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SYMMETRIC DUALITY IN COMPLEX SPACES OVER CONES

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Abstract: Duality theory plays an important role in optimization theory. It has been extensively used for many theoretical and computational problems in mathematical programming. In this paper duality results are established for first and second order Wolfe and Mond-Weir type symmetric dual programs over general polyhedral cones in complex spaces. Corresponding duality relations for nondifferentiable case are also stated. This work will also remove inconsistencies in the earlier work from the literature.

Keywords: Symmetric Duality, Polyhedral Cones, Pseudo-Convex.

MSC: 90C26, 90C27, 49N15.

1. INTRODUCTION

In 1966, Levinson [21], extending some theorems of linear programming to complex space, introduced complex mathematical programming. Since then a great deal of work has been done in this field. His duality results were extended to quadratic programs by Hanson and Mond [16]. Ben-Israel [5] studied duality for linear programs over general polyhedral cones. Abrams and Ben-Israel [3] surveyed contemporary work in complex mathematical programming and discussed some applications.

The complex version of well known Kuhn-Tucker necessary/sufficient optimality conditions and duality in nonlinear programming were obtained by Abrams and Ben-Israel [2] and Abrams [1], while Craven and Mond [8, 9] studied Fritz John conditions. In [7] Craven and Mond proved the converse and symmetric duality theorems for complex space. Some work in complex mathematical programming may be seen in [1-3,7-9,11,16-23].

Kaul and Sharma [18] presented a pair of differentiable symmetric dual nonlinear programs over special polyhedral cones and established weak duality theorem under convexity/concavity assumptions.They have also given an example to show that the strong duality theorem is not true. Some work on symmetric duality can be seen in [6,10,12-14,25,26].

Mishra and Rueda [25], assuming F-convexity established duality theorems for Wolfe and Mond-Weir type first and second order symmetric dual programs in complex space. Their four pairs of primal and dual problems contain the constraints $z \geq 0$ and $v \geq 0$ respectively, which appear to have no meaning for complex vector variables z and v. Also, in complex mathematical programming the constraints are always defined over polyhedral cones and their polar cones [1-3, 7-9, 11, 15-23] but the constraints of the primal and dual problems studied in Mishra and Rueda [25] do not fit into cones.

In the present article, we have studied the primal and dual models of Kaul and Sharma [18] over general polyhedral cones. The paper is divided into ten sections. Section 2 contains some basic notations and definitions, used in the paper. Weak duality theorems for Wolfe type first and second order symmetric dual programs are obtained in Sections 3 and 5 respectively. The duality relations for Mond-Weir type first and second order models are discussed in Sections 4 and 8 respectively. The anomalies contained in Mishra and Rueda [25] are given in section 4. Some special cases are discussed in Section 6. In Section 7, we make two important observations regarding the strong duality theorem. One is that the example in [18] does not satisfy the nonsingularity assumption required for the strong duality theorem and so seems to be inappropriate. Secondly, unlike the statement in [25], we could not obtain the proof of the strong duality theorem on the lines of Dantzig et al.[10]. In sections 9 and 10, we state weak duality theorems for Wolfe and Mond-Weir type nondifferentiable first and second order primal and dual pairs. This work will remove inconsistencies in the earlier work in Mishra and Rueda [25] and Mishra [24]

2. PRELIMINARIES

Let $C^n(R^n)$ denote the n-dimensional complex (real) vector space and $C^{m\times n}(R^{m\times n})$ the set of $m \times n$ complex (real) matrices. For $A = (a_{ij}) \in C^{m\times n}$, $\bar{A} = (\bar{a}_{ij})$ is the conjugate of A. $A^T = (a_{ji})$ is the transpose of A and $A^H = \bar{A}^T = (\bar{a}_{ji})$ is the conjugate transpose of A. For $x \in C^n$, $Re\ x = (Re\ x_j) \in R^n$ is the real part of x, $arg\ x = (arg\ x_j) \in R^n$ is argument of x. Let $R^n_+ = \{x \in R^n : x_j \geq 0 (j = 1, 2, \cdots, n)\}$ be non-negative orthant of R^n.

For a nonempty set $S \subset C^n$,

$$S^* = \{y \in C^n : x \in S \Rightarrow Re(y^H x) \geq 0\}$$

denotes the dual (or positive polar) of S. For a complex function $f : C^n \times C^n \times C^m \times C^m \mapsto C$ analytic in the $2n$-variables (w^1, w^2) at the point $(z_\circ, \bar{z}_\circ) \in C^n \times C^n$, the gradients are given as

$$\nabla_z f(z_\circ, \bar{z}_\circ, \eta) = \left[\frac{\partial f}{\partial w_i^1}(z_\circ, \bar{z}_\circ, \eta)\right],\ i = 1, 2, \ldots, n$$

$$\nabla_{\bar{z}} f(z_\circ, \bar{z}_\circ, \eta) = \left[\frac{\partial f}{\partial w_i^2}(z_\circ, \bar{z}_\circ, \eta)\right],\ i = 1, 2, \ldots, n,$$

where $\eta = (w_1, w_2) \in C^m \times C^m$.

Definition 2.1. [1]. A nonempty set $S \subset C^n$ is said to be a polyhedral cone if for some positive integer k and $A \in C^{n\times k}$, $S = AR^k_+ = \{Ax : x \in R^k_+\}$, i.e., S is generated by finitely many vectors (the columns of A).

Definition 2.2. [1, 17]. The real part of f is said to be convex at $(z_\circ, \bar{z_\circ})$ with respect to R_+ for fixed $(w, \bar{w}) \in C^m \times C^m$, if

$$Re\ [f(z, \bar{z}, w, \bar{w}) - f(z_\circ, \bar{z}_\circ, w, \bar{w})] \geq Re[(z - z_\circ)^T \nabla_z f(z_\circ, \bar{z}_\circ, w, \bar{w}) + (z - z_\circ)^H \nabla_{\bar{z}} f(z_\circ, \bar{z}_\circ, w, \bar{w})]$$

for all $z \in C^n$.

Definition 2.3. [1]. The real part of f is said to be pseudoconvex at $(z_\circ, \bar{z}_\circ)$ with respect to R_+ for fixed $(w, \bar{w}) \in C^m \times C^m$, if

$$Re\ [(z - z_\circ)^T \nabla_z f(z_\circ, \bar{z_\circ}, w, \bar{w}) + (z - z_\circ)^H \nabla_{\bar{z}} f(z_\circ, \bar{z_\circ}, w, \bar{w})] \geq 0$$

$$\Rightarrow Re\ f(z, \bar{z}, w, \bar{w}) \geq Re\ f(z_\circ, \bar{z}_\circ, w, \bar{w})$$

for all $z \in C^n$.

Definition 2.4. The real part of f is said to be pseudoinvex at $(z_\circ, \bar{z}_\circ)$ with respect to R_+ for fixed $(w, \bar{w}) \in C^m \times C^m$, if there exists a function $\eta : C^n \times C^n \mapsto C^n$ such that

$$Re\ [\eta^T(z, z_\circ) \nabla_z f(z_\circ, \bar{z_\circ}, w, \bar{w}) + \eta^H(z, z_\circ) \nabla_{\bar{z}} f(z_\circ, \bar{z_\circ}, w, \bar{w})] \geq 0$$

$$\Rightarrow Re\ f(z, \bar{z}, w, \bar{w}) \geq Re\ f(z_\circ, \bar{z}_\circ, w, \bar{w})$$

for all $z \in C^n$.

Definition 2.5. A functional $F : C^n \times C^n \times C^n \mapsto R$ is said to be sublinear in the third component, if for any $z, u \in C^n$
(A) $F(z, u, a_1 + a_2) \leq F(z, u, a_1) + F(z, u, a_2)$, for any $a_1, a_2 \in C^n$
(B) $F(z, u, \alpha a) = \alpha\, F(z, u, a)$, for any $\alpha \in R_+$ and $a \in C^n$.

Definition 2.6. [25]. The real part of f is said to be F-convex at $(z_\circ, \bar{z_\circ})$ with respect to R_+ for fixed $(w, \bar{w}) \in C^m \times C^m$, if

$$Re\,[f(z,\bar{z},w,\bar{w}) - f(z_\circ,\bar{z}_\circ,w,\bar{w})] \geq F(z, z_\circ; \overline{\nabla_z f(z_\circ,\bar{z}_\circ,w,\bar{w})} + \nabla_{\bar{z}} f(z_\circ,\bar{z}_\circ,w,\bar{w}))$$

for all $z \in C^n$ and for some arbitrary sublinear functional F.

Definition 2.7. [25]. The real part of f is said to be F-concave at $(w_\circ, \bar{w}_\circ)$ with respect to R_+ for fixed $(z, \bar{z}) \in C^n \times C^n$, if

$$Re\,[f(z,\bar{z},w_\circ,\bar{w}_\circ) - f(z,\bar{z},w,\bar{w})] \geq F(w, w_\circ; -\overline{\nabla_w f(z,\bar{z},w_\circ,\bar{w}_\circ)} - \nabla_{\bar{w}} f(z,\bar{z},w_\circ,\bar{w}_\circ))$$

for all $w \in C^m$ and for some arbitrary sublinear functional F.

Definition 2.8. The real part of f is said to be second order F-convex at $(z_\circ, \bar{z}_\circ)$ with respect to R_+ for fixed $(w, \bar{w}) \in C^m \times C^m$, if

$$\begin{gathered} Re\ [f(z,\bar{z},w,\bar{w}) - f(z_\circ,\bar{z}_\circ,w,\bar{w}) + \frac{1}{2} r_1^T (\nabla_{zz} + \nabla_{z\bar{z}}) \\ f(z_\circ,\bar{z}_\circ,w,\bar{w}) r_1 + \frac{1}{2} r_1^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(z_\circ,\bar{z}_\circ,w,\bar{w}) r_1] \\ \geq F(z, z_\circ; \overline{\nabla_z f(z_\circ,\bar{z}_\circ,w,\bar{w})} + \overline{(\nabla_{zz} + \nabla_{z\bar{z}})} \\ f(z_\circ,\bar{z}_\circ,w,\bar{w}) r_1 + \nabla_{\bar{z}} f(z_\circ,\bar{z}_\circ,w,\bar{w}) + (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(z_\circ,\bar{z}_\circ,w,\bar{w}) r_1) \end{gathered}$$

for all $z \in C^n$, $r_1 \in C^n$ and for some arbitrary sublinear functional F.

Definition 2.9. The real part of f is said to be second order F-concave at $(w_\circ, \bar{w}_\circ)$ with respect to R_+ for fixed $(z, \bar{z}) \in C^n \times C^n$, if

$$\begin{gathered} Re\ [f(z,\bar{z},w_\circ,\bar{w}_\circ) - f(z,\bar{z},w,\bar{w}) - \frac{1}{2} r_2^T (\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w_\circ, \\ \bar{w}_\circ) r_2 - \frac{1}{2} r_2^H (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w_\circ,\bar{w}_\circ) r_2] \\ \geq F(w, w_\circ; -\overline{\nabla_w f(z,\bar{z},w_\circ,\bar{w}_\circ)} - \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w_\circ,\bar{w}_\circ) r_2} - \nabla_{\bar{w}} \\ f(z,\bar{z},w_\circ,\bar{w}_\circ) - (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w_\circ,\bar{w}_\circ) r_2) \end{gathered}$$

for all $w \in C^m$, $r_2 \in C^m$ and for some arbitrary sublinear functional F.

Definition 2.10. [25]. The real part of f is said to be F-pseudoconvex at $(z_\circ, \bar{z}_\circ)$ with respect to R_+ for fixed $(w, \bar{w}) \in C^m \times C^m$, if

$$F(z, z_\circ; \overline{\nabla_z f(z_\circ, \bar{z}_\circ, w, \bar{w})} + \nabla_{\bar{z}} f(z_\circ, \bar{z}_\circ, w, \bar{w})) \geq 0$$
$$\Rightarrow Re\ f(z, \bar{z}, w, \bar{w}) \geq Re\ f(z_\circ, \bar{z}_\circ, w, \bar{w})$$

for all $z \in C^n$ and for some arbitrary sublinear functional F.

Definition 2.11. [25]. The real part of f is said to be F-pseudoconcave at $(w_\circ, \bar{w}_\circ)$ with respect to R_+ for fixed $(z, \bar{z}) \in C^n \times C^n$, if

$$F(w, w_\circ; -\overline{\nabla_w f(z, \bar{z}, w_\circ, \bar{w}_\circ)} - \nabla_{\bar{w}} f(z, \bar{z}, w_\circ, \bar{w}_\circ)) \geq 0$$
$$\Rightarrow Ref(z, \bar{z}, w, \bar{w}) \leq Ref(z, \bar{z}, w_\circ, \bar{w}_\circ)$$

for all $w \in C^m$ and for some arbitrary sublinear functional F.

Definition 2.12. The real part of f is said to be second-order F-pseudoconvex at $(z_\circ, \bar{z}_\circ)$ with respect to R_+ for fixed $(w, \bar{w}) \in C^m \times C^m$, if

$$F(z, z_\circ; \overline{\nabla_z f(z_\circ, \bar{z}_\circ, w, \bar{w})} + \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(z_\circ, \bar{z}_\circ, w, \bar{w}) r_1}$$
$$+ \nabla_{\bar{z}} f(z_\circ, \bar{z}_\circ, w, \bar{w}) + (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(z_\circ, \bar{z}_\circ, w, \bar{w}) r_1) \geq 0$$
$$\Rightarrow Re\ [f(z, \bar{z}, w, \bar{w}) - f(z_\circ, \bar{z}_\circ, w, \bar{w}) + \frac{1}{2} r_1^T (\nabla_{zz} + \nabla_{z\bar{z}}) f(z_\circ, \bar{z}_\circ, w, \bar{w}) r_1$$
$$+ \frac{1}{2} r_1^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(z, \bar{z}_\circ, w, \bar{w}) r_1] \geq 0$$

for all $z \in C^n, r_1 \in C^n$ and for some arbitrary sublinear functional F.

Definition 2.13. The real part of f is said to be second-order F-pseudoconcave at $(w_\circ, \bar{w}_\circ)$ with respect to R_+ for fixed $(z, \bar{z}) \in C^n \times C^n$, if

$$F(w, w_\circ; -\overline{\nabla_w f(z, \bar{z}, w_\circ, \bar{w}_\circ)} - \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z, \bar{z}, w_\circ, \bar{w}_\circ) r_2}$$
$$- \nabla_{\bar{w}} f(z, \bar{z}, w_\circ, \bar{w}_\circ) - (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z, \bar{z}, w_\circ, \bar{w}_\circ) r_2) \geq 0$$
$$\Rightarrow Re\ [f(z, \bar{z}, w_\circ, \bar{w}_\circ) - f(z, \bar{z}, w, \bar{w}) - \frac{1}{2} r_2^T (\nabla_{ww} + \nabla_{w\bar{w}}) f(z, \bar{z}, w_\circ, \bar{w}_\circ) r_2$$
$$- \frac{1}{2} r_2^H (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z, \bar{z}, w_\circ, \bar{w}_\circ) r_2] \geq 0$$

for all $w \in C^m,\ r_2 \in C^m$ and for some arbitrary sublinear functional F.

Lemma 2.1. (Generalized Schwartz Inequality). Let A be a positive semidefinite Hermitian matrix of order n. Then for all $z, w \in C^n$,

$$Re\ (z^H A w) \leq (z^H A z)^{\frac{1}{2}} (w^H A w)^{\frac{1}{2}}.$$

Remark 2.1.

1. If we take $F(z, z_\circ, \xi) = Re\ [(z - z_\circ)^H \xi]$, then the definitions of $F-$ pseudoconvexity/pseudoconcavity reduce to pseudoconvexity/pseudoconcavity.
2. Let $F(z, z_\circ, \xi) = Re\ [\eta^H(z, z_\circ)\xi]$, then the definitions of $F-$ pseudoconvexity/pseudoconcavity become the definitions of $\eta-$ pseudoconvexity/pseudoconcavity.
3. If $r_1 = 0$, then the definitions of second-order $F-$ pseudoconvexity/pseudoconcavity yield $F-$ pseudoconvexity/pseudoconcavity.

3. WOLFE TYPE FIRST ORDER SYMMETRIC DUALITY

In this section, we present the following pair of first order Wolfe type symmetric primal-dual pair over general polyhedral cones in complex spaces and establish weak duality theorems.

Primal (WP1)

Minimize $\phi(z, \bar{z}, w, \bar{w}) = Re[f(z, \bar{z}, w, \bar{w}) - w^H \overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - w^H \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})]$
subject to

$$-\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}) \in S, \tag{1}$$

$$z \in T. \tag{2}$$

Dual (WD1)

Maximize $\psi(u, \bar{u}, v, \bar{v}) = Re[f(u, \bar{u}, v, \bar{v}) - u^H \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} - u^H \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})]$
subject to

$$\overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v}) \in T^*, \tag{3}$$

$$v \in S^* \tag{4}$$

where $f : C^n \times C^n \times C^m \times C^m \mapsto C$ is analytic, S and T are general polyhedral cones in C^m and C^n respectively.

Theorem 3.1 (Weak Duality). Let $(z, \bar{z}, w, \bar{w})$ and $(u, \bar{u}, v, \bar{v})$ be feasible solutions of (WP1) and (WD1) respectively. If $Ref(.,.,v,\bar{v})$ is convex at $(u, \bar{u})$ and $Re\ f(z, \bar{z}, .,.)$ is concave at $(w, \bar{w})$ with respect to R_+, then

$$\phi(z, \bar{z}, w, \bar{w}) \geq \psi(u, \bar{u}, v, \bar{v}).$$

Proof. Using (1) and (4), we have

$$Re[-v^H \overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - v^H \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})] \geq 0. \tag{5}$$

Similarly, the constraints (2) and (3) yield

$$Re[z^H \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + z^H \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})] \geq 0. \tag{6}$$

Adding the inequalities (5) and (6), we get

$$Re[-v^H\overline{\nabla_w f(z,\bar z,w,\bar w)}-v^H\nabla_{\bar w} f(z,\bar z,w,\bar w)+z^H\overline{\nabla_z f(u,\bar u,v,\bar v)}+z^H\nabla_{\bar z} f(u,\bar u,v,\bar v) \geq 0. \tag{7}$$

Further, convexity of $Ref(.,.,v,\bar v)$ at $(u,\bar u)$ gives

$$Re[f(z,\bar z,v,\bar v)-f(u,\bar u,v,\bar v)] \geq Re[(z-u)^T\nabla_z f(u,\bar u,v,\bar v)+(z-u)^H\nabla_{\bar z} f(u,\bar u,v,\bar v)] \tag{8}$$

and by concavity of $Ref(z,\bar z,.,.)$ at $(w,\bar w)$, we have

$$Re[f(z,\bar z,v,\bar v)-f(z,\bar z,w,\bar w)] \leq Re[(v-w)^T\nabla_w f(z,\bar z,w,\bar w)+(v-w)^H\nabla_{\bar w} f(z,\bar z,w,\bar w)]. \tag{9}$$

It follows from (8) and (9) that
$Re[f(z,\bar z,w,\bar w)-f(u,\bar u,v,\bar v)]$

$$\begin{aligned}\geq Re[(z-u)^T\nabla_z f(u,\bar u,v,\bar v)+(z-u)^H\nabla_{\bar z} f(u,\bar u,v,\bar v)\\ -(v-w)^T\nabla_w f(z,\bar z,w,\bar w)\\ -(v-w)^H\nabla_{\bar w} f(z,\bar z,w,\bar w)].\end{aligned} \tag{10}$$

Now,
$\phi(z,\bar z,w,\bar w)-\psi(u,\bar u,v,\bar v)$

$$= Re[f(z,\bar z,w,\bar w)-w^H\overline{\nabla_w f(z,\bar z,w,\bar w)}-w^H\nabla_{\bar w} f(z,\bar z,w,\bar w)$$
$$-f(u,\bar u,v,\bar v)+u^H\overline{\nabla_z f(u,\bar u,v,\bar v)}+u^H\nabla_{\bar z} f(u,\bar u,v,\bar v)]$$

$$\geq Re[(z-u)^T\nabla_z f(u,\bar u,v,\bar v)+(z-u)^H\nabla_{\bar z} f(u,\bar u,v,\bar v)-(v-w)^T\nabla_w f(z,\bar z,w,\bar w)$$
$$-(v-w)^H\nabla_{\bar w} f(z,\bar z,w,\bar w)-w^T\nabla_w f(z,\bar z,w,\bar w)-w^H\nabla_{\bar w} f(z,\bar z,w,\bar w)$$
$$+u^T\nabla_z f(u,\bar u,v,\bar v)+u^H\nabla_{\bar z} f(u,\bar u,v,\bar v)]$$

$$= Re[z^T\nabla_z f(u,\bar u,v,\bar v)+z^H\nabla_{\bar z} f(u,\bar u,v,\bar v)-v^T\nabla_w f(z,\bar z,w,\bar w)-v^H\nabla_{\bar w} f(z,\bar z,w,\bar w)$$

≥ 0 (using (7) and (10)).
Hence

$$\phi(z,\bar z,w,\bar w) \geq \psi(u,\bar u,v,\bar v).$$

Remark 3.1. It may be noted that the second term in the objective functions of the two problems contain w^H and u^H respectively instead of w^T and u^T as in Mishra and Rueda [25]. With the objectives as given in [25] we could not obtain the above weak duality theorem. Their other models also need similar corrections.

Theorem 3.2. (Weak Duality). Let $(z, \bar{z}, w, \bar{w})$ and $(u, \bar{u}, v, \bar{v})$ be feasible solutions of (WP1) and (WD1) respectively. If $Ref(.,., v, \bar{v})$ is F_1-convex at $(u, \bar{u})$ and $Ref(z, \bar{z}, ., .)$ is F_2-concave at $(w, \bar{w})$ with respect to R_+, and

(*i*) $F_1(z, u; \xi) + Re[u^H \xi] \geq 0$ for $\xi \in T^*$,
(*ii*) $F_2(v, w; \eta) + Re[w^H \eta] \geq 0$ for $\eta \in S$,

then
$$\phi(z, \bar{z}, w, \bar{w}) \geq \psi(u, \bar{u}, v, \bar{v}).$$

Proof. Using F_1-convexity of $Re\, f(.,., v, \bar{v})$ at $(u, \bar{u})$ and F_2-concavity of $Re\, f(z, \bar{z}, ., .)$ at $(w, \bar{w})$, we have

$$Re\,[f(z, \bar{z}, v, \bar{v}) - f(u, \bar{u}, v, \bar{v})] \geq F_1(z, u; \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})) \text{ and}$$

$$Re\,[f(z, \bar{z}, w, \bar{w}) - f(z, \bar{z}, v, \bar{v})] \geq F_2(v, w; -\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})).$$

Adding these two inequalities, we get
$$Re\,[f(z, \bar{z}, w, \bar{w}) - f(u, \bar{u}, v, \bar{v})] \geq F_1(z, u; \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})) +$$
$$F_2(v, w; -\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})). \tag{11}$$

On taking
$$\xi = \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v}) \in T^*, \text{ and}$$
$$\eta = -\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}) \in S,$$

the assumptions (*i*) and (*ii*), respectively reduce to

$$F_1(z, u; \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})) \geq Re\,[-u^H \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} - u^H \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})], \tag{12}$$

and

$$F_2(v, w; -\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})) \geq Re\,[w^H \overline{\nabla_w f(z, \bar{z}, w, \bar{w})} + w^H \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})]. \tag{13}$$

Inequality (11) together with (12) and (13) yields

$$Re\,[f(z, \bar{z}, w, \bar{w}) - f(u, \bar{u}, v, \bar{v})] \geq Re\,[-u^H \overline{\nabla_z f(u, \bar{u}, v, \bar{v})}$$
$$-u^H \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v}) + w^H \overline{\nabla_w f(z, \bar{z}, w, \bar{w})} + w^H \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})]. \tag{14}$$

Hence
$$\phi(z, \bar{z}, w, \bar{w}) \geq \psi(u, \bar{u}, v, \bar{v}).$$

4. MOND-WEIR TYPE SYMMETRIC DUALITY

In this section, we present the following first order Mond-Weir type problems in complex spaces over general polyhedral cones and establish weak duality theorems.

Primal (MP1)

Minimize $\phi(z,\bar{z},w,\bar{w}) = Re\ f(z,\bar{z},w,\bar{w})$
subject to

$$-\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) \in S, \tag{15}$$

$$Re\ \{w^T \nabla_w f(z,\bar{z},w,\bar{w}) + w^H \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w})\} \geq 0, \tag{16}$$

$$z \in T. \tag{17}$$

Dual (MD1)

Maximize $\psi(u,\bar{u},v,\bar{v}) = Re\ f(u,\bar{u},v,\bar{v})$
subject to

$$\overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) \in T^*, \tag{18}$$

$$Re\ \{u^T \nabla_z f(u,\bar{u},v,\bar{v}) + u^H \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v})\} \leq 0, \tag{19}$$

$$v \in S^* \tag{20}$$

where $f : C^n \times C^n \times C^m \times C^m \mapsto C$ is analytic and S and T are general polyhedral cones in C^m and C^n, respectively.

Theorem 4.1. (Weak duality). Let $(z,\bar{z},w,\bar{w})$ and $(u,\bar{u},v,\bar{v})$ be feasible solutions of (MP1) and (MD1), respectively. If $Ref(.,.,v,\bar{v})$ is pseudoconvex at $(u,\bar{u})$ and $Ref(z,\bar{z},.,.)$ is pseudoconcave at $(w,\bar{w})$ with respect to R_+ , then

$$\phi(z,\bar{z},w,\bar{w}) \geq \psi(u,\bar{u},v,\bar{v}).$$

Proof. Using (15) and (20), we have

$$Re[v^H\{\overline{\nabla_w f(z,\bar{z},w,\bar{w})} + \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w})\}] \leq 0.$$

which implies

$$Re[v^T \nabla_w f(z,\bar{z},w,\bar{w}) + v^H \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w})] \leq 0.$$

This together with (16) gives

$$Re[(v-w)^T \nabla_w f(z,\bar{z},w,\bar{w}) + (v-w)^H \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w})] \leq 0.$$

Since $Ref(z, \bar{z}, ., .)$ is pseudoconcave at $(w, \bar{w})$, the above inequality yields

$$Re\ \{f(z, \bar{z}, w, \bar{w}) - f(z, \bar{z}, v, \bar{v})\} \geq 0. \tag{21}$$

Similarly, from (17), (18) and (19), we obtain

$$Re[(z-u)^T \nabla_z f(u, \bar{u}, v, \bar{v}) + (z-u)^H \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})] \geq 0,$$

which by pseudoconvexity of $Ref(., ., v, \bar{v})$ at $(u, \bar{u})$ implies

$$Re\ \{f(z, \bar{z}, v, \bar{v}) - f(u, \bar{u}, v, \bar{v})\} \geq 0. \tag{22}$$

Finally, adding (21) and (22), we have

$$\phi(z, \bar{z}, w, \bar{w}) \geq \psi(u, \bar{u}, v, \bar{v}).$$

Hence the result.

Theorem 4.2. (Weak duality). Let $(z, \bar{z}, w, \bar{w})$ and $(u, \bar{u}, v, \bar{v})$ be feasible solutions of (MP1) and (MD1), respectively. Let
(i) $Ref(., ., v, \bar{v})$ be F_1-pseudoconvex at $(u, \bar{u})$ and
(ii) $Ref(z, \bar{z}, ., .)$ be F_2-pseudoconcave at $(w, \bar{w})$
with respect to R_+, where the sublinear functionals $F_1 : C^n \times C^n \times C^n \mapsto R$ and $F_2 : C^m \times C^m \times C^m \mapsto R$ satisfy the following conditions:
(iii) $F_1(z, u; \xi) + Re(u^H \xi) \geq 0$ for all $\xi \in T^*$,
(iv) $F_2(v, w; \eta) + Re(w^H \eta) \geq 0$ for all $\eta \in S$.
Then

$$\phi(z, \bar{z}, w, \bar{w}) \geq \psi(u, \bar{u}, v, \bar{v}).$$

Proof. On taking $\xi = \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v}) \in T^*$, from Hypothesis (iii) and (19), we have

$$[F_1(z, u; \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v}))] \geq 0$$

which by F_1-pseudoconvexity of $Re\ f(., ., v, \bar{v})$ at $(u, \bar{u})$ yields

$$Re\ \{f(z, \bar{z}, v, \bar{v}) - f(u, \bar{u}, v, \bar{v})\} \geq 0. \tag{23}$$

Similarly, taking $\eta = -\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}) \in S$ and using Hypothesis (iv), (16), we get

$$[F_2(v, w; -\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}))] \geq 0$$

which by using F_2-pseudoconcavity of $Re\ f(z, \bar{z}, ., .)$ at $(w, \bar{w})$ gives

$$\{f(z, \bar{z}, w, \bar{w}) - f(z, \bar{z}, v, \bar{v})\} \geq 0. \tag{24}$$

Combining (23) and (24), we get the required result.

Remark 4.1.

For the weak duality theorem in [25], Mishra and Rueda assumed that

$$F_1(z,u;\xi_1+\xi_2)+\text{Re}[u^T\xi_1+u^H\xi_2]\geq 0 \quad \text{for all } \ \xi_1,\xi_2\in C^n, \tag{25}$$

$$F_2(v,w;\eta_1+\eta_2)+\text{Re}[w^T\eta_1+w^H\eta_2]\geq 0 \quad \text{for all } \ \eta_1,\eta_2\in C^m. \tag{26}$$

The above assumptions appear to be inappropriate due to the following :
(i) Under convexity assumptions

$F_1(z,u,\xi_1+\xi_2)=\text{R}e[(z-u)^H(\xi_1+\xi_2)]$, and
$F_2(v,w,\eta_1+\eta_2)=\text{R}e[(v-w)^H(\eta_1+\eta_2)]$,

the inequalities (25) and (26) give

$\text{Re}[z^H(\xi_1+\xi_2)+\xi_1(u^T-u^H)]\geq 0 \quad$ for all $\xi_1,\xi_2\in C^n$.
$\text{Re}[v^H(\eta_1+\eta_2)+\eta_1(w^T-w^H)]\geq 0 \quad$ for all $\eta_1,\eta_2\in C^m$.

These imply that $z=0$ and $v=0$. Thus the primal variable z and the dual variable v must be zero vectors.
(ii) The inequalities (25) and (26) do not reduce to the assumptions for the corresponding weak duality theorem over real nonnegative orthants [18-21].

Remark 4.2.

The assumptions (iii) and (iv) in Theorem 4.2 differ from the assumptions (25) and (26) in the sense that instead of two vectors $\xi_1,\xi_2\in C^n$, we have taken $\xi=\xi_1+\xi_2\in T^*$. Moreover, under convexity assumptions, the Hypotheses (iii) and (iv) reduce to $\text{Re}(z^H\xi)\geq 0$ for all $\xi\in T^*$ and $\text{Re}(v^H\eta)\geq 0$ for all $\eta\in S$, which in turn give $z\in T$ and $v\in S^*$. These are the constraints (17) and (20) of the problems (MP1) and (MD1).

We now state a weak duality theorem under $\eta-$ pseudoconvexity assumptions. Its proof follows on the lines of Theorem 4.2 on taking

$$F_1(z,u,\xi)=\text{Re}[\eta_1^H(z,u)\xi]\text{and}F_2(v,w,\eta)=\text{R}e[\eta_2^H(v,w)\eta].$$

Also, the assumptions (iii) and (iv) of the Theorem 4.2 reduces to $\text{Re}\{(\eta_1(z,u)+u)^H\xi\}\geq 0$ for all $\xi\in T^*$ and $\text{Re}\{(\eta_2(v,w)+w)^H\eta\}\geq 0$ for all $\eta\in S$ which imply

$$\eta_1(z,u)+u\in T \text{ and } \eta_2(v,w)+w\in S^*.$$

Theorem 4.3. (Weak duality). Let $(z,\bar{z},w,\bar{w})$ and $(u,\bar{u},v,\bar{v})$ be feasible solutions of (MP1) and (MD1) , respectively. If

(i) $\text{Re}f(.,.,v,\bar{v})$ is η_1-pseudoconvex at $(u,\bar{u})$ with respect to R_+,
(ii) $\text{Re}f(z,\bar{z},.,.)$ is η_1-pseudoconcave at $(w,\bar{w})$ with respect to R_+,
(iii) $\eta_1(z,u)+u\in T$ and
(iv) $\eta_2(v,w)+w\in S^*$,

then

$$\phi(z,\bar{z},w,\bar{w})\geq\psi(u,\bar{u},v,\bar{v}).$$

5. WOLFE TYPE SECOND ORDER SYMMETRIC DUALITY

In this section, we extend the Wolfe type primal and dual problems presented in Section 3 to second order primal and dual problems, and establish a weak duality theorem.

Primal (WP2)

$$\begin{aligned}\text{Minimize}\ \ \phi(z,\bar{z},w,\bar{w}) = Re\,[f(z,\bar{z},w,\bar{w}) &- w^H\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - w^H\nabla_{\bar{w}} f(z,\bar{z},w,\bar{w})\\ &- w^H\overline{(\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})r_2} - w^H(\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2\\ &- \tfrac{1}{2}r_2^H\overline{(\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})r_2} - \tfrac{1}{2}r_2^H(\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2\end{aligned}$$

subject to

$$\begin{aligned}&[-\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w})\\ &-\overline{(\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})r_2} - (\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2] \in S\end{aligned} \tag{27}$$

$$z \in T. \tag{28}$$

Dual (WD2)

$$\begin{aligned}\text{Maximize}\ \ \psi(u,\bar{u},v,\bar{v}) = Re\,[f(u,\bar{u},v,\bar{v}) &- u^H\overline{\nabla_z f(u,\bar{u},v,\bar{v})} - u^H\nabla_{\bar{z}} f(u,\bar{u},v,\bar{v})\\ &- u^H\overline{(\nabla_{zz}+\nabla_{z\bar{z}})f(u,\bar{u},v,\bar{v})r_1} - u^H(\nabla_{\bar{z}z}+\nabla_{\bar{z}\bar{z}})f(u,\bar{u},v,\bar{v})r_1\\ &- \tfrac{1}{2}r_1^H\overline{(\nabla_{zz}+\nabla_{z\bar{z}})f(u,\bar{u},v,\bar{v})r_1} - \tfrac{1}{2}r_1^H(\nabla_{\bar{z}z}+\nabla_{\bar{z}\bar{z}})f(u,\bar{u},v,\bar{v})r_1]\end{aligned}$$

subject to

$$\begin{aligned}&[\overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v})\\ &+\overline{(\nabla_{zz}+\nabla_{z\bar{z}})f(u,\bar{u},v,\bar{v})r_1} + (\nabla_{\bar{z}z}+\nabla_{\bar{z}\bar{z}})f(u,\bar{u},v,\bar{v})r_1] \in T^*\end{aligned} \tag{29}$$

$$v \in S^*. \tag{30}$$

where, $r_1 \in C^n$ and $r_2 \in C^m$.

Theorem 5.1. (Weak Duality). Let $(z,\bar{z},w,\bar{w})$ and $(u,\bar{u},v,\bar{v})$ be feasible solutions of (WP2) and (WD2). If $Re\ f(.,.,v,\bar{v})$ is second order F_1-convex at $(u,\bar{u})$ and $Ref(z,\bar{z},.,.)$ is second order F_2-concave at $(w,\bar{w})$ with respect to R_+, and

(*i*) $F_1(z,u;\xi) + Re[u^H\xi] \geq 0$ for $\xi \in T^*$,
(*ii*) $F_2(v,w;\eta) + Re[w^H\eta_1 + w^H\eta] \geq 0$ for $\eta \in S$.

then

$$\phi(z, \bar{z}, w, \bar{w}) \geq \psi(u, \bar{u}, v, \bar{v}).$$

Proof. Using second order F_1-convexity of $Ref(.,.,v,\bar{v})$ at $(u,\bar{u})$ and second order F_2-concavity of $Ref(z,\bar{z},.,.)$ at $(w,\bar{w})$, we have

$$\begin{aligned}
&Re[f(z,\bar{z},v,\bar{v}) - f(u,\bar{u},v,\bar{v}) + \frac{1}{2}r_1^H\overline{(\nabla_{zz}+\nabla_{z\bar{z}})f(u,\bar{u},v,\bar{v})r_1} \\
&\qquad + \frac{1}{2}r_1^H(\nabla_{\bar{z}z}+\nabla_{\bar{z}\bar{z}})f(u,\bar{u},v,\bar{v})r_1] \\
&\geq F_1(z,u;\overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \overline{(\nabla_{zz}+\nabla_{z\bar{z}})f(u,\bar{u},v,\bar{v})r_1} \\
&\qquad + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + (\nabla_{\bar{z}z}+\nabla_{\bar{z}\bar{z}})f(u,\bar{u},v,\bar{v})r_1),
\end{aligned}$$

and

$$\begin{aligned}
&Re[f(z,\bar{z},w,\bar{w}) - f(z,\bar{z},v,\bar{v}) \\
&-\frac{1}{2}r_2^H\overline{(\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})r_2} - \frac{1}{2}r_2^H(\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2] \\
&\geq F_2(v,w;-\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \overline{(\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})r_2} \\
&\qquad -\nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) - (\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}}) \\
&\qquad f(z,\bar{z},w,\bar{w})r_2).
\end{aligned}$$

Adding these two inequalities, we get

$$\begin{aligned}
&Re\ [f(z,\bar{z},w,\bar{w}) - f(u,\bar{u},v,\bar{v})] \\
&\geq \quad F_1(z,u;\overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \overline{(\nabla_{zz}+\nabla_{z\bar{z}})f(u,\bar{u},v,\bar{v})r_1} \\
&\quad + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + (\nabla_{\bar{z}z}+\nabla_{\bar{z}\bar{z}})f(u,\bar{u},v,\bar{v})r_1) \\
&\quad + F_2(v,w;-\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \overline{(\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})r_2} \\
&\quad -\nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) - (\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2) \\
&\quad -Re[\tfrac{1}{2}r_1^H\overline{(\nabla_{zz}+\nabla_{z\bar{z}})f(u,\bar{u},v,\bar{v})r_1} + \tfrac{1}{2}r_1^H(\nabla_{\bar{z}z}+\nabla_{\bar{z}\bar{z}})f(u,\bar{u},v,\bar{v})r_1] \\
&+Re[\frac{1}{2}r_2^H\overline{(\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})r_2} + \frac{1}{2}r_2^H(\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2]
\end{aligned} \tag{31}$$

On taking

$$\xi = \overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} + (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1 \in T^*,$$

$$\eta = -\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) - \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2} - (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2 \in S$$

The assumptions (i) and (ii) respectively reduce to

$$F_1(z,u;\overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} + (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1)$$
$$\geq Re[-u^H \overline{\nabla_z f(u,\bar{u},v,\bar{v})} - u^H \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) - u^H \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} - u^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1], \quad (32)$$

and

$$F_2(v,w;-\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) - \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2} - (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2)$$
$$\geq Re[w^H \overline{\nabla_w f(z,\bar{z},w,\bar{w})} + w^H \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) + w^H \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2} + w^H (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2]. \quad (33)$$

Inequality (31) together with (32) and (33) yields

$$\begin{aligned} Re[&f(z,\bar{z},w,\bar{w}) - w^H \overline{\nabla_w f(z,\bar{z},w,\bar{w})} - w^H \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) \\ &- w^H \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2} - w^H (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2 \\ &- \tfrac{1}{2} r_2^H \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2} - \tfrac{1}{2} r_2^H (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2] \\ \geq Re[&f(u,\bar{u},v,\bar{v}) - u^H \overline{\nabla_z f(u,\bar{u},v,\bar{v})} - u^H \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) \\ &- u^H \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} - u^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1 \\ &- \frac{1}{2} r_1^H \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} - \frac{1}{2} r_1^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1]. \quad (34) \end{aligned}$$

Hence,

$$\phi(z,\bar{z},w,\bar{w}) \geq \psi(u,\bar{u},v,\bar{v}).$$

6. PARTICULAR CASES

(i) If

$$S = \{w \in C^m : \ | \, arg \ w \, | \leq \ \alpha\}$$

$$T = \{z \in C^n : \ | \, arg \ z \, | \leq \ \beta\}$$

where $\alpha \in R^m_+, \ \beta \in R^n_+$ satisfying $0\mathbf{1} \leq \alpha \leq \frac{\pi}{2}\mathbf{1}$, $0\mathbf{1} \leq \beta \leq \frac{\pi}{2}\mathbf{1}$, $\mathbf{1}$ is the vector of ones, then the primal and dual programs of Section 3 reduce to the problems studied by Kaul and Sharma [18].

(ii) If we fit the constraints of Wolfe type first order symmetric dual problems studied by Mishra and Rueda [25] into our problems of Section 3, then in addition to S and T as defined in (i) above, $\alpha = (\frac{\pi}{2}, \frac{\pi}{2}, \cdots, \frac{\pi}{2})$, $\beta = (0, 0, \cdots, 0)$ and we get,

Primal (WP)

Minimize $\phi(z, \bar{z}, w, \bar{w}) = Re[f(z, \bar{z}, w, \bar{w}) - w^H \overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - w^H \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})]$

subject to

$$Re(\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} + \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})) \leq 0, \tag{35}$$

$$z \quad \in R^n_+. \tag{36}$$

Dual (WD)

Maximize $\psi(u, \bar{u}, v, \bar{v}) = Re[f(u, \bar{u}, v, \bar{v}) - u^H \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} - u^H \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})]$

subject to

$$Re(\overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})) \geq 0. \tag{37}$$

$$v \quad \in R^m_+. \tag{38}$$

In the above models, the complex variables z and v are required to satisfy (36) and (38) reducing the above models to real mathematical programming problems over nonnegative orthants, which were studied by Chandra et al. [6, Section 3].

(iii) If f is a twice differentiable real valued function, $S = R^m_+, \ T = R^n_+$ and $F_1 = F_2$, then our problems (WP2) and (WD2) reduce to Wolfe type second order symmetric dual problems of Mishra [24, Section 3].

7. STRONG DUALITY

Mishra and Rueda [25] after establishing weak duality theorem have stated that the strong duality theorem can be developed on the lines of Dantzig et al., while Kaul and Sharma [18] have given an example to show that the strong duality theorem does not hold. The example appears to be inappropriate as the nonsingularity assumption on the Hessian matrix is not satisfied and the proof of the strong duality theorem could not be obtained on the lines of Dantzig et al. Below we discuss both these points.

(1) **Example (Kaul and Sharma [18])**

Let $z = x+ix_1 \in C^1$, $w = y+iy_1 \in C^1$, $u = \xi+i\xi_1 \in C^1$ and $v = \eta+i\eta_1 \in C^1$ Let $f(z, \bar{z}, w, \bar{w}) = h(x, y) = -x + y(1-x)^2$, $S = \{w \in C^1 : \ | \, arg \; w \, | \leq \alpha\}$ and $T = \{z \in C^1 : \ | \, arg \; z \, | \leq \beta\}$ with $\alpha = 0$ and $\beta = \frac{\pi}{2}$. Then (WP1) and (WD1) reduce to the following problems:

Primal (WP*)

Minimize $\quad -x$

subject to $\quad -(1-x)^2 \geq 0,$

$\qquad x \geq 0.$

Dual (WD*)

Maximize $\quad \eta(1-\xi)(1+\xi)$

subject to $\quad 1 + 2\eta(1-\xi) \leq 0,$

$\qquad \eta \geq 0.$

Kaul and Sharma observed that $z_0 = 1$ and any w_0 is an optimal solution of (WP*), but such a pair is not even feasible for (WD*). However, the strong duality theorem requires $\partial_{yy} h(z_0, w_0)$ to be negative definite [10 (p. 810), 6 (p. 5)] or nonsingular [14, p. 81], but in the above example $\partial_{yy} h(z_0, w_0) = 0$. Hence this assumption of strong duality is not satisfied and so, the example becomes inappropriate.

(2) **Strong Duality Theorem :** Let $(z_0, \bar{z}_0, w_0, \bar{w}_0)$ be an optimal solution for (WP1) and $(\nabla_{ww} + \overline{\nabla_{w\bar{w}}} + \overline{\nabla_{\bar{w}w}} + \nabla_{\bar{w}\bar{w}}) f(z_0, \bar{z}_0, w_0, \bar{w}_0)$ be positive or negative definite. Further, if the hypotheses of Theorem 3.1 or Theorem 3.2 hold for all feasible solutions of (WP1) and (WD1), then $(z_0, \bar{z}_0, w_0, \bar{w}_0)$ is an optimal solution for (WD1) and the two objectives are equal.

Proof. Since $(z_0, \bar{z}_0, w_0, \bar{w}_0)$ is an optimal solution for (WP1), there exists, $\tau \in R_+$,
$r \in S^*$, $\mu \in T^*$ such that $(\tau, r, \mu) \neq 0$ satisfying the following Fritz John conditions [8]:

$$\tau(\overline{\nabla_z f(z_0, \bar{z}_0, w_0, \bar{w}_0)} + \tau\ \nabla_{\bar{z}} f(z_0, \bar{z}_0, w_0, \bar{w}_0) - \tau\ w_0^H(\nabla_{zw} + \overline{\nabla_{z\bar{w}}})f(z_0, \bar{z}_0, w_0, \bar{w}_0)$$

$$+ r^T(\nabla_{zw} + \overline{\nabla_{z\bar{w}}})f(z_0, \bar{z}_0, w_0, \bar{w}_0) + (r - \tau\ w_0)^H(\overline{\nabla_{\bar{z}w}} + \nabla_{\bar{z}\bar{w}})f(z_0, \bar{z}_0, w_0, \bar{w}_0)$$

$$- \mu^T = 0. \tag{39}$$

$$-\tau\ w_0^H(\nabla_{ww} + \overline{\nabla_{w\bar{w}}})f(z_0, \bar{z}_0, w_0, \bar{w}_0) + r^T(\nabla_{ww} + \overline{\nabla_{w\bar{w}}})f(z_0, \bar{z}_0, w_0, \bar{w}_0)$$

$$+(r - \tau\ w_0)^H(\overline{\nabla_{\bar{w}w}} + \nabla_{\bar{w}\bar{w}})f(z_0, \bar{z}_0, w_0, \bar{w}_0) = 0. \tag{40}$$

$$Re\{r^H\overline{\nabla_w f(z_0, \bar{z}_0, w_0, \bar{w}_0)} + r^H\nabla_{\bar{w}} f(z_0, \bar{z}_0, w_0, \bar{w}_0)\} = 0. \tag{41}$$

$$Re\{\ \mu^H\ z_0\} = 0. \tag{42}$$

To use the positive or negative definite assumption, we need to write the equation (40) as

$$(r - \tau\ w_0)^H(\nabla_{ww} + \overline{\nabla_{w\bar{w}}} + \overline{\nabla_{\bar{w}w}} + \nabla_{\bar{w}\bar{w}})f(z_0, \bar{z}_0, w_0, \bar{w}_0) = 0 \tag{43}$$

to get, $\qquad r = \tau\ w_0$
and we can continue to complete the proof on the lines of Dantzig et al.[10]. However to write equation (40) as (43) we need $r = \bar{r}$, which need not be true. Thus we are unable to complete the proof of the strong duality theorem.
Therefore, the question whether the strong duality theorem between the problems (WP1) and (WD1) and also between the problems considered by Kaul and Sharma [18] holds or not remains unanswered.

8. MOND-WEIR TYPE SECOND ORDER SYMMETRIC DUALITY

Primal (MP2)

Minimize $\quad \phi(z, \bar{z}, w, \bar{w}) = Re\ [f(z, \bar{z}, w, \bar{w}) - \frac{1}{2}r_2^H\overline{(\nabla_{ww} + \nabla_{w\bar{w}})f(z, \bar{z}, w, \bar{w})r_2}$
$\qquad - \frac{1}{2}r_2^H(\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}})f(z, \bar{z}, w, \bar{w})r_2]$

subject to

$$\{-\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}) - \overline{(\nabla_{ww} + \nabla_{w\bar{w}})f(z, \bar{z}, w, \bar{w})r_2}$$

$$-(\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}})f(z, \bar{z}, w, \bar{w})r_2\} \in S, \tag{44}$$

$$Re\ [w^T\nabla_w f(z, \bar{z}, w, \bar{w}) + w^H\nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}) + w^T(\nabla_{ww} + \nabla_{w\bar{w}})f(z, \bar{z}, w, \bar{w})r_2$$

$$+ w^H(\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}})f(z, \bar{z}, w, \bar{w})r_2] \geq 0, \tag{45}$$

$$z \in T. \tag{46}$$

Dual (MD2)

Maximize $\quad \psi(u, \bar{u}, v, \bar{v}) = Re\ [f(u, \bar{u}, v, \bar{v}) - \frac{1}{2}r_1^H\overline{(\nabla_{zz} + \nabla_{z\bar{z}}f(u, \bar{u}, v, \bar{v})r_1}$
$\qquad - \frac{1}{2}r_1^H(\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}})f(u, \bar{u}, v, \bar{v})r_1$

subject to

$$\overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} + (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1 \in T^*, \quad (47)$$

$$Re\ [u^T \nabla_z f(u,\bar{u},v,\bar{v}) + u^H \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + u^T (\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1 + u^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1] \leq 0, \quad (48)$$

$$v \in S^*. \quad (49)$$

where $r_1 \in C^n$ and $r_2 \in C^m$.

Theorem 8.1. (Weak duality). Let $(z,\bar{z},w,\bar{w})$ and $(u,\bar{u},v,\bar{v})$ be feasible solutions of (MP2) and (MD2), respectively . Let with respect to R_+
(i) $Ref(.,.,v,\bar{v})$ be second-order F_1-pseudoconvex at$(u,\bar{u})$
(ii) $Ref(z,\bar{z},.,.)$ be second-order F_2-pseudoconcave at $(w,\bar{w})$,
where the sublinear functionals $F_1 : C^n \times C^n \times C^n \mapsto R$ and $F_2 : C^m \times C^m \times C^m \mapsto R$ satisfy the following conditions:
(iii) $F_1(z,u;\xi) + Re(u^H \xi) \geq 0$ for all $\xi \in T^*$,
(iv) $F_2(v,w;\eta) + Re(w^H \eta) \geq 0$ for all $\eta \in S$.
Then

$$\phi(z,\bar{z},w,\bar{w}) \geq \psi(u,\bar{u},v,\bar{v}).$$

Proof. Let

$$\xi = \overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} + (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1 \in T^*.$$

Then from Hypothesis (iii), we have

$$\{F_1(z,u; \overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} + (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1)\} \geq -Re[u^H \overline{\nabla_z f(u,\bar{u},v,\bar{v})} + u^H \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + u^H \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} + u^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1] \geq 0. \quad \text{(from (48))}$$

Using second-order F_1-pseudoconvexity of $Ref(.,.,v,\bar{v})$ at $(u,\bar{u})$, we have

$$Re[f(z,\bar{z},v,\bar{v}) - f(u,\bar{u},v,\bar{v}) + \frac{1}{2} r_1^H \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1} + \frac{1}{2} r_1^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u,\bar{u},v,\bar{v}) r_1] \geq 0. \quad (50)$$

Similarly,

$$\eta = -\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) - \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2} - (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2 \in S$$

So, Hypothesis (iv) becomes

$$F_2(v,w,-\overline{\nabla_w f(z,\bar{z},w,\bar{w})}-\nabla_{\bar{w}}f(z,\bar{z},w,\bar{w})$$
$$-\overline{(\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})}r_2-(\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2)$$
$$\geq Re[w^H\overline{\nabla_w f(z,\bar{z},w,\bar{w})}+w^H\nabla_{\bar{w}}f(z,\bar{z},w,\bar{w})+w^H\overline{(\nabla_{ww}+\nabla_{w\bar{w}})}$$
$$f(z,\bar{z},w,\bar{w})r_2+w^H(\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2)]$$
$$\geq 0 \qquad \text{(from (45))}.$$

Applying second-order F_2-pseudoconcavity of $Ref(z,\bar{z},.,.)$ at $(w,\bar{w})$, we get

$$Re[f(z,\bar{z},w,\bar{w})-f(z,\bar{z},v,\bar{v})-\frac{1}{2}r_2^H\bar{(}\nabla_{ww}+\nabla_{w\bar{w}})f(z,\bar{z},w,\bar{w})r_2$$
$$-\frac{1}{2}r_2^H(\nabla_{\bar{w}w}+\nabla_{\bar{w}\bar{w}})f(z,\bar{z},w,\bar{w})r_2]\geq 0. \tag{51}$$

Finally, adding (50) and (51), we obtain

$$\phi(z,\bar{z},w,\bar{w})\geq\psi(u,\bar{u},v,\bar{v}).$$

9. WOLFE TYPE NONDIFFERENTIABLE SYMMETRIC DUALITY

In this section, we present the following pairs of first and second order Wolfe type nondifferentiable symmetric dual problems over general polyhedral cones in complex spaces and state weak duality theorems. Their proofs can be developed on the lines of the proofs of Theorems 3.2 and 3.3 using Lemma 2.1.

9.1 First Order Symmetric Duality

Primal (WP1*)

Minimize $\phi(z,\bar{z},w,\bar{w})=Re[f(z,\bar{z},w,\bar{w})+(z^HBz)^{\frac{1}{2}}-w^H\overline{\triangledown_w f(z,\bar{z},w,\bar{w})}-w^H\triangledown_{\bar{w}}$ $f(z,\bar{z},w,\bar{w})]$
subject to

$$-\overline{\triangledown_w f(z,\bar{z},w,\bar{w})}-\triangledown_{\bar{w}}f(z,\bar{z},w,\bar{w})+Cp\ \in\ S, \tag{52}$$

$$p^HCp\leq 1, \tag{53}$$

$$z\ \in\ T. \tag{54}$$

Dual (WD1*)

Maximize $\psi(u,\bar{u},v,\bar{v})=Re[f(u,\bar{u},v,\bar{v})-(v^HCv)^{\frac{1}{2}}-u^H\overline{\triangledown_z f(u,\bar{u},v,\bar{v})}-u^H\triangledown_{\bar{z}}$ $f(u,\bar{u},v,\bar{v})]$ subject to

$$\overline{\triangledown_z f(u,\bar{u},v,\bar{v})}+\triangledown_{\bar{z}}f(u,\bar{u},v,\bar{v})+Bq\ \in\ T^*, \tag{55}$$

$$q^HBq\leq 1, \tag{56}$$

$$v \in S^* \tag{57}$$

where, B and C are positive semidefinite Hermitian matrices of order n and m respectively, $p \in C^m$, $q \in C^n$, f, S and T are the same as in Section 3.

Theorem 9.1. (Weak Duality). Let $(z, \bar{z}, w, \bar{w})$ and $(u, \bar{u}, v, \bar{v})$ be feasible solutions of (WP1*) and (WD1*) respectively. If $Re[f(., ., v, \bar{v}) + (.)^H Bq]$ is F_1-convex at $(u, \bar{u})$ and $Re[f(z, \bar{z}, ., .) - (.)^H Cp]$ is F_2-concave at $(w, \bar{w})$ with respect to R_+, and

(i) $F_1(z, u; \xi) + Re[u^H \xi] \geq 0$ for $\xi \in T^*$,
(ii) $F_2(v, w; \eta) + Re[w^H \eta] \geq 0$ for $\eta \in S$,

then

$$\phi(z, \bar{z}, w, \bar{w}) \geq \psi(u, \bar{u}, v, \bar{v}).$$

9.2 Second order symmetric duality

Primal (WP2*)

Minimize $\phi(z, \bar{z}, w, \bar{w}) = Re\ [f(z, \bar{z}, w, \bar{w}) + (z^H Bz)^{\frac{1}{2}} - w^H \overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - w^H \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w})$

$$- w^H \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z, \bar{z}, w, \bar{w}) r_2} - w^H (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z, \bar{z}, w, \bar{w}) r_2$$

$$- \tfrac{1}{2} r_2^H \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z, \bar{z}, w, \bar{w}) r_2} \tfrac{1}{2} r_2^H (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z, \bar{z}, w, \bar{w}) r_2$$

subject to

$$[-\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}) - \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z, \bar{z}, w, \bar{w}) r_2}$$

$$- (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z, \bar{z}, w, \bar{w}) r_2 + Cp] \in S, \tag{58}$$

$$p^H Cp \leq 1, \tag{59}$$

$$z \in T. \tag{60}$$

Dual (WD2*)

Maximize $\psi(u, \bar{u}, v, \bar{v}) = Re\ [f(u, \bar{u}, v, \bar{v}) - (v^H Cv)^{\frac{1}{2}} - u^H \overline{\nabla_z f(u, \bar{u}, v, \bar{v})} - u^H \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v})$

$$- u^H \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u, \bar{u}, v, \bar{v}) r_1} - u^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u, \bar{u}, v, \bar{v}) r_1$$

$$- \tfrac{1}{2} r_1^H \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u, \bar{u}, v, \bar{v}) r_1} - \tfrac{1}{2} r_1^H (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}}) f(u, \bar{u}, v, \bar{v}) r_1]$$

subject to

$$[\overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v}) + \overline{(\nabla_{zz} + \nabla_{z\bar{z}}) f(u, \bar{u}, v, \bar{v}) r_1}$$

$$+ (\nabla_{\bar{z}z} + \nabla_{\bar{z}\bar{z}})f(u, \bar{u}, v, \bar{v})r_1 + Bq] \in T^*, \tag{61}$$

$$q^H Bq \leq 1, \tag{62}$$

$$v \in S^*. \tag{63}$$

Theorem 9.2. (Weak Duality). Let $(z, \bar{z}, w, \bar{w})$ and $(u, \bar{u}, v, \bar{v})$ be feasible solutions of (WP2*) and (WD2*). If $Re\, [f(., ., v, \bar{v}) + (.)^H Bq]$ is second order F_1-convex at $(u, \bar{u})$ and $Re[f(z, \bar{z}, ., .) - (.)^H Cp]$ is second order F_2-concave at $(w, \bar{w})$ with respect to R_+, and

(i) $F_1(z, u; \xi) + Re[u^H \xi] \geq 0$ for all $\xi \in T^*$,
(ii) $F_2(v, w; \eta) + Re[w^H \eta] \geq 0$ for $\eta \in S$.

then

$$\phi(z, \bar{z}, w, \bar{w}) \geq \psi(u, \bar{u}, v, \bar{v}).$$

10. MOND-WEIR TYPE NONDIFFERENTIABLE SYMMETRIC DUALITY

Now, we present the primal and dual pair of nondifferentiable Mond-Weir type problems in complex spaces over genral polyhedral cones and establish weak duality theorems.

10.1 First-order dual problems

Primal (MP1*)

Minimize $\phi(z, \bar{z}, w, \bar{w}) = (Re\, [f(z, \bar{z}, w, \bar{w}) + (z^H Bz)^{\frac{1}{2}} - w^H Cp]$
subject to

$$-\overline{\nabla_w f(z, \bar{z}, w, \bar{w})} - \nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}) + Cp \in S, \tag{64}$$

$$Re\, [w^T \nabla_w f(z, \bar{z}, w, \bar{w}) + w^H \{\nabla_{\bar{w}} f(z, \bar{z}, w, \bar{w}) - Cp\}] \geq 0, \tag{65}$$

$$p^H Cp \leq 1, \tag{66}$$

$$z \in T. \tag{67}$$

Dual (MD1*)

Maximize $\psi(u, \bar{u}, v, \bar{v}) = Re\, [f(u, \bar{u}, v, \bar{v}) - (v^H Cv)^{\frac{1}{2}} + u^H Bq]$
subject to

$$\overline{\nabla_z f(u, \bar{u}, v, \bar{v})} + \nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v}) + Bq \in T^*, \tag{68}$$

$$Re\, [u^T \nabla_z f(u, \bar{u}, v, \bar{v}) + u^H \{\nabla_{\bar{z}} f(u, \bar{u}, v, \bar{v}) + Bq\}] \leq 0, \tag{69}$$

$$q^H Bq \leq 1, \tag{70}$$

$$v \in S^*. \tag{71}$$

Theorem 10.1. Let $(z, \bar{z}, w, \bar{w})$ and $(u, \bar{u}, v, \bar{v})$ be feasible solutions of (MP1*) and (MD1*), respectively . Let with respect to R_+

(i) $Re\{f(.,.,v,\bar{v}) + (.)^H Bq\}$ be F_1 -pseudoconvex at $(u,\bar{u})$
(ii)$Re\{f(z,\bar{z},.,.) - (.)^H Cp\}$ be F_2-pseudoconcave at $(w,\bar{w})$
where the sublinear functionals $F_1 : C^n \times C^n \times C^n \mapsto R$ and $F_2 : C^m \times C^m \times C^m \mapsto R$ satisfy the following conditions:
(i) $F_1(z,u;\xi) + Re[u^H\xi] \geq 0$ for $\xi \in T^*$, and
(ii) $F_2(v,w;\eta) + Re[w^H\eta] \geq 0$ for $\eta \in S$.
Then

$$\phi(z,\bar{z},w,\bar{w}) \geq \psi(u,\bar{u},v,\bar{v}).$$

Proof. On taking $\xi = \overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + Bq$, from Hypothesis (i), we have

$$\begin{aligned} &F_1(z,u;\overline{\nabla_z f(u,\bar{u},v,\bar{v})} + \nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + Bq) \\ &\quad \geq Re\ [-u^H\overline{\nabla_z f(u,\bar{u},v,\bar{v})} - u^H\{\nabla_{\bar{z}} f(u,\bar{u},v,\bar{v}) + Bq\}] \\ &\quad \geq 0 \qquad \text{(using (69))}, \end{aligned}$$

which by F_1-pseudoconvexity of $Re\ \{f(.,.,v,\bar{v}) + (.)^H Bq\}$ at $(u,\bar{u})$ yields

$$Re\ [f(z,\bar{z},v,\bar{v}) + z^H Bq - f(u,\bar{u},v,\bar{v}) - u^H Bq] \geq 0 \tag{72}$$

On taking $\eta = \overline{-\nabla_w f(z,\bar{z},w,\bar{w})} - \nabla_w f(z,\bar{z},w,\bar{w}) + Cp$, we get from Hypothesis (ii) that

$$\begin{aligned} &F_2(v,w;-\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) + Cp) \\ &\quad \geq Re[w^H\overline{\nabla_w f(z,\bar{z},w,\bar{w})} + w^H\{\nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) - Cp\}] \\ &\quad \geq 0 \qquad \text{(from (65))}, \end{aligned}$$

which by using F_2-pseudoconcavity of $Re\ \{f(z,\bar{z},.,.) - (.)^H Cp\}$ at $(w,\bar{w})$ gives

$$Re\ [f(z,\bar{z},w,\bar{w}) - w^H Cp - f(z,\bar{z},v,\bar{v}) + v^H Cp] \geq 0. \tag{73}$$

Combining (72) and (73), we have

$$Re\ [f(z,\bar{z},w,\bar{w}) - f(u,\bar{u},v,\bar{v}) + z^H Bq - u^H Bq - w^H Cp + v^H Cp] \geq 0.$$

Using Schwartz inequality, (66) and (70), we obtain

$$Re\ [f(z,\bar{z},w,\bar{w}) + (z^H Bz)^{\frac{1}{2}} - w^H Cp] \geq Re[f(u,\bar{u},v,\bar{v}) - (v^H Cv)^{\frac{1}{2}} + u^H Bq],$$

Hence

$$\phi(z,\bar{z},w,\bar{w}) \geq \psi(u,\bar{u},v,\bar{v}).$$

10.2 Second-order dual problems

Primal (MP2*)

$$\begin{aligned} \textit{Minimize}\ \ \phi(z,\bar{z},w,\bar{w}) = Re\ [f(z,\bar{z},w,\bar{w}) + (z^H Bz)^{\frac{1}{2}} - w^H Cp - \tfrac{1}{2} r_2^H \\ \overline{(\nabla_{ww} + \nabla_{w\bar{w}} f(z,\bar{z},w,\bar{w}) r_2} \\ \text{-}\ 1_{\overline{2r_2^H(\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2]}} \end{aligned}$$

noindent subject to

$$\begin{aligned} -\overline{\nabla_w f(z,\bar{z},w,\bar{w})} - \nabla_{\bar{w}} f(z,\bar{z},w,\bar{w}) - \overline{(\nabla_{ww} + \nabla_{w\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2} \\ - (\nabla_{\bar{w}w} + \nabla_{\bar{w}\bar{w}}) f(z,\bar{z},w,\bar{w}) r_2 + Cp] \in S, \end{aligned} \tag{74}$$

$$Re\ [w^T\nabla_w f(z,\bar z,w,\bar w)+w^H\{\nabla_{\bar w} f(z,\bar z,w,\bar w)-Cp\}$$
$$+w^T(\nabla_{ww}+\nabla_{w\bar w})f(z,\bar z,w,\bar w)r_2+w^H(\nabla_{\bar w w}+\nabla_{\bar w\bar w})f(z,\bar z,w,\bar w)r_2]\geq 0, \quad (75)$$
$$p^HCp\leq 1, \quad (76)$$
$$z\in T. \quad (77)$$

Dual (MD2*)

$$Maximize\ \ \psi(u,\bar u,v,\bar v)=Re\ [f(u,\bar u,v,\bar v)-(v^HCv)^{\frac{1}{2}}+u^HBq-\tfrac{1}{2}r_1^H \overline{(\nabla_{zz}+\nabla_{z\bar z}f(u,\bar u,v,\bar v)r_1}$$
$$\text{- } 1\overline{{}_{2r_1^H(\nabla_{\bar z z}+\nabla_{\bar z\bar z})f(u,\bar u,v,\bar v)r_1}}$$

subject to

$$\overline{\nabla_z f(u,\bar u,v,\bar v)}+\nabla_{\bar z} f(u,\bar u,v,\bar v)+\overline{(\nabla_{zz}+\nabla_{z\bar z})f(u,\bar u,v,\bar v)r_1}$$
$$+(\nabla_{\bar z z}+\nabla_{\bar z\bar z})f(u,\bar u,v,\bar v)r_1+Bq\in T^*, \quad (78)$$
$$Re\ [u^T\nabla_z f(u,\bar u,v,\bar v)+u^H\{\nabla_{\bar z} f(u,\bar u,v,\bar v)+Bq\}$$
$$+u^T(\nabla_{zz}+\nabla_{z\bar z})f(u,\bar u,v,\bar v)r_1+u^H(\nabla_{\bar z z}+\nabla_{\bar z\bar z})f(u,\bar u,v,\bar v)r_1]\leq 0, \quad (79)$$
$$q^HBq\leq 1, \quad (80)$$
$$v\in S^*. \quad (81)$$

Theorem 10.2. Let $(z,\bar z,w,\bar w)$ and $(u,\bar u,v,\bar v)$ be feasible solutions of (MP2*) and (MD2*), respectively . Let with respect to R_+
(i) $Re\{f(.,.,v,\bar v)+(.)^HBq\}$ be second-order F_1 -pseudoconvex at $(u,\bar u)$
(ii)$Re\{f(z,\bar z,.,.)-(.)^HCp\}$ be second-order F_2-pseudoconcave at $(w,\bar w)$
where the sublinear functionals $F_1: C^n\times C^n\times C^n\mapsto R$ and $F_2: C^m\times C^m\times C^m\mapsto R$ satisfy the following conditions:
(i) $F_1(z,u;\xi)+Re[u^H\xi]\geq 0$ for $\xi\in T^*$, and
(ii) $F_2(v,w;\eta)+Re[w^H\eta]\geq 0$ for $\eta\in S$.
Then
$$\phi(z,\bar z,w,\bar w)\geq\psi(u,\bar u,v,\bar v).$$

Proof. Follows on the lines of Theorem 8.1 and 10.1.

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