

# KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS AND DUALITY FOR MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING WITH EQUILIBRIUM CONSTRAINTS

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**Abstract:** The purpose of this paper is to study multiobjective semi-infinite programming with equilibrium constraints. Firstly, the necessary and sufficient Karush-Kuhn-Tucker optimality conditions for multiobjective semi-infinite programming with equilibrium constraints are established. Then, we formulate types of Wolfe and Mond-Weir dual problems and investigate duality relations under convexity assumptions. Some examples are given to verify our results.

**Keywords:** Multiobjective Semi-Infinite Programming, Equilibrium Constraints, Constraint Qualifications, Karush-Kuhn-Tucker Optimality Conditions, Mond-Weir Duality, Wolfe Duality.

**MSC:** 90C46, 90C33, 49K10.

## 1. INTRODUCTION

The mathematical programming problems with equilibrium constraints could be served in reformulating many problems from economic equilibria, multilevel games [18], industrial engineering [2], healthcare management [16]. Among many other interesting research, optimality conditions and duality for mathematical programming problems with equilibrium constraints have been considered numerously by many researchers (see, e.g., [3, 6, 9, 11, 15, 22, 24, 26] and the references therein). On the other hand, a simultaneous minimization of a finite number of

objective functions over an infinite number of constraints is called a multiobjective semi-infinite programming problem. Due to semi-infinite programming problems having application in many fields [7], they have attracted a lot of attention from many authors (see, e.g., [4, 5, 8, 12, 13, 14, 27, 28, 29, 30, 31, 32] and the references therein). Recently, semi-infinite programming problems with equilibrium constraints have been presented and investigated. In [21], strong Karush-Kuhn-Tucker (KKT) type sufficient optimality conditions for nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints were established via Clarke subdifferentials. By using convexifiers, the paper [20] established necessary and sufficient optimality conditions and derived weak and strong duality theorems relating to the semi-infinite mathematical programming problems with equilibrium constraints. The Lagrange type dual model and saddle point optimality criteria of semi-infinite mathematical programming problems with equilibrium constraints were discussed in [25]. However, KKT necessary optimality conditions for multiobjective semi-infinite programming problems with equilibrium constraints have not yet been considered in [21]. Moreover, to the best of our knowledge, there is no paper dealing with duality for multiobjective semi-infinite programming problems with equilibrium constraints.

Inspired by the above observations, we concentrate on studying Karush-Kuhn-Tucker optimality conditions and duality results for the multiobjective semi-infinite programming with equilibrium constraints. The organization of this paper is as follows. In Section 2, some basic concepts and preliminaries are recollected. Section 3 is a discussion of the KKT necessary and sufficient optimality conditions for the multiobjective semi-infinite programming problems with equilibrium constraints. Then, we explore Mond-Weir and Wolfe dual problems of the multiobjective semi-infinite programming problems with equilibrium constraints in Section 4. Some examples are given to illuminate the results of the paper.

## 2. PRELIMINARIES

The following notations and definitions will be used throughout the paper. Let  $\mathbb{R}^n$  be a finite-dimensional Euclidean space. The notation  $\langle \cdot, \cdot \rangle$  is used to denote the inner product. By  $B(x, \delta)$  we indicate the open ball centered at  $x$  with radius  $\delta > 0$ . For a given  $\bar{x}$ ,  $\mathcal{U}(\bar{x})$  is the system of the neighborhoods of  $\bar{x}$ . For  $A \subseteq \mathbb{R}^n$ ,  $\text{int}A$ ,  $\text{cl}A$ ,  $\text{aff}A$ ,  $\text{span}A$  and  $\text{co}A$  stand for its interior, closure, affine hull, linear hull, convex hull of  $A$ , respectively (resp). The cone and the convex cone (containing the origin) generated by  $A$  are denoted resp by  $\text{cone}A$ ,  $\text{pos}A$ . It should be noted that, for the given sets  $A_1, A_2$  in  $\mathbb{R}^n$ ,

$$\text{span}(A_1 \cup A_2) = \text{span}A_1 + \text{span}A_2 \quad \text{and} \quad \text{pos}(A_1 \cup A_2) = \text{pos}A_1 + \text{pos}A_2.$$

The negative polar cone, the strictly negative polar cone and the orthogonal complement of  $A$  are defined resp by

$$A^- := \{x^* \in \mathbb{R}^n \mid \langle x^*, x \rangle \leq 0, \forall x \in A\},$$

$$A^s := \{x^* \in \mathbb{R}^n \mid \langle x^*, x \rangle < 0, \forall x \in A \setminus \{0\}\},$$

$$A^\perp := \{x^* \in \mathbb{R}^n \mid \langle x^*, x \rangle = 0, \forall x \in A\}.$$

It is easy to check that  $A^s \subset A^-$  and if  $A^s \neq \emptyset$  then  $\text{cl}A^s = A^-$ . Moreover, the bipolar theorem, see e.g. [1], states that  $A^{--} = \text{cl pos}A$ . For a given nonempty subset  $A$  of  $\mathbb{R}^n$ , the contingent cone [1] of  $A$  at  $\bar{x} \in \text{cl}A$  is

$$T(A, \bar{x}) := \{x \in \mathbb{R}^n \mid \exists \tau_k \downarrow 0, \exists x_k \rightarrow \bar{x}, \forall k \in \mathbb{N}, \bar{x} + \tau_k x_k \in A\}.$$

Note that if  $A$  is a convex set then  $T(A, \bar{x}) = \text{clcone}(A - \bar{x})$ . If  $\langle x^*, x \rangle \geq 0$  for all  $x^* \in A^*$ , where  $A^*$  is a subset of the dual space of  $\mathbb{R}^n$ , we write  $\langle A^*, x \rangle \geq 0$ . The notion  $o(\tau^k)$ , for  $\tau > 0$  and  $k \in \mathbb{N}$ , designates a moving point such that  $o(\tau^k)/\tau^k \rightarrow 0$  as  $\tau \rightarrow 0^+$ . The cardinality of the index set  $I$  is denoted by  $|I|$ . For an index subset  $I \subset \{1, \dots, n\}$ ,  $x_I = 0 (x_I \geq 0)$  stands for  $x_i = 0 (x_i \geq 0, \text{ resp})$  for all  $i \in I$ .

In the line of [21], we consider the following multiobjective semi-infinite programming with equilibrium constraints (P):

$$\begin{aligned} \mathbb{R}_+^m - \min \quad & f(x) = (f_1(x), \dots, f_m(x)) \\ \text{s.t.} \quad & g_t(x) \leq 0, t \in T, \\ & h_i(x) = 0, i = 1, \dots, q, \\ & G_i(x) \geq 0, i = 1, \dots, l, \\ & H_i(x) \geq 0, i = 1, \dots, l, \\ & G_i(x)H_i(x) = 0, i = 1, \dots, l, \end{aligned}$$

where  $f_i (i = 1, \dots, m)$ ,  $g_t (t \in T)$ ,  $h_i (i = 1, \dots, q)$  and  $G_i, H_i (i = 1, \dots, l)$  are continuously differentiable functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ . The index set  $T$  is an arbitrary nonempty set, not necessary finite. Let us denote  $I := \{1, \dots, m\}$ ,  $I_h := \{1, \dots, q\}$  and  $I_l := \{1, \dots, l\}$ . The feasible solution set of (P) is

$$\Omega := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0 (t \in T), h_i(x) = 0 (i \in I_h), G_i(x) \geq 0, H_i(x) \geq 0, G_i(x)H_i(x) = 0 (i \in I_l)\}.$$

Recall some types of efficient solutions, see e.g. [17], of the multiobjective semi-infinite programming as follows.

**Definition 1.** Let  $\bar{x} \in \Omega$ .

- (i)  $\bar{x}$  is a locally (Pareto) efficient solution of (P), denoted by  $\bar{x} \in \text{locE}(P)$ , if there exists a neighborhood  $U \in \mathcal{U}(\bar{x})$  such that

$$f(x) - f(\bar{x}) \notin -\mathbb{R}_+^m \setminus \{0\}, \forall x \in \Omega \cap U.$$

- (ii)  $\bar{x}$  is a locally weakly efficient solution of (P), denoted by  $\bar{x} \in \text{locWE}(P)$ , if there exists a neighborhood  $U \in \mathcal{U}(\bar{x})$  such that

$$f(x) - f(\bar{x}) \notin -\text{int}\mathbb{R}_+^m, \forall x \in \Omega \cap U.$$

If  $U = \mathbb{R}^n$ , the word “locally” is omitted. In this case, the efficient solution sets/the weakly efficient solution sets are denoted by  $E(P)/WE(P)$ . It is straightforward that  $E(P) \subset WE(P)$ .

The notation  $\mathbb{R}_+^{|T|}$  represents the collection of all the functions  $\lambda : T \rightarrow \mathbb{R}$  taking values  $\lambda_t$ 's positive only at finitely many points of  $T$ , and equal to zero at the other points. For a given  $\bar{x} \in \Omega$ , we signify by  $I_g(\bar{x}) := \{t \in T | g_t(\bar{x}) = 0\}$  the index set of all active constraints at  $\bar{x}$ . The set of active constraint multipliers at  $\bar{x} \in \Omega$  is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}_+^{|T|} | \lambda_t g_t(\bar{x}) = 0, \forall t \in T\}.$$

Notice that  $\lambda \in \Lambda(\bar{x})$  if there exists a finite index set  $J \subset I_g(\bar{x})$  such that  $\lambda_t > 0$  for all  $t \in J$  and  $\lambda_t = 0$  for all  $t \in T \setminus J$ . For each  $\bar{x} \in \Omega$ , let us define

$$\begin{aligned} I_{+0}(\bar{x}) &:= \{i \in I_l | G_i(\bar{x}) > 0, H_i(\bar{x}) = 0\}, \\ I_{00}(\bar{x}) &:= \{i \in I_l | G_i(\bar{x}) = 0, H_i(\bar{x}) = 0\}, \\ I_{0+}(\bar{x}) &:= \{i \in I_l | G_i(\bar{x}) = 0, H_i(\bar{x}) > 0\}. \end{aligned}$$

**Definition 2.** *The point  $\bar{x} \in \Omega$  is called a strong stationary point of (P) iff there exists  $(\alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^m \times \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$  with  $\lambda_{I_{+0}}^G = 0, \lambda_{I_{00}}^G \geq 0, \lambda_{I_{00}}^H \geq 0$  and  $\lambda_{I_{0+}}^H = 0$  such that*

$$\sum_{i \in I} \alpha_i \nabla f_i(\bar{x}) + \sum_{t \in T} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(\bar{x}) = 0.$$

For  $\bar{x} \in \Omega$  and  $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ , we define

$$\begin{aligned} I_g^+(\bar{x}) &:= \{t \in I_g(\bar{x}) | \lambda_t^g > 0\}, \\ I_h^+(\bar{x}) &:= \{i \in I_h | \lambda_i^h > 0\}, I_h^-(\bar{x}) := \{i \in I_h | \lambda_i^h < 0\}, \\ I_{00}^{++}(\bar{x}) &:= \{i \in I_{00}(\bar{x}) | \lambda_i^G > 0, \lambda_i^H > 0\}, \\ I_{0+}^+(\bar{x}) &:= \{i \in I_{0+}(\bar{x}) | \lambda_i^G > 0\}, I_{0+}^-(\bar{x}) := \{i \in I_{0+}(\bar{x}) | \lambda_i^G < 0\}, \\ I_{00}^+(\bar{x}) &:= \{i \in I_{00}(\bar{x}) | \lambda_i^G > 0, \lambda_i^H = 0\}, I_{00}^-(\bar{x}) := \{i \in I_{00}(\bar{x}) | \lambda_i^G < 0, \lambda_i^H = 0\}, \\ \hat{I}_{+0}^+(\bar{x}) &:= \{i \in I_{+0}(\bar{x}) | \lambda_i^H > 0\}, \hat{I}_{+0}^-(\bar{x}) := \{i \in I_{+0}(\bar{x}) | \lambda_i^H < 0\}, \\ \hat{I}_{00}^+(\bar{x}) &:= \{i \in I_{00}(\bar{x}) | \lambda_i^G = 0, \lambda_i^H > 0\}, \hat{I}_{00}^-(\bar{x}) := \{i \in I_{00}(\bar{x}) | \lambda_i^G = 0, \lambda_i^H < 0\}. \end{aligned}$$

**Definition 3.** [23] *Let  $X \subset \mathbb{R}^n$  be an open convex set and  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable at  $\bar{x} \in X$ .*

- (i)  $\varphi$  is convex at  $\bar{x}$  if  $\varphi(\lambda\bar{x} + (1-\lambda)x) \leq \lambda\varphi(\bar{x}) + (1-\lambda)\varphi(x), \forall x \in X, \forall \lambda \in [0, 1]$ .
- (ii)  $\varphi$  is strictly convex at  $\bar{x}$  if

$$\varphi(\lambda\bar{x} + (1-\lambda)x) < \lambda\varphi(\bar{x}) + (1-\lambda)\varphi(x), \forall x \in X \setminus \{\bar{x}\}, \forall \lambda \in (0, 1).$$

- (iii)  $\varphi$  is quasiconvex at  $\bar{x}$  if  $\varphi(\lambda\bar{x} + (1-\lambda)x) \leq \max\{\varphi(\bar{x}), \varphi(x)\}, \forall x \in X, \forall \lambda \in [0, 1]$ .

(iv)  $\varphi$  is pseudoconvex at  $\bar{x}$  if, for all  $x \in X$ ,

$$\varphi(x) < \varphi(\bar{x}) \Rightarrow \langle \nabla \varphi(\bar{x}), x - \bar{x} \rangle < 0.$$

(v)  $\varphi$  is strictly pseudoconvex at  $\bar{x}$  if, for all  $x \in X \setminus \{\bar{x}\}$ ,

$$\varphi(x) \leq \varphi(\bar{x}) \Rightarrow \langle \nabla \varphi(\bar{x}), x - \bar{x} \rangle < 0.$$

(vi)  $\varphi$  is convex on  $X$  if  $\varphi$  is convex on each point of  $X$ . Other concepts here introduced can be defined on a set in a similar way.

**Remark 4.** [23] Let  $X \subset \mathbb{R}^n$  be an open convex set and  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable at  $\bar{x} \in X$ .

(i) If  $\varphi$  is convex at  $\bar{x}$ , then

$$\langle \nabla \varphi(\bar{x}), x - \bar{x} \rangle \leq \varphi(x) - \varphi(\bar{x}), \text{ for all } x \in X.$$

(ii) If  $\varphi$  is quasiconvex at  $\bar{x}$ , then, for all  $x \in X$ ,

$$\varphi(x) \leq \varphi(\bar{x}) \Rightarrow \langle \nabla \varphi(\bar{x}), x - \bar{x} \rangle \leq 0.$$

(iii) If  $\varphi$  is convex at  $\bar{x}$  then  $\varphi$  is pseudoconvex at  $\bar{x}$ . If  $\varphi$  is pseudoconvex at  $\bar{x}$  then  $\varphi$  is quasiconvex at  $\bar{x}$ .

**Lemma 5.** [23] Let  $\{C_t | t \in \Gamma\}$  be an arbitrary collection of nonempty convex sets in  $\mathbb{R}^n$  and  $K = \text{pos} \left( \bigcup_{t \in \Gamma} C_t \right)$ . Then, every nonzero vector of  $K$  can be expressed as a non-negative linear combination of  $n$  or fewer linear independent vectors, each belonging to a different  $C_t$ .

**Lemma 6.** [7] Suppose that  $S, T, P$  are arbitrary (possibly infinite) index sets,  $a_s = a(s) = (a_1(s), \dots, a_n(s))$  maps  $S$  onto  $\mathbb{R}^n$ , and so do  $a_t$  and  $a_p$ . Suppose that the set  $\text{co}\{a_s, s \in S\} + \text{pos}\{a_t, t \in T\} + \text{span}\{a_p, p \in P\}$  is closed. Then the following statements are equivalent:

$$I : \begin{cases} \langle a_s, x \rangle < 0, s \in S, S \neq \emptyset \\ \langle a_t, x \rangle \leq 0, t \in T \\ \langle a_p, x \rangle = 0, p \in P \end{cases} \quad \text{has no solution } x \in \mathbb{R}^n;$$

$$II : 0 \in \text{co}\{a_s, s \in S\} + \text{pos}\{a_t, t \in T\} + \text{span}\{a_p, p \in P\}.$$

**Lemma 7.** [10] If  $A$  is a nonempty compact subset of  $\mathbb{R}^n$ , then,

- (i)  $\text{co}A$  is a compact set;
- (ii) if  $0 \notin \text{co}A$ , then  $\text{pos}A$  is a closed cone.

### 3. KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS

In this section, we write the index set  $I_g$  instead of  $I_g(\bar{x})$  for the sake of convenience. The other index sets are expressed similarly.

**Definition 8.** *The linearized cone of (P) at  $\bar{x} \in \Omega$  is*

$$L(\bar{x}) := \{d \in \mathbb{R}^n \mid \langle \nabla g_t(\bar{x}), d \rangle \leq 0 (t \in I_g), \langle \nabla h_i(\bar{x}), d \rangle = 0 (i \in I_h), \langle \nabla G_i(\bar{x}), d \rangle = 0 (i \in I_{0+}), \langle \nabla G_i(\bar{x}), d \rangle \geq 0 (i \in I_{00}), \langle \nabla H_i(\bar{x}), d \rangle \geq 0 (i \in I_{00}), \langle \nabla H_i(\bar{x}), d \rangle = 0 (i \in I_{+0})\}.$$

*By the proof similar to the proof of Lemma 4 in [6], we can prove that  $L(\bar{x})$  is the linearized cone of (P) in the sense of nonlinear programming.*

**Remark 9.** *We can check that*

$$L(\bar{x}) = \left( \bigcup_{t \in I_g} \nabla g_t(\bar{x}) \right)^- \cap \left( \bigcup_{i \in I_h} \nabla h_i(\bar{x}) \right)^\perp \cap \left( \bigcup_{i \in I_{0+}} \nabla G_i(\bar{x}) \right)^\perp \\ \cap \left( \bigcup_{i \in I_{00}} (-\nabla G_i(\bar{x})) \right)^- \cap \left( \bigcup_{i \in I_{00}} (-\nabla H_i(\bar{x})) \right)^- \cap \left( \bigcup_{i \in I_{+0}} \nabla H_i(\bar{x}) \right)^\perp.$$

Now, we establish the KKT necessary optimality condition for locally weakly efficient solutions of (P) under the following constraint qualification:

$$(ACQ) : L(\bar{x}) \subseteq T(\Omega, \bar{x}).$$

**Proposition 10.** *Let  $\bar{x} \in \text{locWE}(P)$ . If (ACQ) holds at  $\bar{x}$  and the set*

$$\Delta := \text{pos} \left( \bigcup_{t \in I_g} \nabla g_t(\bar{x}) \cup \bigcup_{i \in I_{00}} (-\nabla G_i(\bar{x})) \cup -\nabla H_i(\bar{x}) \right) \\ + \text{span} \left( \bigcup_{i \in I_h} \nabla h_i(\bar{x}) \cup \bigcup_{i \in I_{0+}} \nabla G_i(\bar{x}) \cup \bigcup_{i \in I_{+0}} \nabla H_i(\bar{x}) \right)$$

*is closed, then  $\bar{x}$  is a strong stationary point of (P).*

*Proof.* Since  $\bar{x} \in \text{locWE}(P)$ , there exists  $U \in \mathcal{U}(\bar{x})$  such that there is no  $x \in \Omega \cap U$  satisfying

$$f_i(x) < f_i(\bar{x}), \forall i \in I. \tag{1}$$

First, we justify that

$$\left( \bigcup_{i \in I} \nabla f_i(\bar{x}) \right)^s \cap T(\Omega, \bar{x}) = \emptyset. \tag{2}$$

On the contrary, assume that there exists  $d \in \left( \bigcup_{i \in I} \nabla f_i(\bar{x}) \right)^s \cap T(\Omega, \bar{x})$ . Then, it is straightforward that

$$\langle \nabla f_i(\bar{x}), d \rangle < 0, \forall i \in I.$$

By  $d \in T(\Omega, \bar{x})$ , there exist  $\tau_k \downarrow 0$  and  $d_k \rightarrow d$  such that  $\bar{x} + \tau_k d_k \in \Omega$  for all  $k$ . As  $f_i(i \in I)$  is continuously differentiable at  $\bar{x}$ , one has

$$f_i(\bar{x} + \tau_k d_k) = f_i(\bar{x}) + \tau_k \langle \nabla f_i(\bar{x}), d_k \rangle + o(\tau_k \|d_k\|), \forall i \in I.$$

In consequence, for all  $i \in I$ ,

$$\frac{f_i(\bar{x} + \tau_k d_k) - f_i(\bar{x})}{\tau_k} = \langle \nabla f_i(\bar{x}), d_k \rangle + \frac{o(\tau_k \|d_k\|)}{\tau_k \|d_k\|} \cdot \|d_k\| \rightarrow \langle \nabla f_i(\bar{x}), d \rangle < 0, \text{ when } k \rightarrow \infty.$$

Hence, for each  $i \in I$ , there exists  $\bar{k}_i > 0$  such that  $\frac{f_i(\bar{x} + \tau_k d_k) - f_i(\bar{x})}{\tau_k} < 0$ , for all  $k > \bar{k}_i$ . Setting  $\bar{k} := \max\{\bar{k}_i \mid i \in I\}$ , we guarantee the existence of  $k > \bar{k}$  large enough such that  $\bar{x} + \tau_k d_k \in \Omega \cap U$  and

$$f_i(\bar{x} + \tau_k d_k) < f_i(\bar{x}), \forall i \in I,$$

which contradicts (1). Therefore, the fulfillment of (2) follows. We get from (2) and (ACQ) that

$$\begin{aligned} & \left( \bigcup_{i \in I} \nabla f_i(\bar{x}) \right)^s \cap \left( \bigcup_{t \in I_g} \nabla g_t(\bar{x}) \right)^- \cap \left( \bigcup_{i \in I_h} \nabla h_i(\bar{x}) \right)^\perp \cap \left( \bigcup_{i \in I_{0+}} \nabla G_i(\bar{x}) \right)^\perp \\ & \cap \left( \bigcup_{i \in I_{00}} (-\nabla G_i(\bar{x})) \right)^- \cap \left( \bigcup_{i \in I_{00}} (-\nabla H_i(\bar{x})) \right)^- \cap \left( \bigcup_{i \in I_{+0}} \nabla H_i(\bar{x}) \right)^\perp = \emptyset. \end{aligned}$$

This implies that there is no  $d \in \mathbb{R}^n$  such that

$$\begin{cases} \langle \nabla f_i(\bar{x}), d \rangle < 0, & \forall i \in I, \\ \langle \nabla g_t(\bar{x}), d \rangle \leq 0, & \forall t \in I_g, \\ \langle \nabla h_i(\bar{x}), d \rangle = 0, & \forall i \in I_h, \\ \langle \nabla G_i(\bar{x}), d \rangle = 0, & \forall i \in I_{0+}, \\ \langle -\nabla G_i(\bar{x}), d \rangle \leq 0, & \forall i \in I_{00}, \\ \langle -\nabla H_i(\bar{x}), d \rangle \leq 0, & \forall i \in I_{00}, \\ \langle \nabla H_i(\bar{x}), d \rangle = 0, & \forall i \in I_{+0}. \end{cases}$$

In addition, it follows from Lemma 7 that  $\text{co}\{\bigcup_{i \in I} \nabla f_i(\bar{x})\}$  is a compact set, which in turn implies  $\text{co}\{\bigcup_{i \in I} \nabla f_i(\bar{x})\} + \Delta$  is closed. According to Lemma 6, one has

$$0 \in \text{co} \bigcup_{i \in I} \nabla f_i(\bar{x}) + \text{pos} \left( \bigcup_{t \in I_g} \nabla g_t(\bar{x}) \cup \bigcup_{i \in I_{00}} (-\nabla G_i(\bar{x})) \cup -\nabla H_i(\bar{x}) \right)$$

$$+\text{span} \left( \bigcup_{i \in I_h} \nabla h_i(\bar{x}) \cup \bigcup_{i \in I_{0+}} \nabla G_i(\bar{x}) \cup \bigcup_{i \in I_{+0}} \nabla H_i(\bar{x}) \right).$$

This leads that

$$\begin{aligned} 0 \in & \text{co} \bigcup_{i \in I} \nabla f(\bar{x}) + \text{pos} \bigcup_{t \in I_g} \nabla g_t(\bar{x}) + \text{span} \bigcup_{i \in I_h} \nabla h_i(\bar{x}) + \text{pos} \bigcup_{i \in I_{00}} (-\nabla G_i(\bar{x})) \\ & + \text{span} \bigcup_{i \in I_{0+}} \nabla G_i(\bar{x}) + \text{pos} \bigcup_{i \in I_{00}} (-\nabla H_i(\bar{x})) + \text{span} \bigcup_{i \in I_{+0}} \nabla H_i(\bar{x}). \end{aligned}$$

By Lemma 5, we know that there exists  $(\alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^m \times \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$  with  $\lambda_{I_{+0}}^G = 0, \lambda_{I_{00}}^G \geq 0, \lambda_{I_{00}}^H \geq 0$  and  $\lambda_{I_{+0}}^H = 0$  such that

$$\sum_{i \in I} \alpha_i \nabla f_i(\bar{x}) + \sum_{t \in T} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_1} \lambda_i^G \nabla G_i(\bar{x}) - \sum_{i \in I_1} \lambda_i^H \nabla H_i(\bar{x}) = 0.$$

So,  $\bar{x}$  is a strong-stationary point of (P).  $\square$

**Proposition 11.** *Let  $\bar{x} \in \Omega$  be a strong stationary point of (P). Suppose that  $I_{0+}^- \cup \hat{I}_{+0}^- = \emptyset$  and  $g_t(t \in I_g), h_i(i \in I_h^+), -h_i(i \in I_h^-), -G_i(i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++}), -H_i(i \in \hat{I}_{00}^+ \cup \hat{I}_{+0}^+ \cup I_{00}^{++})$  are quasiconvex at  $\bar{x}$ .*

- (i) *If  $f_i(i \in I)$  is pseudoconvex at  $\bar{x}$ , then  $\bar{x}$  is a weakly efficient solution of (P).*
- (ii) *If  $f_i(i \in I)$  is strictly pseudoconvex at  $\bar{x}$ , then  $\bar{x}$  is an efficient solution of (P).*

*Proof.* Since  $\bar{x}$  is a strong stationary point of (P), there exists  $(\alpha, \lambda_J^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^m \times \mathbb{R}_+^{|J|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ , where  $J$  is a finite subset of  $I_g$ , with  $\lambda_{I_{+0}}^G = 0, \lambda_{I_{00}}^G \geq 0, \lambda_{I_{00}}^H \geq 0$  and  $\lambda_{I_{+0}}^H = 0$  such that

$$\sum_{i \in I} \alpha_i \nabla f_i(\bar{x}) + \sum_{t \in J} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_{0+} \cup I_{00}} \lambda_i^G \nabla G_i(\bar{x}) - \sum_{i \in I_{00} \cup I_{+0}} \lambda_i^H \nabla H_i(\bar{x}) = 0. \tag{3}$$

For an arbitrary  $x \in \Omega$ , one gets that  $g_t(x) \leq 0 = g_t(\bar{x})$  for each  $t \in I_g$ . Therefore, by the quasiconvexity at  $\bar{x}$  of  $g_t(t \in I_g)$ , we have

$$\langle \nabla g_t(\bar{x}), x - \bar{x} \rangle \leq 0, \forall t \in J,$$

which in turn together with  $\lambda_J^g \in \mathbb{R}_+^{|J|}$  derives that

$$\left\langle \sum_{t \in J} \lambda_t^g \nabla g_t(\bar{x}), x - \bar{x} \right\rangle \leq 0. \tag{4}$$

We deduce from  $x, \bar{x} \in \Omega$  that  $h_i(x) = h_i(\bar{x}) = 0, \forall i \in I_h$ , and hence,

$$h_i(x) \leq h_i(\bar{x}), \forall i \in I_h^+ \text{ and } -h_i(x) \leq -h(\bar{x}), \forall i \in I_h^-.$$

The above inequalities together with the quasiconvexity at  $\bar{x}$  of  $h_i(i \in I_h^+)$  and  $-h_i(i \in I_h^-)$  ensures that

$$\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_h^+ \quad \text{and} \quad \langle -\nabla h_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_h^-.$$

This, taking into account the definitions of  $I_h^+, I_h^-$ , gives us

$$\left\langle \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}), x - \bar{x} \right\rangle \leq 0. \tag{5}$$

Again, we derive from  $x \in \Omega$  that  $-G_i(x) \leq 0, \forall i \in I_l$ , and thus,  $-G_i(x) \leq -G_i(\bar{x})(i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++})$ . Therefore, by the quasiconvexity of  $-G_i(i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++})$  at  $\bar{x}$ , one yields that

$$\langle -\nabla G_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++},$$

which, along with the definitions of  $I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++}$ , leads that

$$-\left\langle \sum_{i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++}} \lambda_i^G \nabla G_i(\bar{x}), x - \bar{x} \right\rangle \leq 0 \tag{6}$$

Similarly, we can justify that

$$-\left\langle \sum_{i \in \hat{I}_{00}^+ \cup \hat{I}_{+0}^+ \cup I_{00}^{++}} \lambda_i^H \nabla H_i(\bar{x}), x - \bar{x} \right\rangle \leq 0. \tag{7}$$

As  $I_{0+}^- \cup \hat{I}_{+0}^- = \emptyset$ , we infer from (3) - (7) that

$$\begin{aligned} & \left\langle \sum_{i \in I} \alpha_i \nabla f_i(\bar{x}), x - \bar{x} \right\rangle \\ &= - \left\langle \sum_{t \in T} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(\bar{x}), x - \bar{x} \right\rangle \geq 0, \end{aligned} \tag{8}$$

for all  $x \in \Omega$ .

(i) Suppose, to the contrary, that  $\bar{x}$  is not a weakly efficient solution of (P). This leads to the existence of a feasible point  $\tilde{x} \in \Omega$  satisfying

$$f_i(\tilde{x}) < f_i(\bar{x}), \forall i \in I.$$

The fact on  $f_i(\tilde{x}) < f_i(\bar{x})$  for each  $i$  and the pseudoconvexity of  $f_i(i \in I)$  give us the inclusions

$$\langle \nabla f_i(\bar{x}), \tilde{x} - \bar{x} \rangle < 0, i \in I.$$

Combining this with  $\alpha \in \mathbb{R}_+^m$  and  $\sum_{i=1}^m \alpha_i = 1$ , we arrive at

$$\left\langle \sum_{i \in I} \alpha_i \nabla f_i(\bar{x}), \tilde{x} - \bar{x} \right\rangle < 0,$$

contradicting with (8).

(ii) Reasoning by contraposition, assume that  $\bar{x}$  is not an efficient solution. Then there exists a feasible point  $\tilde{x}$  and at least  $i_0 \in I$  fulfilling

$$\begin{cases} f_i(\tilde{x}) \leq f_i(\bar{x}), & \forall i \in I \setminus \{i_0\}, \\ f_{i_0}(\tilde{x}) < f_{i_0}(\bar{x}), \end{cases}$$

and hence,  $\tilde{x} \neq \bar{x}$ . It follows from the fact that  $f_i (i \in I)$  are strictly pseudoconvex and  $x \neq \bar{x}$ , one has

$$\langle \nabla f_i, \tilde{x} - \bar{x} \rangle < 0, \forall i \in I.$$

Using this with  $\alpha \in \mathbb{R}_+^m$  and  $\sum_{i=1}^m \alpha_i = 1$  tells us that

$$\left\langle \sum_{i \in I} \alpha_i \nabla f_i(\bar{x}), \tilde{x} - \bar{x} \right\rangle < 0,$$

which contradicts (8).  $\square$

**Example 12.** Let  $m = 2, n = 2$  and  $l = 1$ . Let us consider the following (P):

$$\begin{aligned} \mathbb{R}_+^2 - \min \quad & f(x) = (f_1(x), f_2(x)) = (x_1^2 + x_2^2 + 2x_1, x_1^2 + 2x_2^2), \\ \text{s.t.} \quad & g_t(x) = tx_1 \leq 0, t \in T = \mathbb{N} = \{1, 2, \dots\}, \\ & G_1(x) = x_1 \geq 0, \\ & H_1(x) = x_1 + x_2 \geq 0, \\ & G_1(x)H_1(x) = x_1(x_1 + x_2) = 0. \end{aligned}$$

Then,  $\Omega = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}$ . For  $\bar{x} = (0, 0) \in \Omega$ , direct calculations give us that

$$T(\Omega, \bar{x}) = \Omega, \nabla f_1(\bar{x}) = \{(2, 0)\}, \nabla f_2(\bar{x}) = \{(0, 0)\}, I_g = T = \mathbb{N},$$

$$\nabla g_t(\bar{x}) = \{(t, 0)\}, t \in T, \left( \bigcup_{t \in I_g} \nabla g_t(\bar{x}) \right)^- = \{x \in \mathbb{R}^2 \mid x_1 \leq 0\},$$

$$I_{+0} = I_{0+} = \emptyset, I_{00} = \{1\}, \nabla G_1(\bar{x}) = \{(1, 0)\}, \nabla H_1(\bar{x}) = \{(1, 1)\},$$

$$\left( \bigcup_{i \in I_{00}} (-\nabla H_i(\bar{x})) \right)^- = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \geq 0\}, \left( \bigcup_{i \in I_{00}} -\nabla G_i(\bar{x}) \right)^- = \{x \in \mathbb{R}^2 \mid x_1 \geq 0\},$$

$$\left( \bigcup_{t \in I_g} \nabla g_t(\bar{x}) \right)^- \cap \left( \bigcup_{i \in I_{00}} (-\nabla H_i(\bar{x})) \right)^- \cap \left( \bigcup_{i \in I_{00}} \nabla G_i(\bar{x}) \right)^- = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}.$$

Hence,

$$\left( \bigcup_{t \in I_g} \nabla g_t(\bar{x}) \right)^- \cap \left( \bigcup_{i \in I_{00}} (-\nabla G_i(\bar{x})) \right)^- \cap \left( \bigcup_{i \in I_{00}} -\nabla H_i(\bar{x}) \right)^- \subset T(\Omega, \bar{x}),$$

leading that (ACQ) holds at  $\bar{x}$ . Moreover,

$$\Delta = \text{pos} \left( \bigcup_{t \in I_g} \nabla g_t(\bar{x}) \cup \bigcup_{i \in I_{00}} (-\nabla G_i(\bar{x})) \cup \bigcup_{i \in I_{00}} (-\nabla H_i(\bar{x})) \right) = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$$

is closed. Due to the fact  $f(x) - f(\bar{x}) \notin -\mathbb{R}_+^2 \setminus \{0\}, \forall x \in \Omega$ , we conclude that  $\bar{x} \in WE(P)$ . Thus, all assumptions in Proposition 10 are fulfilled. Now, let  $\alpha_1 = \alpha_2 = \frac{1}{2}, \lambda_1^G = 2, \lambda_1^H = 0$  and  $\lambda^g : T \rightarrow \mathbb{R}$  be defined by

$$\lambda^g(t) = \begin{cases} 1, & \text{if } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\frac{1}{2}(2, 0) + \frac{1}{2}(0, 0) + \sum_{t \in T} \lambda_t^g(t, 0) - \lambda_1^G(1, 0) - \lambda_1^H(1, 1) = (0, 0),$$

which means that  $\bar{x}$  is a strong stationary point of (P). Notice that, for the above  $\bar{x}$  and  $(\lambda^g, \lambda_1^H, \lambda_1^G)$ , one has

$$I_{00}^{++} = I_{00}^- = \hat{I}_{00}^+ = \hat{I}_{00}^- = \emptyset, I_{00}^+ = \{1\}.$$

Furthermore, we can check that  $g_t(t \in I_g), -G_1(1 \in I_{00}^+)$  are convex at  $\bar{x}$  and  $f_i(i \in I)$  are strictly convex at  $\bar{x}$ . Hence, all assumptions in Proposition 11 (ii) are satisfied. Then, it follows Proposition 11 (ii) that  $\bar{x}$  is an efficient solution of (P).

#### 4. DUALITY

In this section, we consider the Wolfe [33] and Mond-Weir [19] duality schemes for (P). For  $\bar{x} \in \Omega$ , the index sets with respect to  $\bar{x}$  are denoted identically to Section 3. In what follows, for  $u, v \in \mathbb{R}^m$ , we use the notations:

$$u \prec v \Leftrightarrow u_i < v_i \text{ for all } i \in I, \quad u \not\prec v \text{ is the negation of } u \prec v,$$

$$u \preceq v \Leftrightarrow \begin{cases} u_i \leq v_i, & \text{for all } i \in I, \\ u_i < v_i, & \text{for at least one } i_0 \in I, \end{cases} \quad u \not\preceq v \text{ is the negation of } u \preceq v.$$

Note that  $\bar{x} \in \text{loc}E(P)$  ( $\bar{x} \in \text{loc}WE(P)$ ) if there exists  $U \in \mathcal{U}(\bar{x})$  such that there is no  $x \in \Omega \cap U$  satisfying  $f(x) \preceq f(\bar{x})$  ( $f(x) \prec f(\bar{x})$ ).

##### 4.1. The Wolfe type duality

For an arbitrary  $\bar{x} \in \Omega$ ,  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$  with  $\sum_{i \in I} \alpha_i = 1, \lambda_{I_{+0}(\bar{x})}^G = 0, \lambda_{I_{00}(\bar{x})}^G \geq 0, \lambda_{I_{00}(\bar{x})}^H \geq 0$  and  $\lambda_{I_{0+}(\bar{x})}^H = 0$ , we define

$$L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) := f(u)$$

$$+ \left( \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right) e,$$

where  $e := (1, \dots, 1) \in \mathbb{R}^m$ . In this paper, we consider the Wolfe type dual problem as follows:

$$\begin{aligned} (D_W(\bar{x})) : \mathbb{R}_+^m - \max L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \\ \text{s.t. } \sum_{i \in I} \alpha_i \nabla f_i(u) + \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u) = 0, \\ \sum_{i \in I} \alpha_i = 1, \lambda_{I_{+0}(\bar{x})}^G = 0, \lambda_{I_{00}(\bar{x})}^G \geq 0, \lambda_{I_{00}(\bar{x})}^H \geq 0, \lambda_{I_{0+}(\bar{x})}^H = 0, \\ (u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l. \end{aligned}$$

The feasible set of  $(D_W(\bar{x}))$  is defined by

$$\begin{aligned} \Omega_W(\bar{x}) := \left\{ (u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l \mid \right. \\ \left. \sum_{i \in I} \alpha_i = 1, \lambda_{I_{+0}(\bar{x})}^G = 0, \lambda_{I_{00}(\bar{x})}^G \geq 0, \lambda_{I_{00}(\bar{x})}^H \geq 0, \lambda_{I_{0+}(\bar{x})}^H = 0, \right. \\ \left. \sum_{i \in I} \alpha_i \nabla f_i(u) + \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u) = 0 \right\}. \end{aligned}$$

**Definition 13.** Let  $\bar{x} \in \Omega$ .

- (i)  $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_W(\bar{x})$  is a locally efficient solution of  $(D_W(\bar{x}))$ , denoted by  $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \text{loc}E(D_W(\bar{x}))$ , if there exists  $U \in \mathcal{N}(\bar{u})$  such that there is no  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x}) \cap U$  satisfying

$$L(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \preceq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

- (ii)  $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_W(\bar{x})$  is a locally weakly efficient solution of  $(D_W(\bar{x}))$ , denoted by  $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \text{loc}WE(D_W(\bar{x}))$ , if there exists  $U \in \mathcal{N}(\bar{u})$  such that there is no  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x}) \cap U$  fulfilling

$$L(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

If  $U = \mathbb{R}^n$ , the word “locally” is omitted.

**Remark 14.** When  $m = 1$  and  $f_1, g_t(t \in T), h_i(i = 1, \dots, q)$  and  $G_i, H_i(i = 1, \dots, l)$  are continuously differentiable functions,  $D_W(\bar{x})$  becomes the Wolfe type dual model WDSIMPEC( $\bar{x}$ ) in [20].

The following proposition describes weak duality relations between (P) and the dual problem  $(D_W(\bar{x}))$ .

**Proposition 15.** (weak duality) Let  $x \in \Omega$  and  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$ . Suppose that  $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$  and  $g_t(t \in T), h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x})), -G_i(i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})), -H_i(i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$  are convex at  $u$ .

(i) If  $f_i (i \in I)$  are convex at  $u$ , then

$$f(x) \not\leq L(y, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

(ii) If  $f_i (i \in I)$  are strictly convex at  $u$ , then

$$f(x) \not\leq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

*Proof.* For  $x \in \Omega$  and  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$ , one gets

$$g_t(x) \leq 0 (t \in T), h_i(x) = 0 (i \in I_h), G_i(x) \geq 0, H_i(x) \geq 0, G_i(x)H_i(x) = 0 (i \in I_l), \tag{9}$$

and

$$\sum_{i \in I} \alpha_i \nabla f_i(u) + \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u) = 0 \tag{10}$$

with

$$\sum_{i \in I} \alpha_i = 1, \lambda_{I_{+0}(\bar{x})}^G = 0, \lambda_{I_{00}(\bar{x})}^G \geq 0, \lambda_{I_{00}(\bar{x})}^H \geq 0, \lambda_{I_{0+}(\bar{x})}^H = 0. \tag{11}$$

Therefore, we infer from (9), the convexity of  $g_t (t \in T), h_i (i \in I_h^+(\bar{x}), -h_i (i \in I_h^-(\bar{x})), -G_i (i \in I_{0+}^+(x) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})), -H_i (i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$  at  $u$  and the definitions of the index sets that

$$\begin{aligned} g_t(u) + \langle \nabla g_t(u), x - u \rangle &\leq g_t(x) \leq 0, \lambda_t^g \geq 0, \forall t \in T, \\ h_i(u) + \langle \nabla h_i(u), x - u \rangle &\leq h_i(x) = 0, \lambda_i^h > 0, \forall i \in I_h^+(\bar{x}), \\ -h_i(u) + \langle -\nabla h_i(u), x - u \rangle &\leq -h_i(x) = 0, \lambda_i^h < 0, \forall i \in I_h^-(\bar{x}), \\ -G_i(u) + \langle -\nabla G_i(u), x - u \rangle &\leq -G_i(x) \leq 0, \lambda_i^G > 0, \forall i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}), \\ -H_i(u) + \langle -\nabla H_i(u), x - u \rangle &\leq -H_i(x) \leq 0, \lambda_i^H > 0, \forall i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}). \end{aligned}$$

The above inequalities together with  $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$  imply that

$$\begin{aligned} &\sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \\ &+ \left\langle \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u), x - u \right\rangle \leq 0. \end{aligned}$$

It follows from the above inequality and (10) that

$$\left\langle \sum_{i \in I} \alpha_i \nabla f_i(u), x - u \right\rangle$$

$$\begin{aligned}
 &= - \left\langle \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u), x - u \right\rangle \\
 &\geq \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u). \tag{12}
 \end{aligned}$$

(i) Reasoning ad absurdum, suppose that

$$f(x) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H). \tag{13}$$

It follows from (13),  $\alpha \in \mathbb{R}_+^m$  and  $\sum_{i=1}^m \alpha_i = 1$  that  $\langle \alpha, f(x) - L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \rangle < 0$ , which is equivalent to

$$\sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)) - \sum_{i=1}^m \alpha_i \left( \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right) < 0.$$

The above inequality, together with  $\sum_{i=1}^m \alpha_i = 1$ , yields

$$\sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)) < \left( \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right). \tag{14}$$

The convexity of  $f_i (i \in I)$  at  $u$  confirms that

$$\langle \nabla f_i(u), x - u \rangle \leq f_i(x) - f_i(u), \forall i \in I,$$

leading to

$$\left\langle \sum_{i=1}^m \alpha_i \nabla f_i(u), x - u \right\rangle \leq \sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)). \tag{15}$$

We verify from (14) and (15) that

$$\left\langle \sum_{i=1}^m \alpha_i \nabla f_i(u), x - u \right\rangle < \left( \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right),$$

contradicting with (12).

(ii) Reasoning by contraposition, assume that

$$f(x) \preceq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H), \tag{16}$$

We claim that  $x \neq u$ . If otherwise, we use (16) and  $x = u$  to derive that

$$a := - \left( \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right) e \preceq 0. \tag{17}$$

Observe by  $u = x \in \Omega(\bar{x})$  and (11) that

$$\begin{aligned} g_t(u) &= g_t(x) \leq 0, \forall t \in T, \lambda \in \mathbb{R}_+^{|T|}, \\ h_i(u) &= h_i(x) = 0, \forall i \in I_h, \lambda_i^h \in \mathbb{R}, \\ -G_i(u) &= -G_i(x) \leq 0, \forall i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}), \lambda_{I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})}^H \geq 0, \\ -H_i(u) &= -H_i(x) \leq 0, \forall i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}), \lambda_{\hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})}^H \geq 0. \end{aligned}$$

The above inequalities together with  $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$  imply that

$$\sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \leq 0.$$

Hence,  $\alpha_i \geq 0, \forall i \in I$ , contradicts with (17), which in turn leads to  $x \neq u$ . On the other hand, we deduce from (16) and  $\alpha \in \mathbb{R}_+^m$  that  $\langle \alpha, f(x) - L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \rangle \leq 0$ , in other words,

$$\sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)) - \sum_{i=1}^m \alpha_i \left( \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right) \leq 0.$$

Employing this, together with  $\sum_{i=1}^m \alpha_i = 1$ , bring us the inequality

$$\sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)) \leq \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u). \tag{18}$$

Since  $f_i (i \in I)$  are strictly convex at  $u$  and  $x \neq u$ , we have

$$\langle \nabla f_i(u), x - u \rangle < f_i(x) - f_i(u), \forall i \in I,$$

leading that

$$\left\langle \sum_{i=1}^m \alpha_i \nabla f_i(u), x - u \right\rangle < \sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)). \tag{19}$$

It follows from (18) and (19) that

$$\left\langle \sum_{i=1}^m \alpha_i \nabla f_i(u), x - u \right\rangle < \left( \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right),$$

contradicting with (12).  $\square$

**Proposition 16.** (strong duality) *Let  $\bar{x} \in \Omega$  be a locally weakly efficient solution of (P). If (ACQ) holds at  $\bar{x}$  and the set  $\Delta$  is closed, then there exists  $(\bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}_+^m \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$  with  $\sum_{i \in I} \bar{\alpha}_i = 1, \bar{\lambda}_{I_{+0}(\bar{x})}^G = 0, \bar{\lambda}_{I_{00}(\bar{x})}^G \geq 0, \bar{\lambda}_{I_{00}(\bar{x})}^H \geq 0$  and  $\bar{\lambda}_{I_{0+}(\bar{x})}^H = 0$  such that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_W(\bar{x})$  and*

$$f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H).$$

*Assume further that  $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$  and  $g_t(t \in T), h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x})), -G_i(i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})), -H_i(i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$  are convex at  $\bar{x}$ .*

- (i) *If  $f_i(i \in I)$  are convex at  $\bar{x}$ , then  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is a weakly efficient solution of  $D_W(\bar{x})$ .*
- (ii) *If  $f_i(i \in I)$  are strictly convex at  $\bar{x}$ , then  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is an efficient solution of  $D_W(\bar{x})$ .*

*Proof.* In view of Proposition 10, there exists  $(\bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}_+^m \times \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$  with  $\sum_{i \in I} \bar{\alpha}_i = 1, \bar{\lambda}_{I_{+0}(\bar{x})}^G = 0, \bar{\lambda}_{I_{00}(\bar{x})}^G \geq 0, \bar{\lambda}_{I_{00}(\bar{x})}^H \geq 0$  and  $\bar{\lambda}_{I_{0+}(\bar{x})}^H = 0$  such that

$$\sum_{i \in I} \bar{\alpha}_i \nabla f_i(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \bar{\lambda}_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_t} \bar{\lambda}_i^G \nabla G_i(\bar{x}) - \sum_{i \in I_t} \bar{\lambda}_i^H \nabla H_i(\bar{x}) = 0.$$

Since  $\bar{\lambda}^g \in \Lambda(\bar{x})$ , one has  $\bar{\lambda}_t^g g_t(\bar{x}) = 0$  for all  $t \in T$ , and thus,  $\sum_{t \in T} \bar{\lambda}_t^g g_t(\bar{x}) = 0$ .

The fact that  $\bar{x} \in \Omega$  ensures that  $\sum_{i \in I_h} \bar{\lambda}_i^h h_i(\bar{x}) = 0$ . Moreover, as  $\lambda_{I_{+0}(\bar{x})}^G = 0$  and  $G_i(\bar{x}) = 0$  for all  $i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})$ , we know that  $\sum_{i \in I_t} \bar{\lambda}_i^G G_i(\bar{x}) = 0$ .

Analogously, we observe by  $\bar{\lambda}_{I_{0+}(\bar{x})}^H = 0$  and  $H_i(\bar{x}) = 0$  for all  $i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x})$  that  $\sum_{i \in I_t} \bar{\lambda}_i^H H_i(\bar{x}) = 0$ . Thus,  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_W(\bar{x})$  and

$$\sum_{t \in T} \lambda_t g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h h_i(\bar{x}) - \sum_{i \in I_t} \lambda_i^G G_i(\bar{x}) - \sum_{i \in I_t} \lambda_i^H H_i(\bar{x}) = 0,$$

which is nothing else but the following equality  $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ .

(i) Now, arguing by contradiction, let us suppose that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is not a weakly efficient solution of  $D_W(\bar{x})$ . By definition, there exists  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$  such that

$$L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

This shows that

$$f(\bar{x}) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

which contradicts with Proposition 15 (i). So,  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is a weakly efficient solution to  $(D_W(\bar{x}))$ .

(ii) Reasoning to the contrary, let us assume that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is not an efficient solution to  $D_W(\bar{x})$ . Then, it guarantees the existence of  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$  such that

$$L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \preceq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

Consequently,

$$f(\bar{x}) \preceq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

which contradicts with Proposition 15 (ii). So,  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is an efficient solution to  $(D_W(\bar{x}))$ .  $\square$

**Example 17.** Let  $m = n = 2$  and  $l = 1$ . Consider the following (P):

$$\begin{aligned} \mathbb{R}_+^2 - \min \quad & f(x) = (x_1^2 + x_2^2 + 4x_2, x_1 - x_2), \\ \text{s.t.} \quad & g_t(x) = tx_1 \leq 0, t \in T = \mathbb{N}, \\ & G_1(x) = x_1 \geq 0, \\ & H_1(x) = x_1 + x_2 \geq 0, \\ & G_1(x)H_1(x) = x_1(x_1 + x_2) = 0. \end{aligned}$$

Then,  $\Omega = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}$ . For any  $\bar{x} \in \Omega$ ,

$$\begin{aligned} (D_{MW}(\bar{x})) : \mathbb{R}_+^2 - \max \quad & L(u, \alpha, \lambda^g, \lambda^G, \lambda^H) \\ & = (u_1^2 + u_2^2 + 4u_2, u_1 - u_2) + \left( \sum_{t \in T} tu_1 - \lambda_1^G u_1 - \lambda_1^H (u_1 + u_2) \right) (1, 1) \\ \text{s.t.} \quad & \alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1) + \sum_{t \in T} \lambda_t^g(t, 0) - \lambda_1^G(1, 0) - \lambda_1^H(1, 1) = (0, 0), \\ & \alpha_1 + \alpha_2 = 1, \lambda_1^G \begin{cases} = 0, & \text{if } 1 \in I_{+0}(\bar{x}), \\ \geq 0, & \text{if } 1 \in I_{00}(\bar{x}), \\ \in \mathbb{R}, & \text{if } 1 \in I_{0+}(\bar{x}), \end{cases} \lambda_1^H \begin{cases} \in \mathbb{R}, & \text{if } 1 \in I_{+0}(\bar{x}), \\ \geq 0, & \text{if } 1 \in I_{00}(\bar{x}), \\ = 0, & \text{if } 1 \in I_{0+}(\bar{x}), \end{cases} \\ & (u, \alpha, \lambda^g, \lambda_1^G, \lambda_1^H) \in \mathbb{R}^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^{|T|} \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

By taking  $\bar{x} = (0, 0) \in \Omega$ , we invoke from Example 12 that all hypotheses of Proposition 16 (i) are fulfilled. Since  $f(x) - f(\bar{x}) = (x_2^2 + 4x_2, -x_2) \notin -\text{int}\mathbb{R}_+^2, \forall x \in \Omega$ , one has  $\bar{x} \in WE(P)$ . Now, if we select  $\bar{\alpha}_1 = \bar{\alpha}_2 = \frac{1}{2}, \bar{\lambda}_1^G = 0, \bar{\lambda}_1^H = \frac{3}{2}$  and

$$\bar{\lambda}^g(t) = \begin{cases} 1, & \text{if } t = 1, \\ 0, & \text{otherwise,} \end{cases}$$

then we get

$$\frac{1}{2}(0, 4) + \frac{1}{2}(1, -1) + \sum_{t \in T} \bar{\lambda}_t^g(t, 0) - \bar{\lambda}_1^G(1, 0) - \bar{\lambda}_1^H(1, 1) = (0, 0),$$

and,

$$\begin{aligned} I_{0+}(\bar{x}) &= I_{0+}(\bar{x}) = \emptyset, I_{00}(\bar{x}) = \{1\}, \\ \bar{\lambda}_1^H &= 1 \geq 0, \bar{\lambda}_1^G = 0 \geq 0, 1 \in I_{00}(\bar{x}), \end{aligned}$$

which gives the result  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H) \in \Omega_W(\bar{x})$  and  $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$ . Note that, for the above  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$ ,

$$\hat{I}_{00}^+(\bar{x}) = \{1\}, \hat{I}_{00}^-(\bar{x}) = I_{00}^+(\bar{x}) = I_{00}^-(\bar{x}) = I_{00}^{++}(\bar{x}) = \emptyset.$$

Moreover, we can verify that  $f_1, f_2, g_t(t \in T), -H_i(i \in \hat{I}_{00}^+(\bar{x}))$  are convex at  $\bar{x}$ . Hence, Proposition 16 (i) asserts that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$  is a weakly efficient solution to  $(D_W(\bar{x}))$ .

We can check directly that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$  is a weakly efficient solution to  $(D_W(\bar{x}))$  as follows. Firstly, we conclude from  $\bar{x} = (0, 0)$  and  $I_{0+}(\bar{x}) = I_{0+}(\bar{x}) = \emptyset, I_{00}(\bar{x}) = \{1\}$  that

$$\Omega_W(\bar{x}) = \left\{ (u, \alpha, \lambda^g, \lambda_1^G, \lambda_1^H) \in \mathbb{R}_+^2 \times \mathbb{R}^2 \times \mathbb{R}_+^{|T|} \times \mathbb{R} \times \mathbb{R} \mid \alpha_1 + \alpha_2 = 1, \lambda_1^G \geq 0, \lambda_1^H \geq 0 \right. \\ \left. \alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1) + \sum_{t \in T} \lambda_t^g g_t(t, 0) - \lambda_1^G(1, 0) - \lambda_1^H(1, 1) = (0, 0) \right\}.$$

Now, for an arbitrary  $u \in \Omega_W(\bar{x})$ , the convexity of  $g_t(t \in T), -G_i(i \in I_{00}^+(\bar{x})), -H_i(i \in \hat{I}_{00}^+(\bar{x}))$  at  $u$  and the definitions of the index sets deduce the inequalities

$$g_t(u) + \langle (t, 0), \bar{x} - u \rangle \leq g_t(\bar{x}) \leq 0, \lambda_t^g \geq 0, \forall t \in T,$$

$$-G_1(u) + \langle -(1, 0), \bar{x} - u \rangle \leq -G_1(\bar{x}) = 0, \lambda_1^G > 0, \text{ if } 1 \in I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}),$$

$$-H_1(u) + \langle -(1, -1), \bar{x} - u \rangle \leq -H_1(\bar{x}) = 0, \lambda_1^H > 0, \text{ if } 1 \in \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}).$$

We deduce from the above inequalities,  $u \in \Omega_W(\bar{x})$  and  $I_{00}^-(\bar{x}) = \hat{I}_{00}^-(\bar{x}) = \emptyset$  that

$$\langle \alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1), \bar{x} - u \rangle = - \left\langle \sum_{t \in T} \lambda_t^g g_t(t, 0) - \lambda_1^G(1, 0) - \lambda_1^H(1, 1), \bar{x} - u \right\rangle \\ \geq \sum_{t \in T} \lambda_t g_t(u) - \sum_{i \in I_1} \lambda_i^G G_i(u) - \sum_{i \in I_1} \lambda_i^H H_i(u). \tag{20}$$

Reasoning by contraposition, suppose that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$  is not a weakly efficient solution to  $(D_W(\bar{x}))$ . Then there exists  $(u, \alpha, \lambda^g, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$  such that

$$L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H) \prec L(u, \alpha, \lambda^g, \lambda_1^G, \lambda_1^H).$$

This along with  $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H), \alpha \in \mathbb{R}_+^2$  and  $\sum_{i=1}^2 \alpha_i = 1$  gives us that  $\langle \alpha, f(\bar{x}) - L(u, \alpha, \lambda^g, \lambda^G, \lambda^H) \rangle < 0$ , which is equivalent to

$$\sum_{i=1}^2 \alpha_i (f_i(\bar{x}) - f_i(u)) - \left( \sum_{t \in T} \lambda_t g_t(u) - \sum_{i \in I_1} \lambda_i^G G_i(u) - \sum_{i \in I_1} \lambda_i^H H_i(u) \right) < 0.$$

From the above relation together with (20), we derive

$$\sum_{i=1}^2 \alpha_i (f_i(\bar{x}) - f_i(u)) < \langle \alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1), \bar{x} - u \rangle. \tag{21}$$

On the other hand, since  $f_1, f_2$  are convexity at  $u$ , this yields

$$\langle (2u_1, 2u_2 + 4), \bar{x} - u \rangle \leq f_1(\bar{x}) - f_1(u),$$

$$\langle (1, -1), \bar{x} - u \rangle \leq f_2(\bar{x}) - f_2(u),$$

which, taking into account  $\alpha \in \mathbb{R}_+^m$ , justifies that

$$\langle \alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1), \bar{x} - u \rangle \leq \sum_{i=1}^2 \alpha_i(f_i(\bar{x}) - f_i(u)),$$

contradicting with (21).

#### 4.2. The Mond-Weir type duality

For an arbitrary  $\bar{x} \in \Omega$  and  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$  with  $\sum_{i \in I} \alpha_i = 1$ ,  $\lambda_{T \setminus I_g(\bar{x})}^g = 0$ ,  $\lambda_{I_{+0}(\bar{x})}^G = 0$ ,  $\lambda_{I_{00}(\bar{x})}^G \geq 0$ ,  $\lambda_{I_{00}(\bar{x})}^H \geq 0$  and  $\lambda_{I_{0+}(\bar{x})}^H = 0$ , we define

$$\tilde{L}(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) := f(u).$$

Now, we consider the Mond-Weir type dual problem as follows:

$$(D_{MW}(\bar{x})): \max \tilde{L}(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) = f(u)$$

$$\text{s.t. } \sum_{i \in I} \alpha_i \nabla f_i(u) + \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_1} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u) = 0,$$

$$g_t(u) \geq 0 (t \in I_g(\bar{x})), h_i(u) = 0 (i \in I_h(\bar{x})),$$

$$G_i(u) \geq 0 (i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})), H_i(u) \geq 0 (i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x}))$$

$$\sum_{i \in I} \alpha_i = 1, \lambda_{T \setminus I_g(\bar{x})}^g = 0, \lambda_{I_{+0}(\bar{x})}^G = 0, \lambda_{I_{00}(\bar{x})}^G \geq 0, \lambda_{I_{00}(\bar{x})}^H \geq 0, \lambda_{I_{0+}(\bar{x})}^H = 0,$$

$$(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^m \times \mathbb{R}^n \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l.$$

The feasible set of  $(D_{MW}(\bar{x}))$  is defined by

$$\Omega_{MW}(\bar{x}) := \left\{ (u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l \mid \right.$$

$$\left. \sum_{i \in I} \alpha_i \nabla f_i(u) + \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_1} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u) = 0, \right.$$

$$\left. g_t(u) \geq 0 (t \in I_g(\bar{x})), h_i(u) = 0 (i \in I_h(\bar{x})), \right.$$

$$\left. G_i(u) \geq 0 (i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})), H_i(u) \geq 0 (i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x})) \right\}.$$

$$\left. \sum_{i \in I} \alpha_i = 1, \lambda_{T \setminus I_g(\bar{x})}^g = 0, \lambda_{I_{+0}(\bar{x})}^G = 0, \lambda_{I_{00}(\bar{x})}^G \geq 0, \lambda_{I_{00}(\bar{x})}^H \geq 0, \lambda_{I_{0+}(\bar{x})}^H = 0, \right\}.$$

**Definition 18.** Let  $\bar{x} \in \Omega$ .

(i)  $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$  is said to be a locally efficient solution to  $(D_{MW}(\bar{x}))$ , denoted by  $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in locE(D_{MW}(\bar{x}))$ , if there exists  $U \in \mathcal{N}(\bar{u})$  such that there is no  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x}) \cap U$  fulfilling

$$\tilde{L}(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \preceq \tilde{L}(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

(ii)  $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$  is called a locally weakly efficient solution to  $(D_{MW}(\bar{x}))$ , denoted by  $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in locWE(D_{MW}(\bar{x}))$ , if there exists  $U \in \mathcal{N}(\bar{u})$  such that there is no  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x}) \cap U$  satisfying

$$\tilde{L}(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \prec \tilde{L}(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

If  $U = \mathbb{R}^n$ , the word “locally” is dropped.

**Remark 19.** When  $m = 1$  and  $f_1, g_t(t \in T), h_i(i = 1, \dots, q)$  and  $G_i, H_i(i = 1, \dots, l)$  are continuously differentiable functions,  $D_{MW}(\bar{x})$  becomes the Mond-Weir type dual model MWDSIMEC( $\bar{x}$ ) in [20].

**Proposition 20.** (weak duality) Let  $x \in \Omega$  and  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$ . Suppose that  $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$  and  $g_t(t \in T), h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x})), -G_i(i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})), -H_i(i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{+0}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$  are quasiconvex at  $u$ .

(i) If  $f_i(i \in I)$  are pseudoconvex at  $u$ , then

$$f(x) \not\prec \tilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

(ii) If  $f_i(i \in I)$  are strictly pseudoconvex at  $u$ , then

$$f(x) \not\prec \tilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

*Proof.* For  $x \in \Omega$  and  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$ , we have

$$g_t(x) \leq 0(t \in T), h_i(x) = 0(i \in I_h), G_i(x) \geq 0, H_i(x) \geq 0, G_i(x)H_i(x) = 0(i \in I_l), \tag{22}$$

$$\sum_{i \in I} \nabla f_i(u) + \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u) = 0, \tag{23}$$

and

$$g_t(u) \geq 0(t \in I_g(\bar{x})), h_i(u) = 0(i \in I_h),$$

$$G_i(u) \geq 0(i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})), H_i(u) \geq 0(i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x})) \tag{24}$$

with  $\sum_{i \in I} \alpha_i = 1, \lambda_{T \setminus I_g(\bar{x})}^g = 0, \lambda_{I_{+0}(\bar{x})}^G = 0, \lambda_{I_{00}(\bar{x})}^G \geq 0, \lambda_{I_{00}(\bar{x})}^H \geq 0, \lambda_{I_{+0}(\bar{x})}^H = 0$ .

It follows from the above inequalities that

$$\begin{aligned} g_t(x) &\leq 0 \leq g_t(u), \forall t \in I_g(\bar{x}), \\ h_i(x) &= h_i(u) = 0, \forall i \in I_h^+(\bar{x}) \cup I_h^-(\bar{x}), \\ -G_i(x) &\leq 0 \leq -G_i(u), \forall i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}), \\ -H_i(x) &\leq 0 \leq -H_i(u), \forall i \in \hat{I}_{00}^+(\bar{x}) \cup \hat{I}_{+0}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}). \end{aligned}$$

Therefore, we deduce from the quasiconvexity of  $g_t(t \in T)$ ,  $h_i(i \in I_h^+(\bar{x}))$ ,  $-h_i(i \in I_h^-(\bar{x}))$ ,  $-G_i(i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$ ,  $-H_i(i \in \hat{I}_{00}^+(\bar{x}) \cup \hat{I}_{+0}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$  at  $u$  and the definitions of the index sets that

$$\begin{aligned} \langle \nabla g_i(u), x - u \rangle &\leq 0, \lambda_i^g \geq 0, \forall i \in I_g(\bar{x}), \\ \langle \nabla h_i(u), x - u \rangle &\leq 0, \lambda_i^h > 0, \forall i \in I_h^+(\bar{x}), \\ \langle -\nabla h_i(u), x - u \rangle &\leq 0, \lambda_i^h < 0, \forall i \in I_h^-(\bar{x}), \\ \langle -\nabla G_i(u), x - u \rangle &\leq 0, \lambda_i^G > 0, \forall i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}), \\ \langle -\nabla H_i(u), x - u \rangle &\leq 0, \lambda_i^H > 0, \forall i \in \hat{I}_{00}^+(\bar{x}) \cup \hat{I}_{+0}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}). \end{aligned}$$

It follows from the above inequalities,  $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$ ,  $\lambda_{T \setminus I_g(\bar{x})}^g = 0$  and (23) that

$$\begin{aligned} &\left\langle \sum_{i \in I} \alpha_i \nabla f_i(u), x - u \right\rangle \\ &= - \left\langle \sum_{t \in I_g(\bar{x})} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u), x - u \right\rangle \\ &= - \left\langle \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u), x - u \right\rangle \geq 0. \end{aligned} \tag{25}$$

(i) Suppose by contradiction that

$$f(x) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

equivalently,

$$f_i(x) < f_i(u), \forall i \in I.$$

The above inequalities and the pseudoconvexity of  $f_i(i \in I)$  at  $u$  tell us that

$$\langle \nabla f_i(u), x - u \rangle < 0, \forall i \in I,$$

which, along with  $\sum_{i \in I} \alpha_i = 1$ , lead to

$$\left\langle \sum_{i=1}^m \alpha_i \nabla f_i(u), x - u \right\rangle < 0,$$

contradicting with (25).

(ii) Assume by contradiction that

$$f(x) \preceq L(y, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

This is equivalent to saying that

$$\begin{cases} f_i(x) \leq f_i(u), & \forall i \in I, \\ f_{i_0}(x) < f_{i_0}(u), & \text{for at least one } i_0 \in I, \end{cases}$$

which imply  $x \neq u$ . Granting this, we can deduce from the strictly pseudoconvexity of  $f_i (i \in I)$  at  $u$  that

$$\langle \nabla f_i(u), x - u \rangle < 0, \forall i \in I.$$

This, taking into account  $\sum_{i \in I} \alpha_i = 1$ , yields

$$\left\langle \sum_{i=1}^m \alpha_i \nabla f_i(u), x - u \right\rangle < 0,$$

contradicting with (25).  $\square$

**Proposition 21.** (strong duality) Let  $\bar{x} \in \Omega$  be a local weakly efficient solution to (P). If (ACQ) holds at  $\bar{x}$  and the set  $\Delta$  is closed, then there exist  $(\bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}_+^m \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$  with  $\sum_{i \in I} \bar{\alpha}_i = 1, \lambda_{T \setminus I_g(\bar{x})}^g = 0, \bar{\lambda}_{I_{+0}(\bar{x})}^G = 0, \bar{\lambda}_{I_{00}(\bar{x})}^G \geq 0, \bar{\lambda}_{I_{00}(\bar{x})}^H \geq 0$  and  $\bar{\lambda}_{I_{0+}(\bar{x})}^H = 0$  such that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$ . Assume further that  $I_{0+}(\bar{x}) \cup \hat{I}_{+0}(\bar{x}) = \emptyset$  and  $g_t (t \in T), h_i (i \in I_h^+(\bar{x})), -h_i (i \in I_h^-(\bar{x})), -G_i (i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})), -H_i (i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$  are quasi-convex at  $\bar{x}$ .

- (i) If  $f_i (i \in I)$  is pseudoconvex at  $\bar{x}$ , then  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is a weakly efficient solution to  $D_{MW}(\bar{x})$ .
- (ii) If  $f_i (i \in I)$  is strictly pseudoconvex at  $\bar{x}$ , then  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is an efficient solution to  $D_{MW}(\bar{x})$ .

*Proof.* By invoking Proposition 10, there exist  $(\bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}_+^m \times \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$  with  $\sum_{i \in I} \bar{\alpha}_i = 1, \lambda_{I_{+0}(\bar{x})}^G = 0, \bar{\lambda}_{I_{00}(\bar{x})}^G \geq 0, \bar{\lambda}_{I_{00}(\bar{x})}^H \geq 0$  and  $\bar{\lambda}_{I_{0+}(\bar{x})}^H = 0$  such that

$$\sum_{i \in I} \nabla f_i(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \bar{\lambda}_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_l} \bar{\lambda}_i^G \nabla G_i(\bar{x}) - \sum_{i \in I_l} \bar{\lambda}_i^H \nabla H_i(\bar{x}) = 0.$$

Since  $\bar{x} \in \Omega$  and  $\bar{\lambda}^g \in \Lambda(\bar{x})$ , one has  $\lambda_t g_t(\bar{x}) = 0$  and  $g_t(\bar{x}) \leq 0$  for all  $t \in T$ . Hence,  $g_t(\bar{x}) = 0$  for all  $t \in I_g(\bar{x})$  and  $g_t(\bar{x}) < 0$  for all  $t \in T \setminus I_g(\bar{x})$ , which in turn implies that  $\bar{\lambda}_{T \setminus I_g(\bar{x})}^g = 0$ . Again, the fact that  $\bar{x} \in \Omega$  guarantees that  $h_i(\bar{x}) = 0, \forall i \in I_h(\bar{x})$ . In addition, we get from  $G_i(\bar{x}) = 0$  for all  $i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})$  that  $G_i(\bar{x}) \geq 0$  for all  $i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})$ . Similarly, we have  $H_i(\bar{x}) \geq 0$  for all  $i \in I_{+0}(\bar{x}) \cup I_{00}(\bar{x})$ . Thus,  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$  and  $f(\bar{x}) = \tilde{L}(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ .

(i) Arguing by contradiction, suppose that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is not a weakly efficient solution to  $D_{MW}(\bar{x})$ . By denotation, there exists  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$  such that

$$f(\bar{x}) = \tilde{L}(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \prec \tilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

which contradicts with Proposition 20 (i), and thus, completes the proof.

(ii) Suppose to the contrary that  $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$  is not an efficient solution to  $D_{MW}(\bar{x})$ . In other words, there exists  $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$  such that

$$f(\bar{x}) = \tilde{L}(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \preceq \tilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

which contradicts with Proposition 20 (ii). So, we arrive at the conclusion.  $\square$

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