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NEW OPTIMALITY CONDITIONS IN VECTOR CONTINUOUS-TIME PROGRAMMING

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Abstract: In this work vector continuous-time programming problem with inequality constraints is considered. The necessary and sufficient optimality conditions under generalized concavity assumptions are established. The results were formulated using differentiability.

Keywords: Continuous-time Programming, Vector Optimization, Optimality Conditions.

MSC: 90C30, 90C46, 90C29.

1. INTRODUCTION

Vector continuous-time programming problem has been the subject of numerous investigations in the past three decades. This problem was studied in [3, 4, 5, 6]. In [4, 5] a fundamental tool was Gordan's Transposition Theorem given in [7]. In [1], Arutyunov et al. indicated that such a result is incorrect. In [5], Nobakhtian and Pouryayevali established necessary and sufficient conditions for the nonsmooth problem under invexity assumptions, but unfortunately, that article used the result from [11], which is also incorrect (see [1]). In [2], Monte and Oliveira provided new necessary optimality conditions in the type of Karush-Kuhn-Tucker conditions for smooth continuous-time programming problem with scalar valued objective function. The aforementioned conditions are obtained, first for problem with inequality constraints and then for problem with both inequality and equality constraints. The alternative theorem for obtaining these conditions is given in [1]. To apply the alternative theorem, a specific regularity condition must be satisfied. Our aim in this paper is to provide necessary and sufficient optimality conditions for differentiable vector continuous-time problem defined in $L_\infty([0,T];\mathbb{R}^n)$.

The paper is organized in the following way. Some preliminaries about the problem are given in section 2, where some important definitions are stated. In section 3, Karush-Kuhn-Tucker necessary optimality conditions are obtained. In section 4, sufficient optimality conditions are obtained under generalized concavity assumptions.

2. NOTATIONS AND PRELIMINARIES

In this work, we consider the following vector continuous-time problem (VCTP):

$$\begin{aligned} \max \quad & \int_0^T f(t,x(t))\,dt = \left(\int_0^T f_1(t,x(t))\,dt, \ldots, \int_0^T f_k(t,x(t))\,dt\right) \\ \text{s.t.} \quad & g_i(t,x(t)) \geq 0, \quad i \in I = \{1,\ldots,m\} \text{ a.e. in } [0,T], \\ & x \in L_\infty([0,T];\mathbb{R}^n), \end{aligned}$$

where $f_j : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $j \in J = \{1,\ldots,k\}$ and $g_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are given functions and $f_j(t,x(t))$ denotes the j-th component of $f(t,x(t)) \in \mathbb{R}^k$. Here for each $t \in [0,T]$, $x_k(t)$ is the kth component of $x(t) \in \mathbb{R}^n$. All integrals are given in the Lebesgue sense. B denotes the open unit ball with centre at the origin, independently of the space or dimension.

Let

$$\Omega = \{x \in L_\infty([0,T];\mathbb{R}^n) : g_i(t,x(t)) \geq 0, \quad i \in I, \text{ a.e. in } [0,T]\}$$

be the set of feasible solutions for (VCTP). Let $\varepsilon > 0$ and $\hat{x} \in \Omega$. We assume that functions $f_j(t,\cdot)$ and $g_i(t,\cdot)$ are continuously differentiable on $\hat{x}(t) + \varepsilon\bar{B}$ a.e. in $[0,T]$, $j \in J$, $i \in I$. We assume also that functions $f_j(\cdot,x)$, $g_i(\cdot,x)$ are Lebesgue measurable for each x, $j \in J$, $i \in I$, $f_j(\cdot,x(\cdot))$, $g_i(\cdot,x(\cdot))$ are essentially bounded in $[0,T]$ for all $x \in L_\infty([0,T],\mathbb{R}^n)$ and there exist $K_f > 0$ and $K_g > 0$ such that

$$\|\nabla f_j(t,\hat{x}(t))\| \leq K_f, \; j \in J, \text{ a.e in } [0,T],$$

$$\|\nabla g_i(t,\hat{x}(t))\| \leq K_g, \; i \in I, \text{ a.e in } [0,T].$$

The maximization in the initial problem is in the sense of an efficient point.

Definition 1. *A feasible solution $\hat{x}$ for (VCTP) is said to be an efficient solution for (VCTP) if there is no other feasible solution x for (VCTP) such that*

$$\int_0^T f_j(t,x(t))\,dt \geq \int_0^T f_j(t,\hat{x}(t))\,dt, \; j \in J,$$

with at least one strict inequality.

3. KARUSH-KUHN-TUCKER NECESSARY OPTIMALITY CONDITIONS

In this section, we discuss the necessary optimality conditions for (VCTP). Let $r \in J$ and $\hat{x} \in \Omega$. Consider an auxiliary problem:

$$
\mathrm{P}_r(\hat{x}) \qquad \begin{aligned} &\max \int_0^T f_r(t, x(t))\, dt \\ &\text{s.t. } g_i(t, x(t)) \geq 0, \;\; i \in I, \;\text{a.e. in } [0, T], \\ &\qquad f_j(t, x(t)) \geq f_j(t, \hat{x}(t)), \;\; j \in J \setminus \{r\}, \;\text{a.e. in } [0, T]. \end{aligned}
$$

The following lemma shows the connection between (VCTP) and scalar problem $\mathrm{P}_r(\hat{x})$, and plays a key role in proving main result in this section.

Lemma 2. *(Chankong and Haimes [8]) If a point $\hat{x} \in \Omega$ is an efficient solution for* (VCTP)*, then $\hat{x}$ solves $\mathrm{P}_r(\hat{x})$ for all $r \in J$.*

Consider the following scalar problem:

$$
\text{(SCTP)} \qquad \begin{aligned} &\max \int_0^T f_r(t, x(t))\, dt \\ &\text{s.t. } g_i(t, x(t)) \geq 0, \;\; i \in I, \;\text{a.e. in } [0, T], \\ &\qquad x \in L_\infty([0, T]; \mathbb{R}^n). \end{aligned}
$$

Let $b > 0$. We will denote by $I_b(t)$ the index set of b-active constraints at $\hat{x} \in \Omega$, that is,

$$
I_b(t) = \{\, i \in I : 0 \leq g_i(t, \hat{x}(t)) \leq b \,\}, \quad \text{for each } t \in [0, T].
$$

For all $i \in I$, let us define the function $\delta_i^b : [0, T] \to \mathbb{R}$ as

$$
\delta_i^b(t) = \begin{cases} 1, \; i \in I_b(t) \\ 0, \text{ otherwise.} \end{cases}
$$

Let

$$
\begin{aligned} \phi_r(t, x) &= -\int_0^T \nabla f_r(t, \hat{x}(t))^T x\, dt < 0, \\ \phi_i(t, x) &= -\, g_i(t, \hat{x}(t)) - \delta_i^b(t) \nabla g_i(t, \hat{x}(t))^T x \leq 0, \;\; i \in I, \\ &\quad x \in \mathbb{R}^n, \end{aligned} \tag{1}
$$

be a system corresponding to the problem (SCTP) and

$$
\mathcal{I}(t, x) = \big\{ p : \phi_p(t, x) = \max \big\{ \phi_r(t, x), \phi_1(t, x), \ldots, \phi_m(t, x) \big\}, \;\; t \in [0, T], \;\; x \in \mathbb{R}^n.
$$

Definition 3. *(Arutyunov et al. [1]) System* (1) *is said to be regular when there exist a function* $\bar{x}(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$*, real numbers* $R \geq 0$ *and* $\alpha > 0$ *such that for a.e.* $t \in [0,1]$ *and for all* $x \in \mathbb{R}^n$ *with* $\|x - \bar{x}(t)\| \geq R$*, there exists a unit vector* $e = e(t,x) \in \mathbb{R}^n$*, satisfying*

$$\langle \partial_x \phi_p(t,x), e \rangle \geq \alpha \quad \forall p \in \mathcal{I}(t,x),$$

where $\partial_x \phi_p$ *denotes the partial subdifferential of* ϕ_p *at* (t,x) *in the sense of convex analysis. For more information, the reader is referred to Rockafellar [12].*

Remark 4. *We say that* $\hat{x} \in \Omega$ *satisfies the constraint qualification (MFCQ), if there exist* $\bar{\gamma} \in L_\infty([0,T];\mathbb{R}^n)$ *and* $\hat{b} > 0$ *such that, for almost every* $t \in [0,T]$*,*

$$\nabla g_i(t,\hat{x}(t))^T \bar{\gamma}(t) \geq \beta, \;\; i \in I_{\hat{b}}(t),$$

for some $\beta > 0$*.*

Note that (MFCQ) is a continuous-time version of the Mangasarian-Fromovitz constraint qualification.

Lemma 5. *(Monte and Oliveira [2]) Assume that* $\hat{x} \in \Omega$ *satisfies (MFCQ) and solves (SCTP). If the system* (1) *is regular, then there exists* $\hat{v} \in L_\infty([0,T];\mathbb{R}^m)$ *such that, for almost every* $t \in [0,T]$*,*

$$\nabla f_r(t,\hat{x}(t)) + \sum_{i \in I} \hat{v}_i(t) \nabla g_i(t,\hat{x}(t)) = 0,$$

$$\hat{v}_i(t) g_i(t,\hat{x}(t)) = 0, \;\; i \in I,$$

$$\hat{v}_i(t) \geq 0, \;\; i \in I.$$

Now, we give necessary Karush-Kuhn-Tucker optimality conditions for (VCTP).

Theorem 6. *If* $\hat{x}$ *is an efficient solution for (VCTP),* $P_r(\hat{x})$ *satisfies the constraint qualification* $(MFCQ)$ *at* $\hat{x}$ *for some* r *and corresponding system is regular, then there exists* $(\hat{\lambda}, \hat{v}) \in L_\infty([0,T];\mathbb{R}^k \times \mathbb{R}^m)$ *such that, for almost every* $t \in [0,T]$*,*

$$\sum_{j \in J} \hat{\lambda}_j(t) \nabla f_j(t,\hat{x}(t)) + \sum_{i \in I} \hat{v}_i(t) \nabla g_i(t,\hat{x}(t)) = 0, \tag{2}$$

$$\hat{v}_i(t) g_i(t,\hat{x}(t)) = 0, \;\; \hat{v}_i(t) \geq 0, \;\; i \in I, \tag{3}$$

$$\sum_{j \in J} \hat{\lambda}_j(t) = 1, \;\; \hat{\lambda}_j(t) \geq 0, \;\; j \in J. \tag{4}$$

Proof. Since $\hat{x}$ is an efficient solution of (VCTP), then by Lemma 2, $\hat{x}$ solves $\mathrm{P}_r(\hat{x})$ for all $r \in J$. Hence, by Lemma 5 there exist $\hat{w} \in L_\infty([0,T];\mathbb{R}^{k-1})$ and $\hat{u} \in L_\infty([0,T];\mathbb{R}^m)$ such that

$$\nabla f_r(t,\hat{x}(t)) + \sum_{\substack{p \in J \\ p \neq r}} \hat{w}_p(t) \nabla f_p(t,\hat{x}(t)) + \sum_{j \in J} \hat{u}_i(t) \nabla g_i(t,\hat{x}(t)) = 0 \quad \text{a.e. in } [0,T], \tag{5}$$

$$\hat{u}_i(t) g_i(t, \hat{x}(t)) = 0, \quad i \in I, \text{ a.e. in } [0, T], \tag{6}$$

$$\hat{u}_i(t) \geq 0, \ \ i \in I, \text{ a.e. in } [0, T], \tag{7}$$

$$\hat{w}_p(t) \geq 0, \ \ p \in J \setminus \{r\}, \text{ a.e. in } [0, T]. \tag{8}$$

Now, multiplying all terms in (5) and (6) by $\frac{1}{1+\sum_{\substack{p\in J\\ p\neq r}} \hat{w}_p(t)}$, and setting

$$\hat{\lambda}_r(t) = \frac{1}{1 + \sum_{\substack{p\in J\\ p\neq r}} \hat{w}_p(t)}, \ \ t \in [0, T],$$

$$\hat{\lambda}_p(t) = \frac{\hat{w}_p(t)}{1 + \sum_{\substack{p\in J\\ p\neq r}} \hat{w}_p(t)}, \ \ p \in J \setminus \{r\}, \ t \in [0, T],$$

and

$$\hat{v}_i(t) = \frac{\hat{u}_i(t)}{1 + \sum_{\substack{p\in J\\ p\neq r}} \hat{w}_p(t)}, \ \ i \in I, \ t \in [0, T],$$

we conclude that conditions (2)-(4) hold. □

Remark 7. *Assume that $g_i(t, \cdot)$ is a concave function almost everywhere in $[0, T]$, $i \in I$. We say that (VCTP) satisfies the constraint qualification (SCQ), if there exist $x \in \Omega$ and $\hat{b} > 0$ such that, for almost every $t \in [0, T]$,*

$$g_i(t, x(t)) \geq \beta, \ \ i \in I_{\hat{b}}(t),$$

for some $\beta > 0$.

Note that (SCQ) is a continuous-time version of the Slater constraint qualification. In [2], Monte and Oliveira showed that (SCQ) is a sufficient condition for (MFCQ) under concavity assumption.

Corollary 8. *Let $\hat{x}$ be an efficient solution for (VCTP). Assume that $g_i(t, \cdot)$ is a concave function almost everywhere in $[0, T]$, $i \in I$. If $P_r(\hat{x})$ satisfies the constraint qualification (SCQ) for some r and corresponding system is regular, then there exists $(\hat{\lambda}, \hat{v}) \in L_\infty([0, T]; \mathbb{R}^k \times \mathbb{R}^m)$ such that, for almost every $t \in [0, T]$,*

$$\sum_{j\in J} \hat{\lambda}_j(t) \nabla f_j(t, \hat{x}(t)) + \sum_{i\in I} \hat{v}_i(t) \nabla g_i(t, \hat{x}(t)) = 0, \tag{9}$$

$$\hat{v}_i(t) g_i(t, \hat{x}(t)) = 0, \ \ \hat{v}_i(t) \geq 0, \ \ i \in I, \tag{10}$$

$$\sum_{j\in J} \hat{\lambda}_j(t) = 1, \ \ \hat{\lambda}_j(t) \geq 0, \ \ j \in J. \tag{11}$$

4. KARUSH-KUHN-TUCKER SUFFICIENT OPTIMALITY CONDITIONS

In this section we will present sufficient optimality criteria of the Karush-Kuhn-Tucker type for (VCTP). The proofs of the main theorems in this section, will be based primarily on the generalized concavity assumptions imposed on the functions involved, and will not require the regularity condition and application of any theorems for scalar problems. We assume that the definitions of quasiconcave, pseudoconcave and strictly pseudoconcave functions are known to the reader. For these, the reader is referred to [9, 10].

Theorem 9. *Assume that there exist a feasible solution $\hat{x}$ for (VCTP) and $(\hat{\lambda}, \hat{v}) \in L_\infty([0,T]; \mathbb{R}^k \times \mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,*

$$\sum_{j\in J} \hat{\lambda}_j(t)\nabla f_j(t, \hat{x}(t)) + \sum_{i\in I} \hat{v}_i(t)\nabla g_i(t, \hat{x}(t)) = 0, \tag{12}$$

$$\hat{v}_i(t) g_i(t, \hat{x}(t)) = 0, \ \hat{v}_i(t) \geq 0, \ i \in I, \tag{13}$$

$$\sum_{j\in J} \hat{\lambda}_j(t) = 1, \ \hat{\lambda}_j(t) > 0, \ j \in J. \tag{14}$$

If the function $\sum_{j\in J} \hat{\lambda}_j(t) f_j(t, \cdot)$ is pseudoconcave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$, and $\sum_{i\in I} \hat{v}_i g_i(t, \cdot)$ is quasiconcave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$, then $\hat{x}$ is an efficient solution for (VCTP).

Proof. From $x \in \Omega$ and (13), we have

$$\hat{v}_i(t) g_i(t, x(t)) \geq \hat{v}_i(t) g_i(t, \hat{x}(t)) = 0, \quad i \in I, \quad \forall x \in \Omega \ \text{ a.e. in } [0,T],$$

i.e.

$$\sum_{i\in I} \hat{v}_i(t) g_i(t, x(t)) \geq \sum_{i\in I} \hat{v}_i(t) g_i(t, \hat{x}(t)), \ \ \forall x \in \Omega \ \text{ a.e. in } [0,T]. \tag{15}$$

Since $\sum_{i\in I} \hat{v}_i(t) g_i(t, \cdot)$ is quasiconcave at $x(t) = \hat{x}(t)$ a.e. in $[0,T]$, (15) yields

$$\sum_{i\in I} \hat{v}_i(t)\nabla g_i(t, \hat{x}(t))^T (x(t) - \hat{x}(t)) \geq 0, \ \ \forall x \in \Omega, \ \text{a.e. in } [0,T]. \tag{16}$$

From (12) and (16), we obtain

$$\sum_{j\in J} \hat{\lambda}_j(t)\nabla f_j(t, \hat{x}(t))^T (x(t) - \hat{x}(t)) \leq 0, \ \ \forall x \in \Omega, \ \text{a.e. in } [0,T].$$

Since $\sum_{j\in J} \hat{\lambda}_j(t) f_j(t, \cdot)$ is pseudoconcave at $\hat{x}(t)$ a.e. in $[0,T]$,

$$\sum_{j\in J} \hat{\lambda}_j(t) f_j(t, \hat{x}(t)) \geq \sum_{j\in J} \hat{\lambda}_j(t) f_j(t, x(t)), \ \ \forall x \in \Omega, \ \ \text{a.e. in } [0,T].$$

Integrating the previous inequality from 0 to T, we have

$$\int_0^T \sum_{j\in J} \hat{\lambda}_j(t) f_j(t,\hat{x}(t))\,dt \ge \int_0^T \sum_{j\in J} \hat{\lambda}_j(t) f_j(t,x(t))\,dt, \quad \forall x \in \Omega. \tag{17}$$

Let us assume that $\hat{x}$ is not efficient for (VCTP). Then there exists some point $\bar{x} \in \Omega$ such that

$$f_j(t,\bar{x}(t))\,dt \ge f_j(t,\hat{x}(t))\,dt \quad \text{for all } j \in J, \text{ a.e. in } [0,T],$$

and for at least one index i is

$$f_i(t,\bar{x}(t))\,dt > f_i(t,\hat{x}(t))\,dt \text{ a.e. in } [0,T].$$

Because every $\hat{\lambda}_j(t)$, $j \in J$ was assumed to be positive a.e. in $[0,T]$, we obtain

$$\int_0^T \sum_{j\in J} \hat{\lambda}_j(t) f_j(t,\bar{x}(t))\,dt > \int_0^T \sum_{j\in J} \hat{\lambda}_j(t) f_j(t,\hat{x}(t))\,dt.$$

This is a contradiction with (17), thus $\hat{x}$ must be an efficient solution for (VCTP). □

Theorem 10. *Assume that there exist a feasible solution $\hat{x}$ for (VCTP) and $(\hat{\lambda},\hat{v}) \in L_\infty([0,T];\mathbb{R}^k \times \mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,*

$$\sum_{j\in J} \hat{\lambda}_j(t) \nabla f_j(t,\hat{x}(t)) + \sum_{i\in I} \hat{v}_i(t) \nabla g_i(t,\hat{x}(t)) = 0, \tag{18}$$

$$\hat{v}_i(t) g_i(t,\hat{x}(t)) = 0, \;\; \hat{v}_i(t) \ge 0, \;\; i \in I, \tag{19}$$

$$\sum_{j\in J} \hat{\lambda}_j(t) = 1, \;\; \hat{\lambda}_j(t) > 0, \;\; j \in J. \tag{20}$$

If the function $\sum_{j\in J} \hat{\lambda}_j f_j(t,\cdot)$ is quasiconcave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$ and $\sum_{i\in I} \hat{v}_i g_i(t,\cdot)$ is strictly pseudoconcave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$, then $\hat{x}$ is an efficient solution for (VCTP).

Proof. From $x \in \Omega$ and (19), we have

$$\hat{v}_i(t) g_i(t,x(t)) \ge \hat{v}_i(t) g_i(t,\hat{x}(t)) = 0, \;\; i \in I, \;\; \forall x \in \Omega \;\; \text{a.e. in } [0,T],$$

i.e.

$$\sum_{i\in I} \hat{v}_i(t) g_i(t,x(t)) \ge \sum_{i\in I} \hat{v}_i(t) g_i(t,\hat{x}(t)), \;\; \forall x \in \Omega \text{ a.e. in } [0,T]. \tag{21}$$

Since

$$\sum_{i\in I} \hat{v}_i(t) g_i(t,\cdot)$$

is strictly pseudoconcave at $x(t) = \hat{x}(t)$ a.e. in $[0,T]$, (21) yields

$$\sum_{i\in I} \hat{v}_i(t)\nabla g_i(t,\hat{x}(t))^T(x(t)-\hat{x}(t)) > 0, \quad \forall x \in \Omega, \tag{22}$$

such that $x(t) \neq \hat{x}(t)$ a.e. in $[0,T]$. From (18) and (22), we obtain

$$\sum_{j\in J} \hat{\lambda}_j(t)\nabla f_j(t)(t,\hat{x}(t))^T(x(t)-\hat{x}(t)) < 0, \quad \forall x \in \Omega,$$

such that $x(t) \neq \hat{x}(t)$ a.e. in $[0,T]$.
Since $\sum_{j\in J} \hat{\lambda}_j(t) f_j(t,\cdot)$ is quasiconcave at $\hat{x}(t)$ a.e. in $[0,T]$,

$$\sum_{j\in J} \hat{\lambda}_j(t) f_j(t,\hat{x}(t)) > \sum_{j\in J} \hat{\lambda}_j(t) f_j(t,x(t)), \quad \forall x \in \Omega, \text{ such that } x(t) \neq \hat{x}(t),$$

a.e. in $[0,T]$. Therefore, we have

$$\sum_{j\in J} \hat{\lambda}_j(t) f_j(t,\hat{x}(t)) \geq \sum_{j\in J} \hat{\lambda}_j(t) f_j(t,x(t)), \quad \forall x \in \Omega, \text{ a.e. in } [0,T].$$

Integrating the previous inequality from 0 to T, we obtain

$$\int_0^T \sum_{j\in J} \hat{\lambda}_j(t) f_j(t,\hat{x}(t))\,dt \geq \int_0^T \sum_{j\in J} \hat{\lambda}_j(t) f_j(t,x(t))\,dt \quad \forall x \in \Omega.$$

So, as in the proof of Theorem 9, we conclude that $\hat{x}$ is an efficient solution for (VCTP). $\square$

Following the same approach, we obtain sufficient conditions for (VCTP) without complementary slackness condition.

Theorem 11. *Let $\hat{x}$ be a feasible solution for (VCTP) and A denotes the index set of all the binding inequality constraints at $\hat{x}(t)$, i.e.,*
$A = \{\, i \in I \,:\, g_i(t,\hat{x}(t)) = 0 \quad a.e.\ in\ [0,T]\,\}$. Assume that for each $i \in A$, $g_i(t,\cdot)$ is quasiconcave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$ and there exists $(\hat{\lambda},\hat{v}) \in L_\infty([0,T];\mathbb{R}^k \times \mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,

$$\sum_{j\in J} \hat{\lambda}_j(t)\nabla f_j(t,\hat{x}(t)) + \sum_{i\in I} \hat{v}_i(t)\nabla g_i(t,\hat{x}(t)) = 0, \tag{23}$$

$$\hat{v}_i(t) \geq 0, \ i \in A, \tag{24}$$

$$\hat{v}_i(t) = 0, \ i \in I \setminus A, \tag{25}$$

$$\sum_{j\in J} \hat{\lambda}_j(t) = 1, \ \hat{\lambda}_j(t) > 0, \quad j \in J. \tag{26}$$

If the function $\sum_{j\in J} \hat{\lambda}_j(t) f_j(t,\cdot)$ is pseudoconcave in its second argument at $\hat{x}(t)$ almost everywhere in $[0,T]$, then $\hat{x}$ is an efficient solution for (VCTP).

Proof. For any feasible x,

$$g_i(t, x(t)) \geq g_i(t, \hat{x}(t)) = 0, \ i \in A, \ \text{ a.e. in } [0, T].$$

By the quasiconcavity $g_i(t, \cdot)$ at $\hat{x}(t)$, $i \in A$ a.e. in $[0, T]$, we have

$$\nabla \hat{g}_i(t, \hat{x}(t))^T (x(t) - \hat{x}(t)) \geq 0, \ i \in A, \ \text{a.e. in } [0, T].$$

Since $\hat{v}_i(t) \geq 0$, $i \in A$, a.e. in $[0, T]$, we obtain

$$\sum_{i \in A} \hat{v}_i(t) \nabla g_i(t, \hat{x}(t))^T (x(t) - \hat{x}(t)) \geq 0, \ \ \forall x \in \Omega, \ \text{a.e. in } [0, T]. \tag{27}$$

From (23) and (27), we have

$$\sum_{j \in J} \hat{\lambda}_j(t) \nabla f_j(t, \hat{x}(t))^T (x(t) - \hat{x}(t)) \leq 0, \ \ \forall x \in \Omega, \ \text{a.e. in } [0, T]. \tag{28}$$

Since $\sum_{j \in J} \hat{\lambda}_j(t) f_j(t, \cdot)$ is pseudoconcave in its second argument at $\hat{x}(t)$ almost everywhere in $[0, T]$, we obtain

$$\sum_{j \in J} \hat{\lambda}_j f_j(t, \hat{x}(t)) \geq \sum_{j \in J} \hat{\lambda}_j f_j(t, x(t)), \ \ \forall x \in \Omega, \ \text{a.e. in } [0, T].$$

It follows

$$\int_0^T \sum_{j \in J} \hat{\lambda}_j(t) f_j(t, \hat{x}(t))\, dt \geq \int_0^T \sum_{j \in J} \hat{\lambda}_j(t) f_j(t, x(t))\, dt \quad \forall x \in \Omega.$$

Hence, $\hat{x}$ must be an efficient solution for (VCTP). □

5. CONCLUSIONS

This paper addressed the vector continuous-time problems. The main auxiliary results employed in the derivation of the necessary optimality criteria are a new version of the Karush–Kuhn–Tucker-type optimality conditions for scalar problem and scalarization method. Sufficient optimality conditions were given in (strictly) generalized concavity concept. Obtaining duality results is going to be a topic of future works. It would be of interest to see how the similar approach can be extended to examine optimality conditions and duality as well as their applications on the nonsmooth vector continuous-time problems.

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