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NEW COMPLEXITY ANALYSIS OF FULL NESTEROV-TODD STEP INFEASIBLE INTERIOR POINT METHOD FOR SECOND-ORDER CONE OPTIMIZATION

Behrouz KHEIRFAM
Department of Applied Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran
b.kheirfam@azaruniv.edu

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Abstract: We present a full Nesterov-Todd (NT) step infeasible interior-point algorithm for second-order cone optimization based on a different way to calculate feasibility direction. In each iteration of the algorithm we use the largest possible barrier parameter value θ. Moreover, each main iteration of the algorithm consists of a feasibility step and a few centering steps. The feasibility step differs from the feasibility step of the other existing methods. We derive the complexity bound which coincides with the best known bound for infeasible interior point methods.

Keywords: Second-order Cone Optimization, Infeasible Interior-point Method, Full Nesterov-Todd Atep, Polynomial Complexity.

MSC: 90C51.

1. INTRODUCTION

A second order cone optimization (SOCO) problem is a linear optimization problem over a cross product of second order convex cones. The second order cone in $\mathbf{R}^n$ is given by

$$\mathcal{L}^n := \{x \in \mathbf{R}^n : x_1^2 \geq \sum_{i=2}^{n} x_i^2, x_1 \geq 0\}.$$

We consider the following standard primal and dual SOCO problems

$$\min\{c^T x: \ Ax = b, \ x \in \mathcal{K}\}, \qquad (P)$$

$$\max\{b^T y: \ A^T y + s = c, \ s \in \mathcal{K}\}, \qquad (D)$$

where $\mathcal{K} \subseteq \mathbf{R}^n$ is the Cartesian product of several second-order cones, i.e.,

$$\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2 \times \cdots \times \mathcal{K}^N,$$

with $\mathcal{K}^j = \mathcal{L}^{n_j}$ for each $j(j = 1, 2, \ldots, N)$ and $n = \sum_{j=1}^N n_j$. Furthermore, $x = (x^1; x^2; \ldots; x^N), s = (s^1; s^2; \ldots; s^N)$ with $x^j, s^j \in \mathcal{K}^j$, $c = (c^1; c^2; \ldots; c^N)$ with $c^j \in \mathbf{R}^{n_j}$, $A = (A^1; A^2; \ldots; A^N)$ with $A^j \in \mathbf{R}^{m \times n_j}$ and $b \in \mathbf{R}^m$. Without loss of generality, we assume that the matrix A has full row rank, i.e., $\text{rank}(A) = m$.

In recent years, there have been extensive investigations concerning the analysis of interior-point methods (IPMs) for SOCO. Nesterov and Todd [13] considered linear cone optimization problems in which the cone is self-scaled. It has become clear later that self-scaled cones are precisely the symmetric cones. Adler and Alizadeh [1] studied the relationship between semidefinite optimization (SDO) and SOCO problems and presented a unified approach to these problems. Schmieta and Alizadeh [16] extended the analysis of the Monteiro-Zhang family of interior point algorithms from SDO to all symmetric cones using Jordan algebraic techniques. Wang and Bai [18] proposed a full Nesterov-Todd (NT)-step primal-dual path-following interior-point algorithm for SOCO based on Darvay's technique [3].

The methods mentioned above are feasible IPMs, which start with a strictly feasible interior point. All the points generated by feasible IPMs are also strictly feasible. In practice, it is sometimes difficult to obtain an initial strictly feasible point. Infeasible IPMs (IIPMs) do not require that the starting point is feasible, but only that it is in the interior of the cone. The first IIPMs were proposed by Lustig [10]. Global convergence was shown by Kojima et al. [8], whereas Zhang [21] and Mizuno [12] presented polynomial iteration complexity results for variants of this algorithm. For studying more details about IIPMs one can refer to [19]. In 2006, Roos [15] proposed an IIPM to solve LO problem. This method use only full steps (instead of damped steps) unlike the classical IIPMs [8, 12, 21]. Kheirfam [5, 6] presented variants of this algorithm for SDO. Kheirfam and Mahdavi-Amiri [7] generalized it to full NT-step IIPM for linear complementarity problem over symmetric cone (SCLCP). Gu et al. extended it to SOCO [20].

In this paper, we use another definition for the feasibility step, and present a full NT-step IIPM for SOCO. We analyze our algorithm and prove that the iteration bound coincides with the best known bound for IIPMs, providing an interesting analysis along the way.

The rest of the paper is organized as follows: In Section 2 we briefly review some properties of the second-order cone and its associated Euclidean Jordan algebra. In Section 3 we firstly present our algorithm, and then give some results which show that the NT-process converges quadratically. Section 4 is devoted to the analysis of the feasibility step, which is the main part of this paper. The analysis presented in this section differs from the analysis of the existing methods. We derive the complexity bound for algorithm, which coincides with the best known bound for IIPMs. Finally, we end the paper with some concluding remarks in Section 5.

2. PRELIMINARIES

In this section, we briefly state some properties of second order cones and the associated Jordan algebra [17, 18].

Let $\mathbf{R}^n$ denote the set of vectors with n components. For any two vectors $x^j, s^j \in \mathbf{R}^{n_j}$, the bilinear map $\circ$ is defined by

$$x^j \circ s^j = \Big((x^j)^T s^j; x_1^j s_2^j + s_1^j x_2^j; \ldots; x_1^j s_{n_j}^j + s_1^j x_{n_j}^j\Big), \; j = 1, 2, \ldots, N.$$

One can easily verify that $(\mathbf{R}^{n_j}, \circ)$, for each $j = 1, 2, \ldots, N$, is a Euclidean Jordan algebra with $e^j = (1; 0; \ldots; 0)$, $j = 1, 2, \ldots, N$ as an identity element. The matrix of the linear map $s^j \to x^j \circ s^j$ for each $j = 1, 2, \ldots, N$, with respect to the standard basis is a symmetric matrix as

$$L(x^j) := \begin{bmatrix} x_1^j & (x_{2:n_j}^j)^T \\ x_{2:n_j}^j & x_1^j E_{n_j - 1} \end{bmatrix},$$

where $x_{2:n_j}^j = (x_2^j; \ldots; x_{n_j}^j)$ and E_{n_j-1} denotes the $(n_j - 1) \times (n_j - 1)$ identity matrix. For $x^j \in \mathbf{R}^{n_j}$, $j = 1, 2, \ldots, N$, define

$$P(x^j) := 2L(x^j)^2 - L((x^j)^2),$$

where $L(x^j)^2 = L(x^j)L(x^j)$. The map $P(x^j)$ is called the quadratic representation of $\mathbf{R}^{n_j}$. We define the algebra $(\mathbf{R}^n, \circ)$ as a direct product of the Euclidean Jordan algebras $(\mathbf{R}^{n_j}, \circ)$ by

$$x \circ s = (x^1 \circ s^1; x^2 \circ s^2; \ldots; x^N \circ s^N) \tag{1}$$

and the identity element $e = (e^1; e^2; \ldots; e^N)$. Then, the matrices $L(x)$ and $P(x)$ of $\mathbf{R}^n$ can be adjusted to

$$L(x) = \mathrm{diag}\Big(L(x^1), L(x^2), \ldots, L(x^N)\Big), \;\; P(x) = \mathrm{diag}\Big(P(x^1), P(x^2), \ldots, P(x^N)\Big).$$

Let $\lambda_{\max}(x^j)$ and $\lambda_{\min}(x^j)$ denote the maximal and minimal eigenvalues of $L(x^j)$, $j = 1, 2, \ldots, N$, respectively, namely,

$$\lambda_{\max}(x^j) := x_1^j + \|x_{2:n_j}^j\|, \lambda_{\min}(x^j) := x_1^j - \|x_{2:n_j}^j\|.$$

Then we have

$$\lambda_{\max}(x) = \max\{\lambda_{\max}(x^j) : 1 \le j \le N\}, \;\; \lambda_{\min}(x) = \min\{\lambda_{\min}(x^j) : 1 \le j \le N\}.$$

It readily follows that

$$x \in \mathcal{K} \Leftrightarrow \lambda_{\min}(x) \geq 0, \text{ and } x \in \operatorname{int}\mathcal{K} \Leftrightarrow \lambda_{\min}(x) > 0,$$

where $\operatorname{int}\mathcal{K}$ denotes the interior of $\mathcal{K}$. Furthermore,

$$\operatorname{tr}(x) = \sum_{j=1}^{N} \operatorname{tr}(x^j) = \sum_{j=1}^{N} \left(\lambda_{\max}(x^j) + \lambda_{\min}(x^j)\right) := \sum_{j=1}^{2N} \lambda_j(x),$$

and

$$\|x\|_F = \sqrt{\sum_{j=1}^{N} \|x^j\|_F^2}, \ \det(x) = \prod_{j=1}^{N} \det(x^j),$$

where $\|x^j\|_F = \sqrt{tr(x^j \circ x^j)} = \sqrt{\lambda_{\max}(x^j)^2 + \lambda_{\min}(x^j)^2}$ and $\det(x^j) = \lambda_{\max}(x^j)\lambda_{\min}(x^j)$, for $j = 1, 2, \ldots, N$.

Lemma 2.1. (Lemma 14 in [2]) *Let $x \in \operatorname{int}\mathcal{K}$ and z is any vector in R^n such that $x + z \succ_{\mathcal{K}} 0$. Then*

$$\sum_{j=1}^{2N} \frac{1}{\lambda_j(x+z)} \leq \sum_{j=1}^{2N} \frac{1}{\lambda_j(x) + \lambda_j(z)}.$$

Proposition 2.2. (Proposition 2.2 in [9]) *For each $x \in \operatorname{int}\mathcal{K}$, $P(x)$ is an automorphism of $P(x)\operatorname{int}\mathcal{K} = \operatorname{int}\mathcal{K}$. Furthermore, $P(x)$ is positive semidefinite (positive definite) for each $x \in \mathcal{K}(\operatorname{int}\mathcal{K})$.*

Lemma 2.3. (Lemma 3.2 in [4]) *Let $x, s \in \operatorname{int}\mathcal{K}$. Then, there exists a unique $w \in \operatorname{int}\mathcal{K}$ such that $x = P(w)s$. Moreover,*

$$w = P(x^{\frac{1}{2}})\left(P(x^{\frac{1}{2}})s\right)^{-\frac{1}{2}}\left[= P(s^{-\frac{1}{2}})\left(P(s^{\frac{1}{2}})x\right)^{\frac{1}{2}}\right].$$

Due to Proposition 2.2, $P(w)$ is an automorphism. The point w is the so-called NT-scaling point of x and s. The definition of the trace function, together with (1), implies that, for any $x, s \in \mathbf{R}^n$,

$$\langle x, s\rangle := tr(x \circ s) = 2x^T s.$$

In what follows, we list some results which will be needed latter on, when dealing with the analysis of our algorithm.

Lemma 2.4. (Lemma 30 in [16]) *Let $x, s \in \operatorname{int}\mathcal{K}$. Then*

$$\|P(x)^{\frac{1}{2}}s - e\|_F \leq \|x \circ s - e\|_F.$$

Lemma 2.5. (Proposition 3.2.4 in [17]) *Let $x, s \in \operatorname{int}\mathcal{K}$. If w is the scaling point of x and s, then*

$$\left(P(x^{\frac{1}{2}})s\right)^{\frac{1}{2}} \sim P(w^{\frac{1}{2}})s.$$

Lemma 2.6. (Proposition 21 in [16]) *Let $x, s, u \in \mathrm{int}\mathcal{K}$. Then*
(*i*) $P(x^{\frac{1}{2}})s \sim P(s^{\frac{1}{2}})x$.
(*ii*) $P\Big((P(u)x)^{\frac{1}{2}}\Big)P(u^{-1})s \sim P(x^{\frac{1}{2}})s$.

Lemma 2.7. (Lemma 2.6 in [20]) *Let $x, s \in \mathrm{int}\mathcal{K}, u = P(x)^{\frac{1}{2}}s$ and $z = x \circ s \in \mathrm{int}\mathcal{K}$. Then we have*

$$\left\|u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right\|_F \leq \left\|z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right\|_F.$$

Lemma 2.8. (Lemma 6.1 in [18]) *Let $x(\alpha) := x + \alpha\Delta x$ and $s(\alpha) := s + \alpha\Delta s$ for $0 \leq \alpha \leq 1$, and $x, s \in \mathrm{int}\mathcal{K}$. If one has*

$$\det(x(\alpha) \circ s(\alpha)) > 0, \;\; \forall\, \alpha : \; 0 \leq \alpha \leq \bar{\alpha},$$

then $x(\bar{\alpha})$ and $s(\bar{\alpha}) \in \mathrm{int}\mathcal{K}$.

3. THE STATEMENT of THE ALGORITHM

We assume that (P) and (D) have an optimal solution (x^*, y^*, s^*) such that $\mathrm{tr}(x^* \circ s^*) = 0$. We start usually with assuming that the initial iterates x^0, y^0 and s^0 are

$$x^0 = s^0 = \xi e, \;\; y^0 = 0, \;\; \mu^0 = \xi^2,$$

where μ^0 is the initial duality gap and ξ is such that

$$x^* + s^* \preceq_{\mathcal{K}} \xi e.$$

3.1. *The Feasible SOCO Problem*

The perturbed optimality conditions for (P) and (D) are

$$\begin{aligned} Ax = b, &\quad x \in \mathcal{K}, \\ A^T y + s = c, &\quad s \in \mathcal{K}, \\ x \circ s = \mu e, & \end{aligned} \tag{2}$$

where $\mu > 0$ is a parameter. The natural way to define a search direction is to follow the Newton approach and to linearize the third equation in (2), which leads to the system

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ x \circ \Delta s + s \circ \Delta x &= \mu e - x \circ s. \end{aligned} \tag{3}$$

Due to the fact that x and s do not operator commute in general, i.e., $L(x)L(s) \neq L(s)L(x)$, this system does not always have a unique solution. This difficulty can be solved by applying a scaling scheme. It goes as follows. Let $x, s, u \in \mathrm{int}\mathcal{K}$, then $x \circ s = \mu e$ if and only if $P(u)x \circ P(u^{-1})s = \mu e$ (Lemma 28 in [16]). Now replacing the third equation in (2) by $P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})s = \mu e$, where w is the NT-scaling point of x and s as defined in Lemma 2.3, and then applying Newton's method, we obtain

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ P(w^{-\frac{1}{2}})\Delta x \circ P(w^{\frac{1}{2}})s + P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})\Delta s &= \mu e - P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})s. \end{aligned} \tag{4}$$

We define

$$v := \frac{P(w^{-\frac{1}{2}})x}{\sqrt{\mu}}\Big[= \frac{P(w^{\frac{1}{2}})s}{\sqrt{\mu}}\Big], \tag{5}$$

and

$$\bar{A} := \sqrt{\mu}AP(w^{\frac{1}{2}}), \;\; d_x := \frac{P(w^{-\frac{1}{2}})\Delta x}{\sqrt{\mu}}, \;\; d_s := \frac{P(w^{\frac{1}{2}})\Delta s}{\sqrt{\mu}}. \tag{6}$$

Using the above notations, the system (4) transforms to

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\frac{\Delta y}{\mu} + d_s &= 0, \\ d_x + d_s &= v^{-1} - v. \end{aligned} \tag{7}$$

The new search directions d_x and d_s are obtained by solving (7), so Δx and Δs can be computed via (6). Due to the first two equations of (7), d_x belongs to the null space of $\bar{A}$ and d_s belongs to the row space of $\bar{A}$, hence we conclude that $d_x^T d_s = 0$, i.e., d_x and d_s are orthogonal. Thus, from the third equation of (7) and the orthogonality of d_x and d_s, we obtain

$$\|d_x + d_s\|_F^2 = \|d_x\|_F^2 + \|d_s\|_F^2 = \|v^{-1} - v\|_F^2.$$

This implies that d_x and d_s are both zero if and only if $v^{-1} - v = 0$. In this case, x and s satisfy $x \circ s = \mu e$, implying that x and s are the μ-centers [16]. Hence, we can use the norm $\|v^{-1} - v\|_F$ as a quantity to measure closeness to the μ-centers. Let us define

$$\delta(x, s; \mu) \equiv \delta(v) := \frac{1}{2}\|v^{-1} - v\|_F. \tag{8}$$

3.2. *The Perturbed Problem*

For any ν with $0 < \nu \leq 1$, we consider the perturbed problem

$$\min \Big\{(c - \nu r_c^0)^T x : \; b - Ax = \nu r_b^0, \; x \in \mathcal{K}\Big\}, \qquad (P_\nu)$$

and its dual problem

$$\max \Big\{(b - \nu r_b^0)^T y : \; c - A^T y - s = \nu r_c^0, \; s \in \mathcal{K}\Big\}, \qquad (D_\nu)$$

where

$$r_b^0 := b - Ax^0, \quad r_c^0 := c - A^T y^0 - s^0.$$

Note that if $\nu = 1$, then $x = x^0$ and $(y, s) = (y^0, s^0)$ yield strictly feasible solutions of (P_ν) and (D_ν), respectively.

Lemma 3.1. (Lemma 4.1 in [20]) *Let the original problems,* (P) *and* (D)*, be feasible. Then for each* ν *with* $0 < \nu \le 1$*, the perturbed problems* (P_ν) *and* (D_ν) *are strictly feasible.*

Assuming that (P) and (D) are both feasible, it follows from Lemma 3.1 that the problems (P_ν) and (D_ν) are strictly feasible, for each $\nu \in (0, 1]$. This means that the following system

$$\begin{aligned} b - Ax &= \nu r_b^0, \quad x \in \mathcal{K} \\ c - A^T y - s &= \nu r_c^0, \quad s \in \mathcal{K} \\ x \circ s &= \mu e, \end{aligned} \tag{9}$$

has a unique solution, for any $\mu > 0$. For $\nu \in (0, 1]$ and $\mu = \nu\mu^0 = \nu\xi^2$, we denote this unique solution as $(x(\mu, \nu), y(\mu, \nu), s(\mu, \nu))$, where $x(\mu, \nu)$ is the μ-center of (P_ν) and $(y(\mu, \nu), s(\mu, \nu))$ is the μ-center of (D_ν). Due to the fact that the parameters μ and ν will always be in a one-to-one correspondence, according to $\mu = \nu\mu^0 = \nu\xi^2$. For the sake of simplicity, we denote $x(\mu) = x(\mu, \nu)$, $y(\mu) = y(\mu, \nu)$ and $s(\mu) = s(\mu, \nu)$. By taking $\nu = 1$, one has $(x(1), y(1), s(1)) = (x^0, y^0, s^0) = (\xi e, 0, \xi e)$ and $x^0 \circ s^0 = \mu^0 e$. Hence, we initially have $\delta(x, s; \mu) = 0$. In the sequel, we assume that at the start of each iteration, just before the μ-and ν-update, $\delta(x, s; \mu) \le \tau$, where τ is a positive threshold value. This certainly holds at the start of the first iteration.

Now, we describe one main iteration of our algorithm. The algorithm begins with an infeasible interior-point (x, y, s) such that (x, y, s) is feasible for the perturbed problems (P_ν) and (D_ν), with $\mu = \nu\mu^0$ and such that $x^T s = N\mu$ and $\delta(x, s; \mu) \le \tau$. We reduce ν to $\nu^+ = (1 - \theta)\nu$, with $\theta \in (0, 1)$, and find new iterate (x^+, y^+, s^+) that is feasible for the perturbed problems (P_{ν^+}) and (D_{ν^+}), and such that $(x^+)^T s^+ = N\mu$ and $\delta(x^+, s^+; \mu^+) \le \tau$. Every iteration consists of a feasibility step and a few centering steps. The feasibility step serves to get iterates (x^f, y^f, s^f) that are strictly feasible for the perturbed problems with $\nu^+ := (1 - \theta)\nu$, and close to their μ-centers such that $\delta(x^f, s^f; \mu^+) \le \frac{1}{\sqrt[4]{2}}$. Since the triple (x^f, y^f, s^f) is strictly feasible for (P_{ν^+}) and (D_{ν^+}), we perform a few centering steps starting at (x^f, y^f, s^f), targeting at the μ^+-centers of (P_{ν^+}) and (D_{ν^+}), and obtain iterates (x^+, y^+, s^+) that are feasible for (P_{ν^+}) and (D_{ν^+}) such that $\delta(x^+, s^+; \mu^+) \le \tau$. This process is repeated until the algorithm terminates. For the feasibility step in [20] search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$ are defined by the system

$$\begin{aligned} A\Delta^f x &= \theta\nu r_b^0 \\ A^T\Delta^f y + \Delta^f s &= \theta\nu r_c^0 \\ P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})\Delta^f s + P(w^{\frac{1}{2}})s \circ P(w^{-\frac{1}{2}})\Delta^f x &= \\ & \mu e - P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})s. \end{aligned} \tag{10}$$

After the feasibility step, the iterates are given by

$$x^f := x + \Delta^f x, \quad y^f := y + \Delta^f y, \quad s^f := s + \Delta^f s, \tag{11}$$

It can be easily understood that if (x, y, s) is feasible for the perturbed problems (P_ν) and (D_ν), then after the feasibility step, the iterates satisfy the feasibility conditions for (P_{ν^+}) and (D_{ν^+}). Assume that before the step $\delta(x, s; \mu) \le \tau = \frac{1}{16}$

holds. Here, we want to investigate how large θ can be so that it guarantees that after the feasibility step the iterates x^f, y^f and s^f are nonnegative and moreover, $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt[4]{2}}$, where $\mu^+ = (1-\theta)\mu$. In a centering step, the search directions $\Delta x, \Delta y$ and Δs are the usual primal-dual NT directions defined by the system (4). Denoting the iterates after a centering step as x^+, y^+ and s^+, we recall from [20] the following results.

Lemma 3.2. *If $\delta := \delta(v) < 1$, then the full NT-step is strictly feasible, and $(x^+)^T s^+ = N\mu$ and*

$$\delta(x^+, s^+; \mu) \leq \frac{\delta^2}{\sqrt{2(1-\delta^4)}}.$$

Corollary 3.3. *If $\delta(v) \leq \frac{1}{\sqrt[4]{2}}$, then $\delta(x^+, s^+; \mu) \leq \delta^2$, showing that the iterates are in the neighborhood of the quadratic convergence of the proximity measure of the iterates.*

4. NEW FEASIBILITY STEP

In this paper, we use another definition for the feasibility step by replacing the third equation of (10) by the equation

$$P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})\Delta^f s + P(w^{\frac{1}{2}})s \circ P(w^{-\frac{1}{2}})\Delta^f x = -\theta P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})s. \tag{12}$$

4.1. Analysis of the New Feasibility Step

The important and hard part of the analysis is to prove quadratic convergence property of feasibility step. The feasibility step generates new iterates x^f, y^f and s^f that satisfy the feasibility conditions for (P_{ν^+}) and (D_{ν^+}). A crucial element in the analysis is to find the largest value θ so that the iterates (x^f, y^f, s^f) satisfy $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt[4]{2}}$, i.e., that the new iterates are within the region where the Newton process targeting at the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent. After the feasibility step, we perform centering steps in order to get iterates (x^+, y^+, s^+) that satisfy $(x^+)^T s^+ = N\mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$, where $\tau \geq 0$. A more formal description of the algorithm is given as follows:

Algorithm 1 : *A full − Newton step IIPM*

Input :
accuracy parameter $\epsilon > 0$;
barraier update parameter $\theta,\ 0 < \theta < 1$;
and threshold parameter $\tau > 0$;
parameter $\xi > 0$.
begin
$x := \xi e;\ y := 0;\ s := \xi e;\ \mu := \mu^0 = \xi^2;\ \nu = 1$;
while $\max\left(x^T s, \|b - Ax\|, \|c - A^T y - s\|\right) > \epsilon$ **do**
feasibility step :
solve (13) and update
$(x, s, y) := (x, s, y) + (\Delta^f x, \Delta^f s, \Delta^f y)$;
μ − update :
Determine the largest value θ such that
$\left(4N\theta\rho(\delta)\right)^2 + \left(4N\theta\rho(\delta) + \sqrt{2N}\theta\right)^2 \leq 1.166(1-\theta)$;
where $\rho(\delta) = \delta + \sqrt{\delta^2 + 1}$.
$\mu := (1-\theta)\mu$;
centering step :
while $\delta(x, s; \mu) \geq \tau$ **do**
solve (4) and update
$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$
end while
end while
end.

Now, let us replace the third equation in system (10) by (12) which implies the following system:

$$\begin{aligned} A\Delta^f x &= \theta\nu r_b^0 \\ A^T\Delta^f y + \Delta^f s &= \theta\nu r_c^0 \\ P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})\Delta^f s + P(w^{\frac{1}{2}})s \circ P(w^{-\frac{1}{2}})\Delta^f x &= -\theta P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})s. \end{aligned} \tag{13}$$

Let us define

$$d_x^f := \frac{1}{\sqrt{\mu^+}} P(w^{-\frac{1}{2}})\Delta^f x, \quad d_s^f := \frac{1}{\sqrt{\mu^+}} P(w^{\frac{1}{2}})\Delta^f s, \tag{14}$$

where w is the NT-scaling point of x and s. Using (5), (11), and (14), we may write

$$\begin{aligned} x^f &= x + \Delta^f x = \sqrt{\mu}P(w^{\frac{1}{2}})(v + \sqrt{1-\theta}d_x^f), \\ s^f &= s + \Delta^f s = \sqrt{\mu}P(w^{-\frac{1}{2}})(v + \sqrt{1-\theta}d_s^f). \end{aligned} \tag{15}$$

By using the third equation of system (13), we derive that

$$\begin{aligned}(v+\sqrt{1-\theta}d_x^f)\circ(v+\sqrt{1-\theta}d_s^f) &= v^2+\sqrt{1-\theta}v\circ(d_x^f+d_s^f)+(1-\theta)d_x^f\circ d_s^f\\ &= (1-\theta)(v^2+d_x^f\circ d_s^f).\end{aligned} \tag{16}$$

Since $P(w^{\frac{1}{2}})$ and $P(w^{-\frac{1}{2}})$ are automorphisms of $\mathrm{int}\mathcal{K}$, x^f and s^f will belong to $\mathrm{int}\mathcal{K}$ if and only if $v+\sqrt{1-\theta}d_x^f$ and $v+\sqrt{1-\theta}d_s^f$ belong to $\mathrm{int}\mathcal{K}$.

Lemma 4.1. *The iterates (x^f, y^f, s^f) are strictly feasible if $v^2+d_x^f\circ d_s^f\in\mathrm{int}\mathcal{K}$.*

Proof. Introduce a step length α with $\alpha\in[0,1]$ and define

$$v_x(\alpha)=v+\alpha\sqrt{1-\theta}d_x^f,\quad v_s(\alpha)=v+\alpha\sqrt{1-\theta}d_s^f.$$

Therefore, by the third equation of system (13), we have

$$\begin{aligned}v_x(\alpha)\circ v_s(\alpha) &= (v+\alpha\sqrt{1-\theta}d_x^f)\circ(v+\alpha\sqrt{1-\theta}d_s^f)\\ &= v^2+\alpha\sqrt{1-\theta}v\circ(d_x^f+d_s^f)+\alpha^2(1-\theta)d_x^f\circ d_s^f\\ &= (1-\alpha\theta)v^2+\alpha^2(1-\theta)d_x^f\circ d_s^f.\end{aligned}$$

If $v^2+d_x^f\circ d_s^f\in\mathrm{int}\mathcal{K}$, then we have $d_x^f\circ d_s^f\succ_{\mathcal{K}}-v^2$. Substituting this into the above relation, we get

$$v_x(\alpha)\circ v_s(\alpha)\succ_{\mathcal{K}}(1-\alpha)(1+\alpha-\alpha\theta)v^2.$$

If $0\le\alpha\le 1$, then we have $v_x(\alpha)\circ v_s(\alpha)\succ_{\mathcal{K}}0$. Thus $\det(v_x(\alpha)\circ v_s(\alpha))>0$. Lemma 2.8 implies that $v_x(1)=v+\sqrt{1-\theta}d_x^f\in\mathrm{int}\mathcal{K}$ and $v_s(1)=v+\sqrt{1-\theta}d_s^f\in\mathrm{int}\mathcal{K}$. This proves the lemma. □

For the proof of our main result in this subsection, which is Lemma 4.5, we need the following two lemmas.

Lemma 4.2. (See Lemma II.60 in [14]) *Let $\rho(\delta):=\delta+\sqrt{1+\delta^2}$. Then*

$$\frac{1}{\rho(\delta)}\le\lambda_i(v)\le\rho(\delta),\ i=1,2,\cdots,2N.$$

Lemma 4.3. (Lemma A.1 in [11]) *For $i=1,2,\dots,n$, let $f_i:\mathbb{R}_+\to\mathbb{R}$ denote a convex function. Then, for any nonzero vector $z\in\mathbb{R}_+^n$, the following inequality*

$$\sum_{i=1}^n f_i(z_i)\le\frac{1}{\mathbf{1}^Tz}\sum_{j=1}^n z_j\left(f_j(\mathbf{1}^Tz)+\sum_{i\neq j}f_i(0)\right)$$

holds, where **1** *is the all-one vector.*

In the sequel, we denote

$$\omega := \frac{1}{2}\sqrt{\|d_x^f\|_F^2 + \|d_s^f\|_F^2},$$

which implies $\|d_x^f\|_F \leq 2\omega$ and $\|d_s^f\|_F \leq 2\omega$. Moreover, we have

$$\|d_x^f \circ d_s^f\|_F \leq \frac{1}{2}(\|d_x^f\|_F^2 + \|d_s^f\|_F^2) = 2\omega^2, \tag{17}$$

$$|\lambda_i(d_x^f \circ d_s^f)| \leq \|d_x^f \circ d_s^f\|_F \leq 2\omega^2, \quad i = 1, 2, \ldots, 2N. \tag{18}$$

We proceed by deriving an upper bound for $\delta(x^f, s^f; \mu^+)$. Recall from definition (8) that

$$\delta(x^f, s^f; \mu^+) := \delta(v^f) = \frac{1}{2}\left\|(v^f)^{-1} - v^f\right\|_F, \tag{19}$$

where $v^f := \frac{1}{\sqrt{\mu(1-\theta)}}P(w^f)^{-\frac{1}{2}}x^f\Big[= \frac{1}{\sqrt{\mu(1-\theta)}}P(w^f)^{\frac{1}{2}}s^f\Big]$ and w^f is the NT-scaling point of x^f and s^f.

Lemma 4.4. *One has*

$$\sqrt{1-\theta}v^f \sim \Big[P(v + \sqrt{1-\theta}d_x^f)^{\frac{1}{2}}(v + \sqrt{1-\theta}d_s^f)\Big]^{\frac{1}{2}}.$$

Proof. It follows from the definition v^f and Lemma 2.5 that

$$\sqrt{\mu(1-\theta)}v^f = P(w^f)^{\frac{1}{2}}s^f \sim \Big(P(x^f)^{\frac{1}{2}}s^f\Big)^{\frac{1}{2}}.$$

Due to (15) and Lemma 2.6, with $u = w^{\frac{1}{2}}$, we may write

$$\begin{aligned} P(x^f)^{\frac{1}{2}}s^f &= \mu P\Big(P(w^{\frac{1}{2}})(v + \sqrt{1-\theta}d_x^f)\Big)^{\frac{1}{2}} P(w^{-\frac{1}{2}})(v + \sqrt{1-\theta}d_s^f) \\ &\sim \mu P(v + \sqrt{1-\theta}d_x^f)^{\frac{1}{2}}\Big(v + \sqrt{1-\theta}d_s^f\Big). \end{aligned}$$

Thus we obtain

$$\sqrt{\mu(1-\theta)}v^f \sim \sqrt{\mu}\Big[P(v + \sqrt{1-\theta}d_x^f)^{\frac{1}{2}}\Big(v + \sqrt{1-\theta}d_s^f\Big)\Big]^{\frac{1}{2}}.$$

From this the lemma follows. □

The above lemma implies that

$$(v^f)^2 \sim P\Big(\frac{v + \sqrt{1-\theta}d_x^f}{\sqrt{1-\theta}}\Big)^{\frac{1}{2}}\Big(\frac{v + \sqrt{1-\theta}d_s^f}{\sqrt{1-\theta}}\Big).$$

By Lemma 2.7, one can easily verify that

$$\begin{aligned} 4\delta(v^f)^2 = \left\|v^f - (v^f)^{-1}\right\|_F^2 &= \left\|u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right\|_F^2 \\ &\leq \left\|z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right\|_F^2, \end{aligned}$$

where

$$x = \frac{v + \sqrt{1-\theta}d_x^f}{\sqrt{1-\theta}},\ s = \frac{v + \sqrt{1-\theta}d_s^f}{\sqrt{1-\theta}},\ u = P(x)^{\frac{1}{2}}s,\ z = x \circ s.$$

Moreover, again by using (16), we derive

$$(1-\theta)z = \left(v + \sqrt{1-\theta}d_x^f\right) \circ \left(v + \sqrt{1-\theta}d_s^f\right) = (1-\theta)(v^2 + d_x^f \circ d_s^f).$$

So we have

$$4\delta(v^f)^2 \leq \left\|z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right\|_F^2 = \mathrm{tr}(z) + \mathrm{tr}(z^{-1}) - 2\mathrm{tr}(e),\ \ z = v^2 + d_x^f \circ d_s^f. \tag{20}$$

Lemma 4.5. *If* $v^2 + d_x^f \circ d_s^f \in \mathrm{int}\mathcal{K}$ *and* $1 - 2\rho(\delta)^2\omega^2 > 0$, *then*

$$4\delta(v^f)^2 \leq 4\delta(v)^2 + 2\omega^2 + \frac{2\rho(\delta)^4\omega^2}{1 - 2\rho(\delta)^2\omega^2}.$$

Proof. Using (20) and Lemma 2.1, we obtain

$$\begin{aligned}
4\delta(v^f)^2 &\leq \mathrm{tr}(v^2 + d_x^f \circ d_s^f) + \mathrm{tr}((v^2 + d_x^f \circ d_s^f)^{-1}) - 2\mathrm{tr}(e)\\
&= \sum_{i=1}^{2N} \lambda_i(v^2 + d_x^f \circ d_s^f) + \sum_{i=1}^{2N} \frac{1}{\lambda_i(v^2 + d_x^f \circ d_s^f)} - 2\mathrm{tr}(e)\\
&\leq \sum_{i=1}^{2N} \left(\lambda_i(v)^2 + \lambda_i(d_x^f \circ d_s^f)\right) + \sum_{i=1}^{2N} \frac{1}{\lambda_i(v)^2 + \lambda_i(d_x^f \circ d_s^f)} - 2\mathrm{tr}(e)\\
&\leq \sum_{i=1}^{2N} \left(\lambda_i(v)^2 + |\lambda_i(d_x^f \circ d_s^f)|\right) + \sum_{i=1}^{2N} \frac{1}{\lambda_i(v)^2 - |\lambda_i(d_x^f \circ d_s^f)|} - 2\mathrm{tr}(e)\\
&\leq \sum_{i=1}^{2N} \left(\left(\lambda_i(v)^2 + |\lambda_i(d_x^f \circ d_s^f)|\right) + \frac{1}{\lambda_i(v)^2 - |\lambda_i(d_x^f \circ d_s^f)|} - 2\right).
\end{aligned}$$

For each i, taking $y_i = |\lambda_i(d_x^f \circ d_s^f)|$, we define

$$f_i(y_i) := \lambda_i(v)^2 + y_i + \frac{1}{\lambda_i(v)^2 - y_i} - 2, \quad i = 1, 2, \ldots, 2N.$$

Using Lemma 4.2, hypothesis $1 - 2\rho(\delta)^2\omega^2 > 0$ and (18), we get

$$\lambda_i(v)^2 - y_i > 0,\ i = 1, 2, \ldots, 2N.$$

This implies that $f_i(y_i)$ is convex in y_i. Therefore, by Lemma 4.3,

$$\begin{aligned}
4\delta(v^f)^2 &\leq \sum_{j=1}^{2N} f_j(y_j) \leq \frac{1}{\mathbf{1}^T y}\sum_{j=1}^{2N} y_j\Big(f_j(\mathbf{1}^T y) + \sum_{i\neq j} f_i(0)\Big)\\
&\leq \frac{1}{\mathbf{1}^T y}\sum_{j=1}^{2N} y_j\Big[\Big(\lambda_j(v)^2 + 2\omega^2 + \frac{1}{\lambda_j(v)^2 - 2\omega^2} - 2\Big)\\
&\qquad\qquad + \sum_{i\neq j}\Big(\lambda_i(v)^2 + \frac{1}{\lambda_i(v)^2} - 2\Big)\Big].
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sum_{i\neq j}\Big(\lambda_i(v)^2 + \frac{1}{\lambda_i(v)^2} - 2\Big) &= \sum_{i=1}^{2N}\Big(\lambda_i(v)^2 + \frac{1}{\lambda_i(v)^2} - 2\Big) - \Big(\lambda_j(v)^2 + \frac{1}{\lambda_j(v)^2} - 2\Big)\\
&= \sum_{i=1}^{2N}\Big(\lambda_i(v) - \frac{1}{\lambda_i(v)}\Big)^2 - \Big(\lambda_j(v)^2 + \frac{1}{\lambda_j(v)^2} - 2\Big)\\
&= 4\delta(v)^2 - \Big(\lambda_j(v)^2 + \frac{1}{\lambda_j(v)^2} - 2\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
4\delta(v^f)^2 &\leq 4\delta(v)^2 + \frac{1}{\mathbf{1}^T y}\sum_{j=1}^{2N} y_j\Big[\Big(\lambda_j(v)^2 + 2\omega^2 + \frac{1}{\lambda_j(v)^2 - 2\omega^2} - 2\Big)\\
&\qquad\qquad - \Big(\lambda_j(v)^2 + \frac{1}{\lambda_j(v)^2} - 2\Big)\Big]\\
&= 4\delta(v)^2 + 2\omega^2 + \frac{1}{\mathbf{1}^T y}\sum_{j=1}^{2N} y_j \frac{2\omega^2}{\lambda_j(v)^2\big(\lambda_j(v)^2 - 2\omega^2\big)}\\
&\leq 4\delta(v)^2 + 2\omega^2 + \frac{2\omega^2}{\frac{1}{\rho(\delta)^2}\Big(\frac{1}{\rho(\delta)^2} - 2\omega^2\Big)}\\
&= 4\delta(v)^2 + 2\omega^2 + \frac{2\rho(\delta)^4\omega^2}{1 - 2\rho(\delta)^2\omega^2}.
\end{aligned}$$

This completes the proof. $\square$

From Section 3.2, we know that the algorithm begins with a feasible point (x, y, s) for the perturbed pair (P_ν) and (D_ν) such that $\delta(v) := \delta(x, s; \mu) \leq \tau = \frac{1}{16}$. Because we need that (x^f, y^f, s^f) lies in the quadratic convergence neighborhood with respect to the μ^+-center of the perturbed pair (P_{ν^+}) and (D_{ν^+}), i.e., $\delta(v^f) \leq \frac{1}{\sqrt[4]{2}}$, it follows from Lemma 4.5 that it suffices to have

$$4\delta(v)^2 + 2\omega^2 + \frac{2\rho(\delta)^4\omega^2}{1 - 2\rho(\delta)^2\omega^2} \leq 2\sqrt{2}. \tag{21}$$

From the fact that the left-hand side of (21) is increasing in $\delta(v)$, it follows that the inequality (21) holds if (by substituting $\delta(v)$ with its upper bound $\tau = \frac{1}{16}$)

$$2\omega^2 + \frac{2\rho(\frac{1}{16})^4\omega^2}{1 - 2\rho(\frac{1}{16})^2\omega^2} \leq \frac{128\sqrt{2} - 1}{64}. \tag{22}$$

After performing some elementary calculations, we obtain

$$\omega \leq 0.54. \tag{23}$$

4.2. An Upper Bound For ω

One can easily check that the system (13), which defines the search directions $\Delta^f x, \Delta^f y$ and $\Delta^f s$, can be expressed in terms of the scaled search directions d_x^f and d_s^f as follows:

$$\begin{aligned} \bar{A}d_x^f &= \tfrac{\theta}{\sqrt{1-\theta}}\nu r_b^0, \\ \bar{A}^T\frac{\Delta^f y}{\mu\sqrt{1-\theta}} + d_s^f &= \tfrac{\theta}{\sqrt{1-\theta}}\tfrac{\nu}{\sqrt{\mu}}P(w^{\frac{1}{2}})r_c^0, \\ d_x^f + d_s^f &= -\tfrac{\theta}{\sqrt{1-\theta}}v, \end{aligned} \tag{24}$$

where

$$\bar{A} = \sqrt{\mu}AP(w^{\frac{1}{2}}).$$

Let us denote the null space of the matrix $\bar{A}$ as $\mathcal{L}$. So,

$$\mathcal{L} := \left\{\xi \in R^n : \bar{A}\xi = 0\right\}.$$

Clearly, from the first equation in (24), the affine space

$$\left\{\xi \in R^n : \bar{A}\xi = \frac{\theta}{\sqrt{1-\theta}}\nu r_b^0\right\},$$

equals $d_x^f + \mathcal{L}$. Note that due to a well-know result from linear algebra, the row space of $\bar{A}$ equals the orthogonal complement $\mathcal{L}^\perp$ of $\mathcal{L}$, i.e.,

$$\mathcal{L}^\perp = \left\{\bar{A}^T\vartheta : \vartheta \in R^m\right\}.$$

From the second equation in (24), the affine space

$$\left\{\frac{\theta}{\sqrt{1-\theta}}\frac{\nu}{\sqrt{\mu}}P(w^{\frac{1}{2}})r_c^0 + \bar{A}^T\vartheta : \vartheta \in R^m\right\},$$

equals $d_s^f + \mathcal{L}^\perp$. Since $\mathcal{L}^\perp \cap \mathcal{L} = \{0\}$, it follows that the affine spaces $d_s^f + \mathcal{L}^\perp$ and $d_x^f + \mathcal{L}$ meet in a unique point. We call this point q. So q is uniquely determined by the system

$$\begin{aligned} \bar{A}q &= \tfrac{\theta}{\sqrt{1-\theta}}\nu r_b^0, \\ \bar{A}^T\vartheta + q &= \tfrac{\theta}{\sqrt{1-\theta}}\tfrac{\nu}{\sqrt{\mu}}P(w^{\frac{1}{2}})r_c^0. \end{aligned} \tag{25}$$

The proof of the following lemma is similar to the proof of Lemma 4.6 in [15].

Lemma 4.6. *One has*

$$4\omega^2 \leq \|q\|_F^2 + \Big(\|q\|_F + \sqrt{\frac{2N\theta^2}{1-\theta}}\Big)^2.$$

By using Lemma 4.6 in order to have $\omega \leq 0.54$, it suffices if

$$\|q\|_F^2 + \Big(\|q\|_F + \sqrt{\frac{2N\theta^2}{1-\theta}}\Big)^2 \leq 1.166. \tag{26}$$

4.3. Upper Bound For $\|q\|_F$

Recall that the vector q is the unique solution of the system (25), where $\bar{A} = \sqrt{\mu}AP(w^{\frac{1}{2}})$, with w denoting the scaling point of x and s, as defined in Lemma 2.3. So we have

$$\begin{aligned} \sqrt{\mu}AP(w^{\frac{1}{2}})q &= \tfrac{\theta}{\sqrt{1-\theta}}\nu r_b^0, \\ \sqrt{\mu}P(w^{\frac{1}{2}})A^T\vartheta + q &= \tfrac{\theta}{\sqrt{1-\theta}}\tfrac{\nu}{\sqrt{\mu}}P(w^{\frac{1}{2}})r_c^0. \end{aligned} \tag{27}$$

For the moment, let us write

$$D := P(w^{\frac{1}{2}}), \quad r_b := \frac{\theta}{\sqrt{1-\theta}}\nu r_b^0, \quad r_c := \frac{\theta}{\sqrt{1-\theta}}\nu r_c^0.$$

Then the system (27) is equivalent to

$$\begin{aligned} \sqrt{\mu}ADq &= r_b, \\ \sqrt{\mu}DA^T\vartheta + q &= \tfrac{1}{\sqrt{\mu}}Dr_c. \end{aligned} \tag{28}$$

By using similar arguments as in Lemma 4.7 in [15], we obtain the following result:

$$\sqrt{\mu^+}\|q\|_F \leq \theta\nu\sqrt{\|D(s^0-s^*)\|_F^2 + \|D^{-1}(x^0-x^*)\|_F^2}. \tag{29}$$

Using that $P(w^{\frac{1}{2}})$ is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle$ and $P(w)e = w^2$, we may write

$$\begin{aligned} \|D(s^0-s^*)\|_F^2 &= \Big\langle D(s^0-s^*), D(s^0-s^*)\Big\rangle = \Big\langle D^2(s^0-s^*), (s^0-s^*)\Big\rangle \\ &= \Big\langle D^2(s^0-s^*), \xi e\Big\rangle - \Big\langle D^2(s^0-s^*), \xi e - (s^0-s^*)\Big\rangle \\ &\leq \Big\langle D^2(s^0-s^*), \xi e\Big\rangle = \Big\langle s^0-s^*, D^2\xi e\Big\rangle = \xi\Big\langle s^0-s^*, w^2\Big\rangle \\ &= \xi\Big\langle \xi e, w^2\Big\rangle - \xi\Big\langle \xi e - (s^0-s^*), w^2\Big\rangle \leq \xi\Big\langle \xi e, w^2\Big\rangle = \xi^2\mathrm{tr}(w^2). \end{aligned}$$

In the same way it follows that

$$\|D^{-1}(x^0 - x^*)\|_F^2 \leq \xi^2 \mathrm{tr}(w^{-2}).$$

Substitution of the last two inequalities and $\mu = \nu\mu^0 = \nu\xi^2$ into (29) gives

$$\|q\|_F \leq \frac{\theta}{\sqrt{1-\theta}} \sqrt{\nu \mathrm{tr}(w^2 + w^{-2})}.$$

Finally, by using $\mathrm{tr}(w^2 + w^{-2}) \leq \frac{\mathrm{tr}(x+s)^2}{\mu\lambda_{\min}(v)^2}$ (Lemma 4.7 in [20]) and $\mathrm{tr}(x+s) \leq 4N\xi$ (Lemma 4.9 in [20]), we obtain

$$\|q\|_F \leq \frac{\theta}{\sqrt{1-\theta}} \frac{\mathrm{tr}(x+s)}{\xi\lambda_{\min}(v)} \leq \frac{4N\theta}{\sqrt{1-\theta}\lambda_{\min}(v)} \leq \frac{4N\theta}{\sqrt{1-\theta}}\rho(\delta),$$

where the last inequality follows by Lemma 4.2.

4.4. Value For θ

In order to have (26), it suffices if

$$\Big(\frac{4N\theta}{\sqrt{1-\theta}}\rho(\delta)\Big)^2 + \Big(\frac{4N\theta}{\sqrt{1-\theta}}\rho(\delta) + \sqrt{\frac{2N\theta^2}{1-\theta}}\Big)^2 \leq 1.166,$$

which is simplified as follows:

$$\Big(4N\theta\rho(\delta)\Big)^2 + \Big(4N\theta\rho(\delta) + \sqrt{2N}\theta\Big)^2 \leq 1.166(1-\theta). \tag{30}$$

With a value of θ that satisfies the above inequality, we are sure that when starting with $\delta(x, s; \mu) \leq \tau$, after the feasibility step we have $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt[4]{2}}$. Choosing $\delta = \frac{1}{16}$, one may easily verify that if $\theta = \frac{1}{7N}$, then the inequality (30) is satisfied.

4.5. Complexity Analysis

We have seen that if at the start of an iteration the iterate satisfies $\delta(x, s; \mu) \leq \tau$, with $\tau = \frac{1}{16}$, then after the feasibility step with $\theta = \frac{1}{7N}$, the iterate is strictly feasible and satisfies $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt[4]{2}}$.

After the feasibility step, we perform a few centering steps in order to get the iterate (x^+, y^+, s^+) which satisfies $\delta(x^+, s^+; \mu^+) \leq \tau$. By Corollary 3.3, after k centering steps, we will have the iterate (x^+, y^+, s^+) that is still feasible for (P_{ν^+}) and (D_{ν^+}) and satisfies

$$\delta(x^+, s^+; \mu^+) \leq \Big(\frac{1}{\sqrt[4]{2}}\Big)^{2^k}.$$

From this, one easily deduces that $\delta(x^+, s^+; \mu^+) \leq \tau$ will hold after at most

$$2 + \log_2\Big(\log_2 \frac{1}{\tau}\Big), \tag{31}$$

centering steps. According to (31), and since $\tau = \frac{1}{16}$, at most 4 centering steps suffice to get the iterate (x^+, y^+, s^+) that satisfies $\delta(x^+, s^+; \mu^+) \leq \tau$. So, each main iteration consists of one feasibility step and at most 4 centering steps.

In each main iteration both the duality gap and the norms of the residual vectors are reduced by the factor $1 - \theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\{2N\xi^2, \|r_b^0\|_F, \|r_c^0\|_F\}}{\epsilon}.$$

Due to $\theta = \frac{1}{7N}$ and the fact that we need at most 5 inner iterations per main iteration, we may state the main result of the paper.

Theorem 4.7. *If (P) and (D) are feasible and $\xi > 0$ is such that $x^* + s^* \leq \xi e$ for some optimal solution x^* of (P) and (y^*, s^*) of (D), then after at most*

$$35N \log \frac{\max\{2N\xi^2, \|r_b^0\|_F, \|r_c^0\|_F\}}{\epsilon},$$

inner iterations, the algorithm finds an ϵ-optimal solution of (P) and (D).

5. CONCLUDING REMARKS

In this paper, we presented and analyzed an infeasible full-NT step IPM for second-order cone optimization based on a new feasibility direction. The obtained complexity bound in Theorem 4.7 coincides with the best-known iteration complexity for small update methods. However, we have given a different way to calculate feasibility direction and its convergence analysis.

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