

s.t.

$$[1,2]\,x_1 + [1.5,3]\,x_2 \le [4,6]$$
$$[1,3]\,x_1 + [2.5,3.5]\,x_2 \le [12,12]$$
$$x_1, x_2 \ge 0.$$

Also, by using model (13), we have:

$$\min \quad \begin{cases} 2\,x_1 - 2x_2,\, 6x_1 + 3x_2,\, 1.5\,x_1 - 4.5x_2,\, 6x_1 - 3x_2, \\ 4\,x_1 - 8x_2,\, 8\,x_1 - 6x_2,\, 10\,x_1 - 12.5x_2,\, 20\,x_1 - 10x_2, \\ 1/3\,x_1{}^2 - 1/3x_2{}^2,\, 3x_1{}^2 + 2.25/3x_2{}^2, \\ 0.25/3\,x_1{}^2 - 2.25/3x_2{}^2,\, 4/3\,x_1{}^2 - 1/3x_2{}^2, \\ 1/3\,x_1{}^2 - 4/3x_2{}^2,\, 4/3\,x_1{}^2 - 2.25/3x_2{}^2, \\ 4/3\,x_1{}^2 - 6.25/3x_2{}^2,\, 16/3\,x_1{}^2 - 4/3x_2{}^2, \end{cases}$$

s.t.

$$[1,2]\,x_1 + [1.5,3]\,x_2 \le [4,6]$$
$$[1,3]\,x_1 + [2.5,3.5]\,x_2 \le [12,12]$$
$$x_1, x_2 \ge 0$$

Since constraints are interval, by using the BWC method we convert into two sub-models with the optimistic and pessimistic constraints and then we solve sub-models as follows.

Sub-model 1

$$\min \begin{cases} 2\,x_1 - 2x_2, 6x_1 + 3x_2, 1.5\,x_1 - 4.5x_2, 6x_1 - 3x_2, 4\,x_1 - 8x_2, \\ 8\,x_1 - 6x_2, 10\,x_1 - 12.5x_2, 20\,x_1 - 10x_2, 1/3\,x_1{}^2 - 1/3x_2{}^2, \\ 3x_1{}^2 + 2.25/3x_2{}^2, 0.25/3\,x_1{}^2 - 2.25/3x_2{}^2, \\ 4/3\,x_1{}^2 - 1/3x_2{}^2, 1/3\,x_1{}^2 - 4/3x_2{}^2, 4/3\,x_1{}^2 - 2.25/3x_2{}^2, \\ 4/3\,x_1{}^2 - 6.25/3x_2{}^2, 16/3\,x_1{}^2 - 4/3x_2{}^2 \end{cases}$$

s.t.

$$x_1 + 1.5x_2 \le 6$$
$$x_1 + 2.5x_2 \le 12$$
$$x_1, x_2 \ge 0$$

The solution obtained is the point $(0, 4)$.

Sub-model 2

$$\min \begin{cases} 2\,x_1 - 2x_2, 6x_1 + 3x_2, 1.5\,x_1 - 4.5x_2, 6x_1 - 3x_2, \\ 4\,x_1 - 8x_2, 8\,x_1 - 6x_2, 10\,x_1 - 12.5x_2, 20\,x_1 - 10x_2, \\ 1/3\,x_1{}^2 - 1/3x_2{}^2, 3x_1{}^2 + 2.25/3x_2{}^2, \\ 0.25/3\,x_1{}^2 - 2.25/3x_2{}^2, 4/3\,x_1{}^2 - 1/3x_2{}^2, \\ 1/3\,x_1{}^2 - 4/3x_2{}^2, 4/3\,x_1{}^2 - 2.25/3x_2{}^2, \\ 4/3\,x_1{}^2 - 6.25/3x_2{}^2, 16/3\,x_1{}^2 - 4/3x_2{}^2 \end{cases}$$

s.t.

$$2x_1 + 3x_2 \le 4$$
$$3x_1 + 3.5x_2 \le 12$$
$$x_1, x_2 \ge 0.$$

The solution obtained is the point $(0, 1.33)$. Therefore, we have the range of the solutions obtained by the BWC method as $([0, 0]\ , [1.33, 4])^T$, that $([0, 0]\ , [1.33, 4])^T$ includes optimal solutions and the best and the worst solutions for model (15).

Solving by using Monte Carlo simulation

Since objective functions and constraints are interval, we generate 1000 random samples with normal distribution for c_{ij}, $i = 1, ..., 4$, $j = 1, 2$ and a_{ij}, b_i, $i = 1, 2$, $j = 1, 2$ in IMOLP model (15), then the IMOLP is converted into the MOLP model. Now, by the weighted sum method, the MOLP model is solved. This process is repeated 1000 times and solved by using matlab. For example, we show generated samples in the interval $a11 = [1, 2]$, $c11 = [1, 3]$, $b1 = [4, 6]$ and values of the objective function with respect to $w_1 = 0.1$, $w_2 = 0.9$.

The results are given in Fig. 7, Fig. 8, Fig. 9, and Fig. 10.
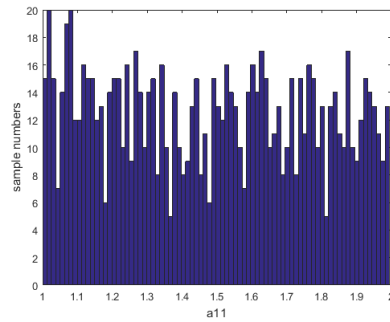
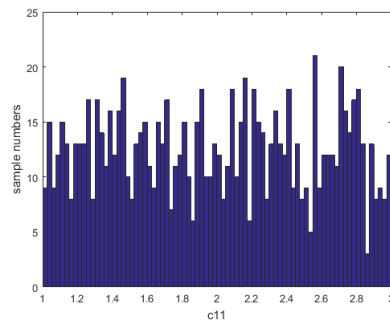Figure 7: Values of $a11$ resulted from Monte Carlo simulation for normal distribution of IMOLP model (15)

Figure 8: Values of $c11$ resulted from Monte Carlo simulation for normal distribution of IMOLP model (15)

The obtained efficient solution with respect to $w_1 = 0.1$ and $w_2 = 0.9$ is the point $(0, 1.8346)$ and lies in the interval $([0,0] , [1.33, 4])^T$, which has beeb obtained by the EVV method. The other efficient solutions are obtained with respect to the other different weights. See Table 3.

Note that if weights change, then we can obtain different efficient solutions that all of them lie in the interval $([0,0] , [1.33, 4])^T$. Indeed, we assessed the solutions obtained by EVV method by using Monte Carlo simulation, and the results are good.
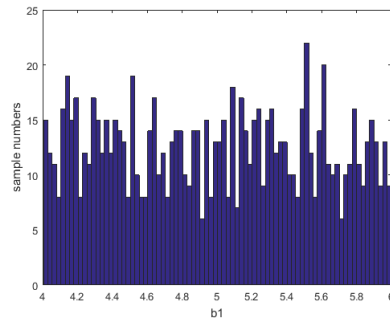
Figure 9: Values of $b1$ resulted from Monte Carlo simulation for normal distribution of IMOLP model (15)
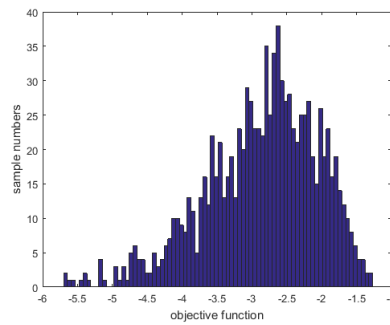

Figure 10: Values of the objective function resulted from Monte Carlo simulation for normal distribution of IMOLP model (15)

Example 6.2 has been discussed in [33]. Note that the results obtained are stated briefly in Table 4.

The solutions obtained in our method are $([0,0]\ ,\ [1.33,\ 4])^T$, and solutions in the Fuzzy set based approach are $(3,3)^T$, $(0,0)^T$.

Table 3: Efficient solutions with respect to different weights

Weights	Efficient solutions
$w_1 = 0.1\,,\ w_2 = 0.9$	(0, 1.8346)
$w_1 = 0.2\,,\ w_2 = 0.8$	(0, 3.4862)
$w_1 = 0.3\,,\ w_2 = 0.7$	(0, 1.8963)
$w_1 = 0.4\,,\ w_2 = 0.6$	(0, 1.8380)

Table 4: Numerical results obtained for example 6.2.

Methods	Efficient solutions
Our method	$(\,[0,0]\,,\ [1.33,\ 4])^T$
Fuzzy set based approach	$(3,3)^T,\ (0,0)^T$

7. CONCLUSION

In this paper, we discuss interval multi-objective linear programming (IMOLP) models. First, we define the expected value and variance operators (EVV operators), and then we study the IMOLP models under these operators. In this method (EVV method), the IMOLP models are converted into the interval linear programming (ILP) models under the EVV operators, which are called EVV-ILP models. Then, Monte Carlo simulation is used to the performance assessment of the obtained efficient solutions by considering uncertainty distribution functions (in this paper, we used the linear uncertainty distribution). The proposed method is applicable for large scale models, too. Although the number of objective functions increases remarkably and some of objective functions have similar solutions, it is suitable for multi-objectives and real-world problems with interval coefficients and uncertainty variables in the objective functions and constraints. Finally, to illustrate the performance of the proposed method, numerical examples are solved.

REFERENCES

[1] Alefeld, G., and Herzberger, J., *Introduction To Interval Computation*, Academic press, New York, 1983.

[2] Batamiz, A., and Allahdadi, M., " Obtaining efficient solutions of interval multi-objective linear programming problems", *International Journal of Fuzzy Systems* 22 (3) (2020) 873–890.

[3] Allahdadi, M., and Khaje Golestane, A., "Monte Carlo simulation for computing the worst value of the objective function in the interval linear programming", *International Journal of Applied and Computational Mathematics*, 2 (2016) 509–518.

[4] Allahdadi, M., and Mishmast Nehi, H., "The optimal solution set of the interval linear programming problems", *Optimization Letters*, 7 (8) (2013) 1893–1911.

[5] Allahdadi, M., Mishmast Nehi, H., Ashayerinasab, H.A., and Javanmard, M., "Improving the modified interval linear programming method by new techniques", *Information Sciences*, 339 (2016) 224–236.

[6] Ashayerinasab, H.A., Mishmast Nehi, H., and Allahdadi, M., " Solving the interval linear programming problem: A new algorithm for a general case", *Expert Systems with Applications*, 93 (2018) 39–49.

[7] Bharati, S.k., and Singh, S.R., " Solution of multiobjective linear programming problems in interval-valued intuitionistic fuzzy environment", *Soft computing*, 23 (1) (2019) 77–84.

[8] Chinneck, J., and Ramadan, K., "Linear programming with interval coefficients", *Journal of the operational research society*, 51 (2) (2000) 209–220.

[9] Dechao, L., Leung, Y., and Weizhi, Wu., "Multiobjective interval linear programming in admissible-order vector space", *Information Science*, 486 (2019) 1–19.

[10] Fan, Y., and Huang, G.H., "A robust two-step method for solving interval linear programming problems within an environmental management context", *Journal of Environmental Informatics*, 19 (1) (2012) 1–9.

[11] Guo, J., Wang, Z., Zheng, M., and Wang, Y., "Uncertain multiobjective redundancy allocation problem of repairable systems based on artificial bee colony algorithm", *Chinese Journal of Aeronautics*, 27 (6) (2014) 1477–1487.

[12] Hladik, M., "Complexity of necessary efficiency in interval linear programming and multi-objective linear programming", *Optimization Letters*, 6 (5) (2012) 893–899.

[13] Hladik, M., "Weak and strong solvability of interval linear systems of equations and inequalities", *Linear Algebra and its Applications*, 438 (11) (2013) 4156–4165.

[14] Hladik, M., "One necessarily efficient solutions in interval multi objective linear programming", *Proceedings of the 25th Mini-EURO Conference Uncertainty and Robustness in Planning and Decision Making URPDM*, April, Coimbra, Portugal, (2010) 1–10.

[15] Huang, G., and Cao, M., " Analysis of solution methods for interval linear programming", *Journal of Environmental Informatics*, 17 (2) (2011) 54–64.

[16] Huang, G., and Moore, R.D., " Grey linear programming, its solving approach, and its application", *International journal of systems science*, 24 (1) (1993) 159–172.

[17] Inuiguchi, M., " Necessary efficiency is partitioned into possible and necessary optimalities", *In: IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, Edmonton, AB, Canada, IEEE (2013), 209–214.

[18] Inuiguchi, M., and Kume, Y., "Goal programming problems with interval coefficients and target intervals", *European Journal of Operational Research*, 52 (3) (1991) 345–360.

[19] Inuiguchi, M., and Sakawa, M., " Possible and necessary efficiency in possibilistic multiobjective linear programming problems and possible efficiency test", *Fuzzy Sets and Systems*, 78 (1996) 321–341.

[20] Inuiguchi, M., and Sakawa, M., "An achievement rate approach to linear programming problems with an interval objective function", *Journal of the Operational Research Society*, 48 (1) (1997) 25–33.

[21] Inuiguchi, M., and Sakawa, M., "Minimax regret solution to linear programming problems with an interval objective function", *European Journal of Operational Research*, 86 (3) (1995) 526–536.

[22] Jiao, D., and Yao, K., "An interest rate model in uncertain environment", *Soft Computing*, 19 (3) (2015) 775–780.

[23] Kolmogorov, A., *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Julius Springer, Berlin, Preis RM 7,50, 1933.

[24] Liu, B., *Theory and practice of uncertain programming*, Springer, Berlin, Heidelberg, 2007.

[25] Liu, B., *Uncertainty theory:A branch of mathematics for modeling Human Uncertainty*, Springer-Verlag, Berlin Heidelberg, 2010.

[26] Liu, B., *Uncertain logic for modeling human language*, Journal of Uncertain Systems, 5 (1) (2011) 3–20.

[27] Liu, B., *Uncertainty theory*, Fifth Edition Uncertainty Theory Laboratory Department of Mathematical Sciences Tsinghua University, Springer-Verlag, Berlin Heidelberg, 2017.

[28] Li, X., and Liu, B., Hybrid logic and uncertain logic", *Journal of Uncertain Systems*, 3 (2) (2009) 83–94.

[29] Li, X., Zhou, J., and Zhao, X., "Travel itinerary problem", *Transportation Research Part B: Methodological*, 91 (2016) 332–343.

[30] Liu, B., and Yao, K., " Uncertain multilevel programming: algorithm and applications", *Computers & Industrial Engineering*, 89 (2015) 235–240.

[31] Oliveira, C., and Antunes, C.H., " Multiple objective linear programming models with interval coefficients–an illustrated overview, *European journal of operational Research*, 181 (3) (2007) 1434–1463.

[32] Peng, Z., and Iwamura, K., " A sufficient and necessary condition of uncertainty distribution", *Journal of Interdisciplinary Mathematics*, 13 (3) (2010) 277–285.

[33] Razavi Hajiagha, H., Amoozad Mahdiraji, H., and Sadat Hashemi, SH., "Multi-objective linear programming with interval coefficients, A fuzzy set based approach", *Kybernetes*, 42 (3) (2013) 482–496.

[34] Rivaz, S., and Yaghoobi, M.A., "Minimax regret solution to multiobjective linear programming problems with interval objective functions coefficients", *Central European Journal of Operations Research*, 21 (3) (2013) 625–649.

[35] Rivaz, S., and Yaghoobi, M.A., " Weighted sum of maximum regrets in an interval MOLP problem", *International Transactions in Operational Research*, 25 (5) (2018) 1659–1676.

[36] Rivaz, S., Yaghoobi, M.A., and Hladik, M., " Using modified maximum regret for finding a necessarily efficient solution in an interval MOLP problem", *Fuzzy Optimization and Decision Making*, 15 (3) (2016) 237–253.

[37] Shaocheng, T., "Interval number and fuzzy number linear programmings", *Fuzzy sets and systems*, 66 (3) (1994) 301–306.

[38] Tierney, L., and Mira, A., "Some adaptive Monte Carlo methods for Bayesian inference", *Statistics in medicine*, 18 (17-18) (1999) 2507–2515.

[39] Wang, Z., Guo, J., Zheng, M., and Yang, Y., "A new approach for uncertain multi objective programming problem based on PE principle", *Journal of Industrial and Management Optimization*, 11 (1) (2015) 13–26.

[40] Wang, Z., Guo, J., Zheng, M., and Wang, Y., "Uncertain multi objective traveling salesman problem", *European Journal of Operational Research*, 241 (2) (2015) 478–489.

[41] Wen, M., Qin, Z., Kang, R., and Yang, Y., "Sensitivity and stability analysis of the additive model in uncertain data envelopment analysis", *Soft Computing*, 19 (7) (2015) 1987–1996.

[42] Yang, X., " On comonotonic functions of uncertain variables", *Fuzzy Optimization and Decision Making*, 12 (1) (2013) 89–98.

[43] Yao, K., and Gao, J., " Uncertain random alternating renewal process with application to interval availability", *IEEE transactions on fuzzy systems*, 23 (5) (2015) 1333–1342.

[44] Zheng, M., Yi, Y., Wang, Z., and Liao, T., "Efficient solution concepts and their application in uncertain multi objective programming", *Applied Soft Computing*, 56 (2017) 557–569.

[45] Zhou, F., Huang, G.H., Chen, G.X., and Guo, H.C., "Enhanced-interval linear programming", *European Journal of Operational Research*, 199 (2) (2009) 323–333.