

VARIANT VACATION QUEUEING SYSTEM WITH BERNOULLI FEEDBACK, BALKING AND SERVER'S STATES-DEPENDENT RENEGING

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Abstract: We consider a single server Markovian feedback queue with variant of multiple vacation policy, balking, server's states-dependent renegeing, and retention of renegeed customers. We obtain the steady-state solution of the considered queue based on the use of probability generating functions. Then, the closed-form expressions of different system characteristics are derived. Finally, we present some numerical results in order to show the impact of the parameters of impatience timers on the performance measures of the system.

Keywords: Queueing Models, Vacation, Impatience, Bernoulli Feedback, Simulation.

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1. INTRODUCTION

Vacation queue modeling is being employed in a large variety of day-to-day congestion issues as well as industrial scenarios including communication and manufacturing systems, transportation systems, call centers, web services and so on. A vast literature has been devoted to vacation queueing models in different frameworks. Research works on single vacation (once the vacation is ended, the server switches to the busy period and stays there waiting for a new arrival) and multiple vacation (the server continues his vacation as long as there is no customers in the system) have a significant developments (cf. Doshi [15], Tian [28], Takagi [26], Tian and Zhang [29], Zhang and Tian [40], Ke *et al.* [18], and Upadhyaya [31]). In recent years, queueing models with variant of multiple vacations have drawn a significant attention, different from the above policies, this new concept considers the case wherein at the vacation completion instant, if the system is still empty, the server is permitted to take a certain number of successive vacations, and once the vacations are ended, the server has to come back to the busy period and remains there, busy or idle, depending on the presence of customers in the system (cf. Banik [6], Yue *et al.* [36], Wang *et al.* [34], and Laxmi and Rajesh [32]).

Impatience (balking and/or reneging) is the most prominent feature in queueing theory. Vacation queueing models with impatient customers are considered to be very appropriate tools in analysing various complex service systems and important industries. Therefore, it has generated a fundamental results with extensive bibliographical references on this area (cf. Zhang *et al.* [41], Altman and Yechaili [2, 3], Ke [16], Ke *et al.* [17], Adan *et al.* [1], Yue *et al.* [37], Ammar [4], Laxmi and Rajesh [33], Bouchentouf *et al.* [9, 10], and Kumar [20]).

In traditional vacation queueing literature with impatient customers, the studies of customer's behavior was always based on the hypothesis that customers' impatience happens only during the absence of the server. This is the case where the customers can see the state of the server. However, in many real-life situations including call center and production systems, it may not be possible to have information on the server's state. Further, a long wait in the queue is another factor which leads to customer's impatience whatever the state of the system (active or on vacation). Despite the rapid growth of the literature about customers's impatience in vacation queueing models, there is very limited literature to deal with the customers' impatience during both vacation and busy periods. The authors can refer to Yue *et al.* [39] and Bouchentouf and Guendouzi [11, 12], and Cherfaoui *et al.* [13].

Customer feedback has an utmost importance in queueing systems at which if the customer is not satisfied with the service, he can return to the system asking for another one. He can retry several times until he gets a satisfactory service. Such queues usually occur in our everyday life. As a concrete example, we cite multiple access telecommunication systems, where the data packet with errors at the destination will be sent over and over again until the data packet is transmitted with success. Queueing models with feedback have been widely studied. The pioneer research work on the subject was done by Takacs [25] who

dealt with an $M/M/1$ queue with feedback, where he determined the stationary process of the queue length as well as a customer waiting distribution in the system. The literature on the related topic is abundant, for a comprehensive overview, the readers may refer to Davignon and Disney [14], Bengtsson [7], Santhakumaran and Thangaraj [23], Atencia *et al.* [5], Upadhyaya [31], Liu and Whitt [21], and Shekhar *et al.* [24].

In regard to the mathematical solution techniques, it is worth pointing out that a distinction can be made between algorithms that allow efficient calculation of the system characteristics numerically; i.e., matrix-analytic approaches; a very powerful numerical technique employed when it is not easy to obtain clear and closed form analytical solutions for queueing problems (see Neuts [22], Blondia and Casals [8]), and analytical approaches that give rise to (exact or approximate) closed-form expressions; i.e., probability generating functions based techniques "special case of the z -transform" (see Takagi [27], Wittevrongel and Bruneel [35]), or approaches based on the maximum entropy approach (see Kouvatsof *et al.* [19]). Via this method, the unbiased distribution of the concerned queueing system is obtained by maximizing the defined entropy function in terms of known performance measures.

In line with the above, we consider in this paper a queueing model under K -variant of multiple vacations, Bernoulli feedback, server' states-dependent renegeing, and retention. The analysis of such a model is complicated. Generally, the problems encountered by processes describing this kind of systems are facing a big challenge. The behavior of the suggested queueing system is studied analytically by means of a generating-functions approach. The probability generating functions (PGF) is a powerful tool for presenting the solution of a difference equations set and solving probability problems. Through the use of PGFs, we can without difficulty convert the discrete sequence of numbers, i.e., probabilities into a function of dummy variable. This results in closed-form expressions for the steady-state distributions of the system. Further, the PGFs are utilized to deduce diverse system characteristics. The obtained expressions are easy to be numerically evaluated. To the best of authors' knowledge, the existing literature mainly focuses on impatient customers during vacation period, whereas, there have been no results presenting the analytic and computational aspects of the $M/M/1$ queueing model with Bernoulli feedback, balking, server's states-dependent renegeing, and retention of renegeed customers under variant of multiple vacation policy. This motivate us to investigate such a queueing model, where the stationary analysis of the queueing model is established via probability generating functions (PGFs) method. Then, various measures of effectiveness including the mean system size, the mean queue length, the mean number of customers served, and average rates of balking and renegeing, are derived in terms of steady-state probabilities. Further, we carried out numerical experiments that can be very beneficial to examine the effects of the parameters of impatience timers on the performance measures in different contexts.

The paper is arranged as follows. Section 2 describes the queueing model. In Section 3, we present the theoretical analysis of the suggested queueing system. In Section 4, we deduce useful system characteristics. Section 5 is devoted to some

particular cases. Section 6 presents numerical computation of the analytical results in order to show the effect of customers's impatience on the performance measures of the queueing system under consideration. Section 7 concludes the paper.

2. MODEL FORMULATION

We analyze a $M/M/1$ Bernoulli feedback queueing system under a variant of a multiple vacation policy with balking, reneging and retention of reneging in which customers arrive at the system according to a Poisson process of rate λ .

During busy period, all customers have i.i.d. service time, assumed to be exponential with parameter μ .

Whenever the system becomes empty at a service completion instant, the server goes on a vacation of random length, which is assumed to be exponentially distributed with rate ϕ .

If at a vacation completion instant, some customers are present in the queue, the server immediately begins the busy period. Otherwise, it will take vacations consecutively until the server has taken a maximum number of vacations (denoted by K -vacations), then the server switches to the busy period and remains idle waiting for a new arrival if the system is still empty.

In addition, we suppose that on arrival a customer either decides to join the queue with probability θ if the number of customers in the system is bigger or equal to one and may balk with probability $\bar{\theta} = 1 - \theta$.

Whenever a customer arrives at the system and finds the server on vacation (respectively, busy), he activates an impatience timer T_0 (respectively, T_1), which follows exponential distribution function with parameter ξ_0 (respectively, ξ_1). If the customer's service has not been finished before the customer's timer expires, the customer may abandon the system. That is, the customer's deadline is effective until the end of his service. This sort of customer behavior occurs often in practice including a situation where a customer's deadline corresponds to a fundamentally irreversible property such as failure or death, at which, the absence of a deadline by the customer may be seen as a permanent phenomenon, and may occur at any time including the time a customer is served. Further, each reneged customer may leave the system without getting service with probability α and may be retained in the queue with probability $\alpha' = 1 - \alpha$.

With probability β' , a customer rejoin the system as a Bernoulli feedback customer to receive another regular service if the initial one is unsatisfactory or incomplete. Otherwise, he leaves the system definitively with probability $\beta = 1 - \beta'$.

We also suppose that inter-arrival times, service times, impatience times, and vacation times are all mutually independent. The service order is supposed to be First-Come-First-Served (FCFS).

3. STEADY-STATE SOLUTION OF THE QUEUEING SYSTEM

Let $N(t)$ denote the number of customers in the system at time t , and let $J(t)$ denote the state of the server at time t , which is defined as follows:

$$J(t) = \begin{cases} j, & \text{the server is taking the } (j+1)^{th} \text{ vacation at time } t \text{ for } j = 0, 1, \dots, K-1, \\ K, & \text{the server is idle or busy at time } t. \end{cases}$$

Figure 1 depicts the state transition diagram of the queueing model under consideration.

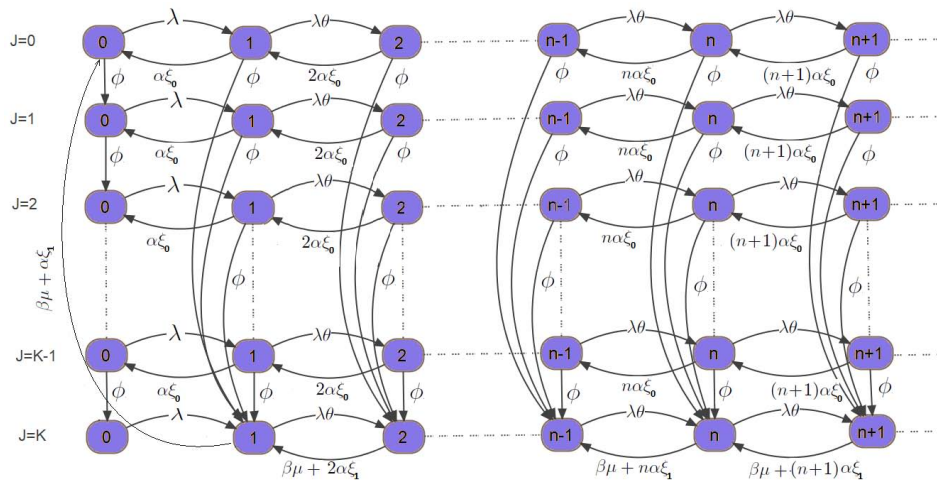


Figure 1: The state-transition diagram.

The pair $\{(N(t); J(t)); t \geq 0\}$ defines a Markovian, continuous-time process where its state space is $\Omega = \{(n, j) : n \geq 0; j = \overline{0, K}\}$.

Let $P_{n,j} = \lim_{t \rightarrow \infty} \mathbb{P}(N(t) = n; J(t) = j, n \geq 0, j = \overline{0, K})$, denote the steady-state probabilities of the process $\{(N(t), J(t)); t \geq 0\}$.

Then, based on the theory of Markov process, it is easy to show that the

steady-state equations of the model are:

$$(\lambda + \phi)P_{0,0} = \alpha\xi_0P_{1,0} + (\beta\mu + \alpha\xi_1)P_{1,K}, \tag{1}$$

$$(\lambda + \phi + \alpha\xi_0)P_{1,0} = \lambda P_{0,0} + 2\alpha\xi_0P_{2,0}, \quad n = 1, \tag{2}$$

$$(\theta\lambda + \phi + n\alpha\xi_0)P_{n,0} = \theta\lambda P_{n-1,0} + (n+1)\alpha\xi_0P_{n+1,0}, \quad n \geq 2, \tag{3}$$

$$(\lambda + \phi)P_{0,j} = \alpha\xi_0P_{1,j} + \phi P_{0,j-1}, \quad j = \overline{1, K-1}, \tag{4}$$

$$(\theta\lambda + \phi + \alpha\xi_0)P_{1,j} = \lambda P_{0,j} + 2\alpha\xi_0P_{2,j}, \quad j = \overline{1, K-1}, \quad n = 1, \tag{5}$$

$$(\theta\lambda + \phi + n\alpha\xi_0)P_{n,j} = \theta\lambda P_{n-1,j} + (n+1)\alpha\xi_0P_{n+1,j}, \quad j = \overline{1, K-1}, \quad n \geq 2, \tag{6}$$

$$\lambda P_{0,K} = \phi P_{0,K-1}, \tag{7}$$

$$(\theta\lambda + \beta\mu + \alpha\xi_1)P_{1,K} = \lambda P_{0,K} + (\beta\mu + 2\alpha\xi_1)P_{2,K} + \phi \sum_{j=0}^{K-1} P_{1,j}, \quad n = 1, \tag{8}$$

$$(\theta\lambda + \beta\mu + n\alpha\xi_1)P_{n,K} = \theta\lambda P_{n-1,K} + (\beta\mu + (n+1)\alpha\xi_1)P_{n+1,K} + \phi \sum_{j=0}^{K-1} P_{n,j}, \quad n \geq 2. \tag{9}$$

Next, the steady-state-probabilities are given in the following theorem.

Theorem 1. *The steady-state-probabilities $P_{n,j}$ of system size are given by*

$$P_{\bullet,j} = A^{j-1}P_{0,0}, \quad j = \overline{0, K-1}, \tag{10}$$

$$P_{\bullet,K} = \Phi(1)P_{0,0}, \tag{11}$$

where

$$P_{0,0} = \left(\frac{1-A^K}{A(1-A)} + \Phi(1) \right)^{-1}, \tag{12}$$

with

$$A = \frac{\phi C_2(1)}{\alpha\xi_0 B},$$

and

$$\Phi(1) = e^{\frac{\lambda\theta}{\alpha\xi_1}} \left\{ \left(\frac{B\xi_0}{C_2(1)\xi_1} + \frac{\phi}{\alpha\xi_1} \left(\frac{1-A^{K-1}}{1-A} \right) \right) H_1(1) + \frac{\phi}{\alpha\xi_1} \left(\frac{A^K}{A\lambda} H_2(1) - \left(\frac{1-A^K}{1-A} \right) H_3(1) \right) \right\},$$

with

$$\begin{aligned}
 B &= \left(1 + \frac{\lambda(1-\theta)C_1(1)}{\alpha\xi_0} \right), \\
 C_1(1) &= \int_0^1 e^{-\frac{\lambda\theta}{\alpha\xi_0}s} (1-s)^{\frac{\phi}{\alpha\xi_0}} ds, \\
 C_2(1) &= \int_0^1 e^{-\frac{\lambda\theta}{\alpha\xi_0}s} (1-s)^{\frac{\phi}{\alpha\xi_0}-1} ds, \\
 H_1(1) &= \int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} (1-s)^{-1} ds, \\
 H_2(1) &= \int_0^1 (\lambda\bar{\theta}s + \beta\mu) s^{\frac{\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} ds, \\
 H_3(1) &= \int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} (1-s)^{-1} \Psi(s) ds, \\
 \Psi(s) &= e^{-\frac{\lambda\theta}{\alpha\xi_0}s} (1-s)^{-\frac{\phi}{\alpha\xi_0}} \left\{ 1 + \frac{\lambda\bar{\theta}}{\alpha\xi_0} C_1(s) - \frac{B}{C_2(1)} C_2(s) \right\}, \\
 C_1(s) &= \int_0^s e^{-\frac{\lambda\theta}{\alpha\xi_0}t} (1-t)^{\frac{\phi}{\alpha\xi_0}} dt, \\
 C_2(s) &= \int_0^s e^{-\frac{\lambda\theta}{\alpha\xi_0}t} (1-t)^{\frac{\phi}{\alpha\xi_0}-1} dt.
 \end{aligned}$$

Proof. We investigate the steady-state probabilities of the system through the use of PGFs. Define the probability generating functions (PGFs) as

$$\begin{aligned}
 G_j(z) &= \sum_{n=0}^{\infty} z^n P_{n,j}, \\
 G'_j(z) &= \frac{d}{dz} G_j(z), \quad j = \overline{0, K}.
 \end{aligned}$$

The normalizing condition is given as

$$\sum_{n=0}^{\infty} \sum_{j=0}^K P_{n,j} = 1.$$

Multiplying equation (3) by z^n , using equations (1)–(2) and summing all possible values of n , we get

$$\alpha\xi_0(1-z)G'_0(z) - [\lambda\theta(1-z) + \phi]G_0(z) = -(\beta\mu + \alpha\xi_1)P_{1,K} + \lambda\bar{\theta}(1-z)P_{0,0}. \tag{13}$$

In the same manner, we obtain from equations (6) and (9), respectively.

$$\alpha\xi_0(1-z)G'_j(z) - [\lambda\theta(1-z) + \phi]G_j(z) = \lambda\bar{\theta}(1-z)P_{0,j} - \phi P_{0,j-1}, \quad j = \overline{1, K-1}, \tag{14}$$

and

$$\begin{aligned}
 \alpha\xi_1 z(1-z)G'_K(z) - (1-z)(\theta\lambda z - \beta\mu)G_K(z) &= (1-z)[\lambda\bar{\theta}z + \beta\mu]P_{0,K} \\
 + z(\beta\mu + \alpha\xi_1)P_{1,K} - \phi z \sum_{j=0}^{K-1} G_j(z) + \phi z \sum_{j=0}^{K-2} P_{0,j}. &\tag{15}
 \end{aligned}$$

For $z \neq 1$, equation (13) can be written as follows :

$$G'_0(z) - \left[\frac{\lambda\theta}{\alpha\xi_0} + \frac{\phi}{\alpha\xi_0(1-z)} \right] G_0(z) = -\frac{\beta\mu + \alpha\xi_1}{\alpha\xi_0(1-z)} P_{1,K} + \frac{\lambda\bar{\theta}}{\alpha\xi_0} P_{0,0}. \tag{16}$$

Multiply both sides of equation (16) by $e^{-\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}$, we get

$$\frac{d}{dz} \left[e^{-\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}} G_0(z) \right] = e^{-\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}} \left[\frac{\lambda\bar{\theta}}{\alpha\xi_0} P_{0,0} - \frac{(\beta\mu + \alpha\xi_1)}{\alpha\xi_0(1-z)} P_{1,K} \right].$$

Then, integrating from 0 to z , we obtain

$$G_0(z) = e^{\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}} \left\{ G_0(0) + \frac{\lambda\bar{\theta}}{\alpha\xi_0} P_{0,0} C_1(z) - \frac{\beta\mu + \alpha\xi_1}{\alpha\xi_0} P_{1,K} C_2(z) \right\}, \tag{17}$$

where

$$C_1(z) = \int_0^z e^{-\frac{\lambda\theta}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}} ds,$$

$$C_2(z) = \int_0^z e^{-\frac{\lambda\theta}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1} ds.$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{n,0} > 0$ and $z = 1$ is the root of denominator of the right hand side of equation (17), we have that $z = 1$ must be the root of the nominator of the right hand side of equation (17). So, we obtain

$$G_0(0) = P_{0,0} = \frac{(\beta\mu + \alpha\xi_1) P_{1,K}}{\alpha\xi_0} C_2(1) - \frac{\lambda\bar{\theta} P_{0,0}}{\alpha\xi_0} C_1(1). \tag{18}$$

Next, equation (18) implies

$$P_{1,K} = \frac{\alpha\xi_0}{(\beta\mu + \alpha\xi_1) C_2(1)} B P_{0,0}, \tag{19}$$

with

$$B = \left[1 + \frac{\lambda}{\alpha\xi_0} \bar{\theta} C_1(1) \right].$$

Substituting equation (19) into equation (17), we obtain

$$G_0(z) = e^{\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}} \left\{ 1 + \frac{\lambda\bar{\theta}}{\alpha\xi_0} C_1(z) - \frac{B}{C_2(1)} C_2(z) \right\} P_{0,0}. \tag{20}$$

Equation (14) can be written as

$$G'_j(z) - \left[\frac{\lambda\theta}{\alpha\xi_0} + \frac{\phi}{\alpha\xi_0(1-z)} \right] G_j(z) = \frac{\lambda\bar{\theta}}{\alpha\xi_0} P_{0,j} - \frac{\phi}{\alpha\xi_0(1-z)} P_{0,j-1}. \tag{21}$$

In a similar manner used for solving equation (14), we get

$$G_j(z) = e^{\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}} \left\{ G_j(0) + \frac{\lambda\bar{\theta}}{\alpha\xi_0}C_1(z)P_{0,j} - \frac{\phi}{\alpha\xi_0}C_2(z)P_{0,j-1} \right\}, \quad j = \overline{1, K-1}. \tag{22}$$

Since $G_j(1) = \sum_{n=0}^{\infty} P_{n,j} > 0$ ($G_j(1) = P_{\bullet,j}$ represents the probability that the server is taking the $(j+1)^{th}$ vacation), and $z = 1$ is the root of denominator of the right hand side of equation (22), we have that $z = 1$ must be the root of the nominator of the right hand side of equation (22). So, we get

$$G_j(0) = P_{0,j} = \frac{\phi C_2(1)}{\alpha\xi_0 B} P_{0,j-1} = A P_{0,j-1}, \quad j = \overline{1, K-1}. \tag{23}$$

Remark 2. It is easy to check that $0 < \phi C_2(1) < \alpha\xi_0$ and $\lambda\bar{\theta}C_1(1) > 0$. Thus, $0 < \phi C_2(1) < \alpha\xi_0 + \lambda\bar{\theta}C_1(1)$. Consequently, we have $0 < A < 1$.

Using equation (23) repeatedly, we find

$$P_{0,j} = A^j P_{0,0}, \quad j = \overline{1, K-1}. \tag{24}$$

Substituting equation (24) into equation (22), we obtain

$$G_j(z) = e^{\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}} A^j \left\{ 1 + \frac{\lambda\bar{\theta}}{\alpha\xi_0}C_1(z) - \frac{B}{C_2(1)}C_2(z) \right\} P_{0,0}, \quad j = \overline{1, K-1}. \tag{25}$$

Using equations (7) and (24), we obtain

$$P_{0,K} = \frac{\phi}{\lambda} A^{K-1} P_{0,0}. \tag{26}$$

Next, equation (15) can be written as

$$G'_K(z) - \left[\frac{\theta\lambda}{\alpha\xi_1} - \frac{\beta\mu}{\alpha\xi_1 z} \right] G_K(z) = \left[\frac{\beta\mu + \alpha\xi_1}{\alpha\xi_1(1-z)} \right] P_{1,K} + \frac{\lambda\bar{\theta}z + \beta\mu}{\alpha\xi_1 z} P_{0,K} + \frac{\phi}{\alpha\xi_1(1-z)} \left[\sum_{j=0}^{K-2} P_{0,j} - \sum_{j=0}^{K-1} G_j(z) \right]. \tag{27}$$

Then, multiplying by $e^{\frac{-\theta\lambda}{\alpha\xi_1}z} \frac{\beta\mu}{\alpha\xi_1}$, we find

$$\frac{d}{dz} \left(e^{\frac{-\theta\lambda}{\alpha\xi_1}z} \frac{\beta\mu}{\alpha\xi_1} G_K(z) \right) = e^{\frac{-\theta\lambda}{\alpha\xi_1}z} \frac{\beta\mu}{\alpha\xi_1} \left\{ \left[\frac{\beta\mu + \alpha\xi_1}{\alpha\xi_1(1-z)} \right] P_{1,K} + \frac{\lambda\bar{\theta}z + \beta\mu}{\alpha\xi_1 z} P_{0,K} + \frac{\phi}{\alpha\xi_1(1-z)} \left[\sum_{j=0}^{K-2} P_{0,j} - \sum_{j=0}^{K-1} G_j(z) \right] \right\}. \tag{28}$$

Then, integrating from 0 to z and using equations (19) and (24)-(26), we obtain

$$G_K(z) = e^{\frac{\lambda\theta}{\alpha\xi_1}z - \frac{\beta\mu}{\alpha\xi_1}} \left\{ \left(\frac{B\xi_0}{C_2(1)\xi_1} + \frac{\phi}{\alpha\xi_1} \left(\frac{1-A^{K-1}}{1-A} \right) \right) H_1(z) + \frac{\phi}{\alpha\xi_1} \left(\frac{A^{K-1}}{\lambda} H_2(z) - \left(\frac{1-A^K}{1-A} \right) H_3(z) \right) \right\} P_{0,0}, \tag{29}$$

where

$$\begin{aligned} H_1(z) &= \int_0^z s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} (1-s)^{-1} ds, \\ H_2(z) &= \int_0^z (\lambda\bar{\theta}s + \beta\mu) s^{\frac{\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} ds, \\ H_3(z) &= \int_0^z s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} \Psi(s) (1-s)^{-1} ds, \\ \Psi(s) &= e^{\frac{\lambda\theta}{\alpha\xi_0}s} (1-s)^{-\frac{\phi}{\alpha\xi_0}} \left\{ 1 + \frac{\lambda\bar{\theta}}{\alpha\xi_0} C_1(s) - \frac{B}{C_2(1)} C_2(s) \right\}. \end{aligned}$$

Thus, we get $G_K(1)$; the probability that the server is busy or idle:

$$G_K(1) = P_{\bullet,K} = \Phi(1)P_{0,0}, \tag{30}$$

where,

$$\begin{aligned} \Phi(1) &= e^{\frac{\lambda\theta}{\alpha\xi_1}} \left\{ \left(\frac{B\xi_0}{C_2(1)\xi_1} + \frac{\phi}{\alpha\xi_1} \left(\frac{1-A^{K-1}}{1-A} \right) \right) H_1(1) + \frac{\phi}{\alpha\xi_1} \left(\frac{A^{K-1}}{\lambda} H_2(1) - \left(\frac{1-A^K}{1-A} \right) H_3(1) \right) \right\}, \\ H_1(1) &= \int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} (1-s)^{-1} ds, \\ H_2(1) &= \int_0^1 (\lambda\bar{\theta}s + \beta\mu) s^{\frac{\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} ds, \\ H_3(1) &= \int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda\theta}{\alpha\xi_1}s} (1-s)^{-1} \Psi(s) ds. \end{aligned}$$

From equations (13)-(14), for $z = 1$, we have

$$P_{\bullet,j} = G_j(1) = A^{j-1}P_{0,0}, \quad j = \overline{0, K-1}. \tag{31}$$

By the definition of $P_{\bullet,j}$ and using the normalizing condition, we get

$$\sum_{j=0}^K P_{\bullet,j} = 1.$$

From equations (30)-(31), we get

$$P_{0,0} = \left(\frac{1-A^K}{A(1-A)} + \Phi(1) \right)^{-1}.$$

This completes the proof. \square

4. PERFORMANCE MEASURES

Now, we present some important performance measures of the queueing model. The mean number of customers in the system when the server is in the state j :

$$E(L_j) = \sum_{n=1}^{\infty} nP_{n,j}, j = \overline{0, K}.$$

The mean system size when the server is on vacation period :

$$E(L_V) = \sum_{j=0}^{K-1} E(L_j) = \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} nP_{n,j} = \sum_{n=1}^{\infty} n \left(\sum_{j=0}^{K-1} P_{n,j} \right).$$

The mean system size when the server is on busy period :

$$E(L_K) = \sum_{n=1}^{\infty} nP_{n,K}.$$

The mean size of the system :

$$E(L) = \sum_{j=0}^K \sum_{n=0}^{\infty} nP_{n,j} = E(L_V) + E(L_K).$$

The probability that the system is in a vacation period :

$$P_V = \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} P_{n,j} = \sum_{j=0}^{K-1} P_{\bullet,j}.$$

The probability that the server is idle and not in vacation period :

$$P_I = P_{0,K}.$$

The probability that the system is busy :

$$P_B = 1 - P_V - P_{0,K}.$$

The mean size of the queue :

$$\begin{aligned} E(L_q) &= \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} nP_{n,j} + \sum_{n=1}^{\infty} (n-1)P_{n,K} \\ &= E(L) - (1 - P_V - P_{0,K}) \\ &= E(L) - P_B. \end{aligned}$$

The expected number of customers served per unit of time :

$$E_{cs} = \beta \mu P_B.$$

The average rate of balking :

$$B_r = \bar{\theta} \lambda \left(\sum_{j=0}^K \sum_{n=0}^{\infty} P_{n,j} \right).$$

The average rate of abandonment of a customer due to impatience :

$$\begin{aligned} R_{ren} &= \alpha \xi_0 \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} n P_{n,j} + \alpha \xi_1 \sum_{n=1}^{\infty} n P_{n,K} \\ &= \alpha \xi_0 E(L_V) + \alpha \xi_1 E(L_K). \end{aligned} \tag{32}$$

The average rate of retention of impatient customers :

$$R_{ret} = \alpha' \xi_0 E(L_V) + \alpha' \xi_1 E(L_K). \tag{33}$$

5. SPECIAL CASES

Case 1: If $\xi_1 = 0$, $K = 1$, $\theta = 1$, $\alpha = 1$ and $\beta = 1$, then, the steady-state probabilities $P_{\bullet,0}$ and $P_{\bullet,1}$ are as:

$$\begin{aligned} P_{\bullet,0} &= \frac{\xi_0}{\phi C(1)} P_{0,0}, \\ P_{\bullet,1} &= \frac{1}{\mu - \lambda} \left(\frac{\lambda \xi_0}{\phi + \xi_0} + \frac{\phi \mu C(1)}{\lambda} \right) P_{0,0}, \end{aligned}$$

where

$$\begin{aligned} P_{0,0} &= (\mu - \lambda) \left(\frac{\lambda \xi_0}{(\xi_0 + \phi) C(1)} + \frac{(\mu - \lambda)}{\phi C(1)} + \frac{\phi \mu}{\lambda} \right)^{-1}, \\ C(1) &= \int_0^1 (1-s)^{\frac{\phi}{\xi_0} - 1} e^{-\frac{\lambda}{\xi_0} s} ds, \end{aligned}$$

which coincide with Equations (5.8) and (5.12) of Altman and Yechiali [2].

Case 2: When $\xi_1 = 0$, $\theta = 1$, $\alpha = 1$ and $\beta = 1$, the steady-state-probabilities of the number of customers in the system have the following form:

$$\begin{aligned} P_{\bullet,j} &= A^{j-1} P_{0,0}, \quad j = \overline{0, K-1}, \\ P_{\bullet,K} &= \frac{\phi}{\mu - \lambda} \left(\frac{\lambda (1 - A^K)}{(\phi + \xi_0) A (1 - A)} + \frac{\mu}{\lambda} A^{K-1} \right) P_{0,0}, \end{aligned}$$

where

$$P_{0,0} = \left\{ \frac{(\mu\phi + (\mu - \lambda)\xi_0)(1 - A^K)}{(\mu - \lambda)(\phi + \xi_0)A(1 - A)} + \frac{\mu\phi A^{K-1}}{\lambda(\mu - \lambda)} \right\}^{-1},$$

with

$$A = \frac{\phi C(1)}{\xi_0},$$

such that

$$C(1) = \int_0^1 e^{-\frac{\lambda}{\xi_0}s} (1 - s)^{\frac{\phi}{\xi_0} - 1} ds.$$

The obtained results match with Equations (26), (33), and (35) in Yue *et al.* [38].

Case 3: If $K = 1$, $\theta = 1$, $\alpha = 1$ and $\beta = 1$, the steady-state probabilities $P_{\bullet,0}$ and $P_{\bullet,1}$ are as:

$$P_{\bullet,0} = \frac{\xi_0}{\phi C_0(1)} P_{00},$$

$$P_{\bullet,1} = \frac{e^{\frac{\lambda}{\xi_1}}}{\xi_1} \left(\xi_0 \frac{C_1(1)}{C_0(1)} - \phi C_2(1) + \frac{\phi\mu}{\lambda} C_3(1) \right) P_{0,0},$$

where

$$P_{0,0} = \left(\frac{\xi_0}{\phi C_0(1)} + \frac{e^{\frac{\lambda}{\xi_1}}}{\xi_1} \left(\xi_0 \frac{C_1(1)}{C_0(1)} - \phi C_2(1) + \frac{\phi\mu}{\lambda} C_3(1) \right) \right)^{-1},$$

with

$$C_0(1) = \int_0^1 (1 - s)^{\frac{\phi}{\xi_0} - 1} e^{-\frac{\lambda}{\xi_0}s} ds,$$

$$C_1(1) = \int_0^1 (1 - s)^{-1} s^{\frac{\mu}{\xi_1}} e^{-\frac{\lambda}{\xi_1}s} ds,$$

$$C_2(1) = \int_0^1 \left(1 - \frac{C_0(t)}{C_0(1)} \right) s^{\frac{\mu}{\xi_1}} (1 - s)^{-\left(\frac{\phi}{\xi_0} + 1\right)} e^{\frac{\lambda}{\xi_0} - \frac{\lambda}{\xi_1}s} ds,$$

$$C_3(1) = \int_0^1 s^{\frac{\mu}{\xi_1} - 1} e^{-\frac{\lambda}{\xi_1}s} ds,$$

which coincide with Equations (44), (43), and (45) in Yue *et al.* [39].

6. NUMERICAL ANALYSIS

To show the applicability of the theoretical results obtained previously, we present some numerical results pointing out the impact of the impatience rates on different performance measures of the considered queueing system. Numerical

ξ_0	ξ_1	P_B	P_V	P_I	$E(L_V)$	$E(L_K)$	$E(L)$	$E(L_q)$	B_r	R_{ren}	R_{ret}
1.00	1.00	0.5228	0.4765	0.0006	0.8160	1.3195	2.1355	1.6127	0.5546	0.9676	0.6451
	2.00	0.4218	0.5774	0.0009	0.9827	0.8843	1.8670	1.4452	0.4913	1.1446	0.7631
	3.00	0.3749	0.6242	0.0008	1.0654	0.7178	1.7832	1.4083	0.4708	1.2564	0.8376
	4.00	0.3494	0.6497	0.0010	1.1033	0.6326	1.7358	1.3865	0.4589	1.3417	0.8944
2.00	1.00	0.4499	0.5475	0.0026	0.5922	1.0804	1.6725	1.2226	0.4413	1.0888	0.7259
	2.00	0.3530	0.6440	0.0031	0.6982	0.6856	1.3838	1.0308	0.3724	1.2370	0.8246
	3.00	0.3122	0.6845	0.0032	0.7403	0.5456	1.2859	0.9736	0.3482	1.3084	0.8723
	4.00	0.2880	0.7087	0.0034	0.7668	0.4703	1.2371	0.9492	0.3359	1.3578	0.9052
3.00	1.00	0.4041	0.5902	0.0056	0.4672	0.9512	1.4184	1.0142	0.3762	1.1691	0.7794
	2.00	0.3129	0.6806	0.0065	0.5408	0.5909	1.1317	0.8187	0.3070	1.3069	0.8713
	3.00	0.2725	0.7207	0.0067	0.5724	0.4598	1.0322	0.7596	0.2822	1.3673	0.9116
	4.00	0.2507	0.7425	0.0068	0.5904	0.3940	0.9844	0.7336	0.2701	1.4065	0.9377
4.00	1.00	0.3748	0.6169	0.0084	0.3873	0.8739	1.2612	0.8864	0.3351	1.2290	0.8193
	2.00	0.2872	0.7031	0.0096	0.4415	0.5357	0.9771	0.6899	0.2661	1.3576	0.9051
	3.00	0.2487	0.7412	0.0101	0.4657	0.4128	0.8784	0.6298	0.2413	1.4130	0.9420
	4.00	0.2274	0.7620	0.0106	0.4787	0.3506	0.8293	0.6019	0.2288	1.4446	0.9630

Table 1: Model characteristics vs. ξ_0 and ξ_1

works have been carried out using MATLAB program. To this end, we put $\lambda = 3$, $\mu = 4$, $\phi = 0.5$, $K = 3$, $\beta = 0.4$ and $\alpha = 0.6$. The obtained results are presented in Table 1 and Figures 2–6.

From the numerical results given in Table 1 and Figures 2–7, we have

- * The monotonicity of P_B , P_V , P_I , $E(L_K)$, $E(L)$, $E(L_q)$, B_r , R_{ren} , and R_{ret} with regard to ξ_0 is similar to the monotonicity of P_B , P_V , P_I , $E(L_K)$, $E(L)$, $E(L_q)$, B_r , R_{ren} , and R_{ret} with regard to ξ_1 . However, $E(L_V)$ increases with the increasing of ξ_1 and decreases with ξ_0 (see Figure 5).

- * As intuitively expected, the increasing of the impatience rates during both vacation and busy periods generate a decrease in the mean number of customers in the queue $E(L_q)$ as well as in the system $E(L)$ (see Figures 2–3). Consequently, the probability that the server is idle during busy period monotonically increase. This leads to a decrease in the average rate of balking B_r .

- * As it should be, the increasing of the impatience rates ξ_0 and ξ_1 implies a diminution in the mean number of customers in the system during vacation $E(L_V)$ and busy $E(L_K)$ period, respectively (see Figures 5–6). This implies a decreasing in the probability of busy period P_B and an increasing in the vacation period P_V (see Figure 7).

- * Obviously, the increase of ξ_0 and ξ_1 implies an increase in the average rate of renegeing R_{ren} (see Figure 4). In this situation, the system uses certain persuasive mechanism in order to convince customers not to leave the system; R_{ret} monotonically increases with ξ_0 and ξ_1 .

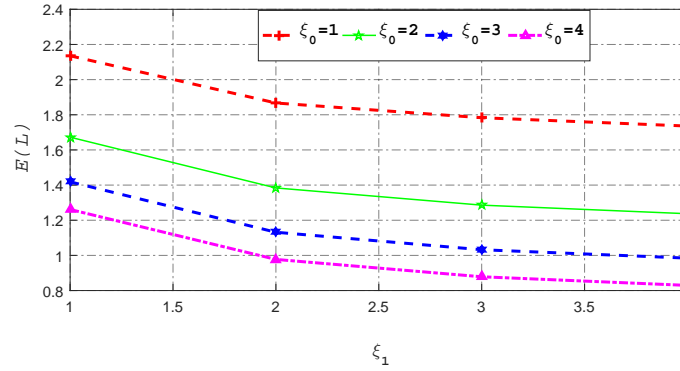


Figure 2: $E(L)$ vs. ξ_0 and ξ_1 .

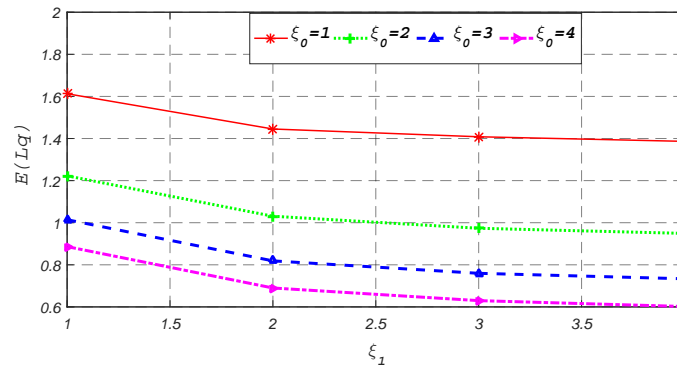


Figure 3: $E(Lq)$ vs. ξ_0 and ξ_1 .

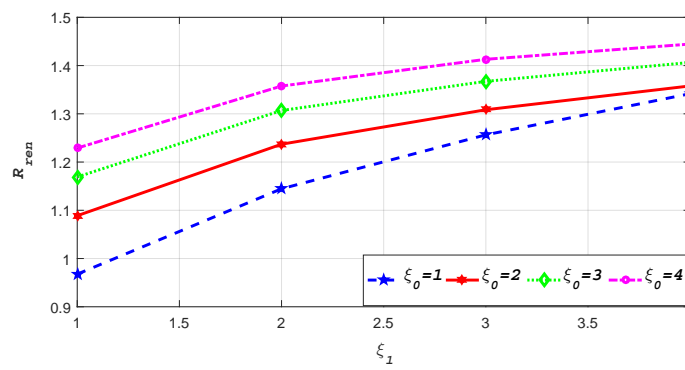


Figure 4: R_{ren} vs. ξ_0 and ξ_1 .

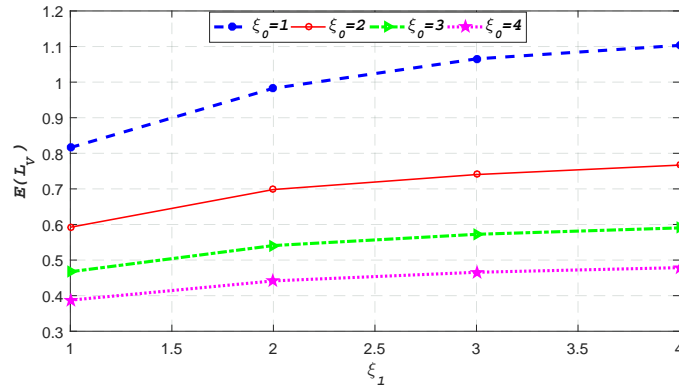


Figure 5: $E(L_V)$ vs. ξ_0 and ξ_1 .

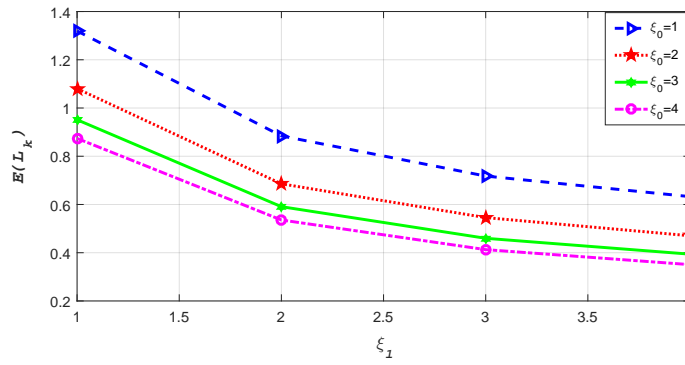


Figure 6: $E(L_K)$ vs. ξ_0 and ξ_1 .

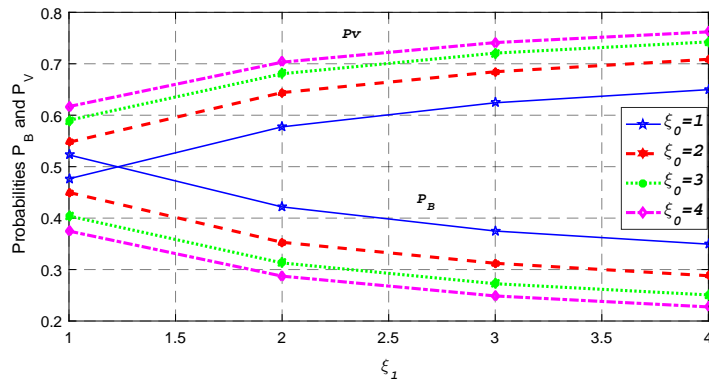


Figure 7: P_B and P_V vs. ξ_0 and ξ_1 .

7. CONCLUSION

In this paper, we analyzed an $M/M/1$ Bernoulli feedback queue with balking, reneging which depends on the state of the server, and retention of reneged customers under K -variant vacation policy. The steady-state probabilities of the queueing system have been obtained, using probability generating functions (PGFs). Then, important system characteristics have been derived. An illustrative numerical example is presented to confirm the theoretical results. Our queueing system can be considered as a generalized version of different existing queueing models presented by Altman and Yechiali [2], Yue *et al.* [37], and Yue *et al.* [39]. Other variations can be done on the considered queueing system, e.g., the queueing model can be extended to a state dependent arrival, state dependent service, and state dependent vacation.

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