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HIGHER ORDER SYMMETRIC DUALITY FOR MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS OVER CONES

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Abstract: This article studies a pair of higher order nondifferentiable symmetric fractional programming problem over cones. First, higher order cone convex function is introduced. Then using the properties of this function, duality results are set up, which give the legitimacy of the pair of primal dual symmetric model.

Keywords: Higher Order Symmetric Duality, Higher Order (Φ, ρ)- convexity, Fractional Programs, Nondifferentiable Programs, Generalized Convexity.

MSC: 90C29,90C30,90C32,90C46.

1. INTRODUCTION

In mathematical programming, symmetric programs are those programs in which primal is the dual of the dual. In other words, the programming problems in which the dual of the dual is primal again, are the symmetric programming problems. Linear programs naturally fall into this category of programs. But for nonlinear programs, such occurrence is quite exceptional.
Dorn [7] first studied symmetric quadratic programs, and Dantzig [6] formulated symmetric nonlinear programs and established weak and strong duality theorems.

Bazaraa and Goode [1] generalized the formulation of symmetric duality introduced in [6] to include the case where the inequality constraints are defined via convex cones and their polars. Mond [19] studied nonsmooth functions called support function of a compact convex set thus introducing non-differentiable symmetric primal dual pairs. Gulati et al. [10] formulated multiobjective symmetric type programs and gave duality results for Wolfe and Mond-Weir type symmetric dual multiobjective programming problems. The symmetric programs in which the objective function is a ratio of two functions, namely fractional programs, were given by Chandra et al. [4]. The notion was further extended to a multiobjective fractional symmetric program by Weir [24]. Another class of fractional symmetric programs are studied by Jayswal and Jha [13]. Kim et al. [17] studied multiobjective symmetric program with cone constraints, which was later extended to a non-differentiable multiobjective program involving cones in [16].

As it is known that dual gives a bound on the value of the primal program, the second and higher order duals give further tighter bounds due to the addition of parameters. So they help in finding better approximation to the value of the primal problem. Bector and Chandra [2] introduced second order symmetric dual program for pseudobonvex and pseudoboncave functions. The multiobjective counterpart was studied by Yang and Hou [25], which was further extended to a symmetric program over cone constraints for second order cone convex functions in [18]. Gulati and Mehndiratta [11] considered a non-differentiable multiobjective symmetric dual pair involving arbitrary cones, thus generalizing the existing classes.

Talking about higher order duality, Gulati and Gupta [9] first studied higher order duality for a symmetric program. Then, Chen [5] discussed about higher order multiobjective non-differentiable symmetric program. Gupta et al. [12] introduced higher order (F, α, ρ, d)- convex functions and studied Wolfe and Mond-Weir type dual symmetric models, whereas Suneja and Louhan [21] studied higher order symmetric programs with cone invexity and cone constraints. Recently, higher order multiobjective non-differentiable fractional symmetric programs with cone constraints are studied in [8, 23]. Some higher order programs are also discussed in [15].

In this paper, motivated by the the work of Dubey and Gupta [8], we study fractional vector optimization problems in which constraints are defined over cones and the ordering of the objectives is described with respect to some closed convex cones. This aspect of symmetric programs is not studied so far. The class of functions used in this direction is higher order (Φ, ρ)- cone convex function. Lastly, we formulate and prove weak, strong, and converse duality theorems.

2. PRELIMINARIES AND DEFINITIONS

The preference among the alternatives must conform to the decision maker's inclinations. So, a suitable domination structure is defined to find an optimized solution of a mathematical program. This leads to the study of mathematical programming problems over arbitrary cones. Let $\mathbb{R}^k$ be a $k-$ dimensional Euclidean space and $\mathbb{R}^k_+$ denote its nonnegative orthant. Let K be a closed convex pointed cone in $\mathbb{R}^k$ with non-empty interior. Consider a general vector optimization problem in which ordering is defined with respect to the convex cone K:

$$\begin{aligned} (VP) \qquad & \text{K} - \text{Minimize} \quad f(x) \\ & x \in S_0 \subseteq \mathbb{R}^n. \end{aligned}$$

Where S_0 is the set of feasible solutions and $S_0 \subseteq X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^k$.

Definition 1. *A point $u \in S_0$ is a weak efficient solution of (VP) if $\nexists\ x \in S_0$ such that $f(u) - f(x) \in \operatorname{int} K$. A point $u \in S_0$ is an efficient solution of (VP) if $\nexists\ x \in S_0$ such that $f(u) - f(x) \in K \setminus \{0\}$.*

Now we give definition of generalized convex functions named as $K - (\Phi, \rho)$ convex functions. First we give a brief overview of (Φ, ρ) convexity and its generalizations. The concept of (Φ, ρ) convexity was set forth by Caristi et al. [3] to extend the class of (F, ρ) convex functions. The following definitions have made grounds for the definition introduced in this paper.
Consider a vector valued function $f = (f_1, f_2, ..., f_k) : X \to \mathbb{R}^k$ differentiable on X, so we have component functions f_i given by $f_i : X \to \mathbb{R}$ for $i = 1, 2, ..., k$ and a vector $\rho = (\rho_1, \rho_2, ..., \rho_k) \in \mathbb{R}^k$.
For a natural number n and a set $X \subseteq \mathbb{R}^n$ consider a functional $\Phi : X \times X \times \mathbb{R}^{n+1} \to \mathbb{R}$. Any element of $\mathbb{R}^{n+1}$ takes the form (a, b), where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Definition 2. *Φ is convex on $\mathbb{R}^{n+1}$ if, for fixed $x, u \in X$, the following holds:*

$$\Phi(x, u; (\lambda(\xi_1, \rho_1) + (1 - \lambda)(\xi_2, \rho_2))) \leqq \lambda\Phi(x, u; (\xi_1, \rho_1)) + (1 - \lambda)\Phi(x, u; (\xi_2, \rho_2)),$$

for all $\xi_1, \xi_2 \in \mathbb{R}^n$, $\rho_1, \rho_2 \in \mathbb{R}$, and $\lambda \in [0, 1]$. Throughout this paper, we assume that $\Phi(x, u; (0, r)) \geq 0$ for $x, u \in X$, and $r \in \mathbb{R}_+$.

The (Φ, ρ) convex functions introduced in [3] are defined as follows.

Definition 3. *The scalar valued functions $f_i : X \to \mathbb{R}$, is (Φ, ρ_i) convex on X if*

$$f_i(x) - f_i(u) \geqq \Phi(x, u; (\nabla f_i(u), \rho_i)),\ \forall\, x \in X$$

and some $\rho_i \in \mathbb{R}$.

This class of functions were extended to cone (Φ, ρ) convex functions by Kapoor [14] who gave the following definition.

Definition 4. *[14] $f : X \to \mathbb{R}^k$ is $K - (\Phi, \rho)$ convex at u on X if for every $x \in X$, the following holds:*

$$f(x) - f(u) - \Phi(x, u, (\nabla f(u), \rho)) \in K.$$

In this, vector Φ is given by
$\Phi(x, u; (\nabla f(u), \rho)) = (\Phi(x, u; (\nabla f_1(u), \rho_1), \Phi(x, u; (\nabla f_2(u), \rho_2), ..., \Phi(x, u; (\nabla f_k(u), \rho_k)).$

The higher-order convex functions were defined by Tripathy and Devi [22] in the next definition.

Definition 5. *A scalar function f_i given by $f_i : X \to \mathbb{R}$ is higher-order (Φ, ρ_i)-invex at $u \in X$ with respect to $h_i(h_i : X \times \mathbb{R}^n \to \mathbb{R})$ if there exist real functional Φ and scalar $\rho_i \in \mathbb{R}$ such that for all $x \in X$ we have*

$$f_i(x) - f_i(u) - h_i(u,p) + p^T \nabla_p h_i(u,p) \geq \Phi(x, u; (\nabla f_i(u) + \nabla_p h_i(u,p), \rho_i)).$$

Now we define higher-order $K - (\Phi, \rho)$- convex functions. For this, assume f, Φ, ρ as defined above and $F : X \times \mathbb{R}^n \to \mathbb{R}^k$ be a differentiable function.

Definition 6. *A function f is higher-order $K - (\Phi, \rho)$- convex at $u \in X$ with respect to F and p where $p = (p_1, p_2, ..., p_k)$ and each $p_i \in \mathbb{R}^n$, if there exist real functional Φ and ρ such that for all $x \in X$ we have*

$$\begin{pmatrix} f_1(x) - f_1(u) - F_1(u,p_1) + p_1^T \nabla_{p_1} F_1(u,p_1) - \Phi(x, u, (\nabla f_1(u) + \nabla_{p_1} F_1(u,p_1), \rho_1)) \\ f_2(x) - f_2(u) - F_2(u,p_2) + p_2^T \nabla_{p_2} F_2(u,p_2) - \Phi(x, u, (\nabla f_2(u) + \nabla_{p_2} F_2(u,p_2), \rho_2)) \\ -- \\ -- \\ f_k(x) - f_k(u) - F_k(u,p_k) + p_k^T \nabla_{p_k} F_k(u,p_k) - \Phi(x, u, (\nabla f_k(u) + \nabla_{p_k} F_k(u,p_k), \rho_k)) \end{pmatrix} \in K$$

Special Cases:.

1. In this definition if $k = 1$ and $K = \mathbb{R}_+$ then we have the higher-order (Φ, ρ) convex functions defined by Tripathy and Devi [22].
2. If $p = 0$ or no approximation functions are used, then we have $K - (\Phi, \rho)$ convex functions defined by Kapoor [14]. In addition to this, if $k = 1$ and $K = \mathbb{R}_+$ then the function reduce to (Φ, ρ) convex functions defined by Caristi et al. [3].

A class of higher-order (Φ, ρ) convex functions is also discussed in [20].

Definition 7. *For a cone K, the positive polar cone(or dual cone) of K, denoted by $K^\star$, is defined as*

$$K^\star = \{y \in \mathbb{R}^n \,:\, x^T y \geqq 0 \,\forall\, x \in K\}$$

We consider a non-differentiable problem in this paper. The non-differentiable part is due to support function and we briefly discuss the related notions below.

Definition 8. *If C is a compact and convex subset of $\mathbb{R}^n$, then support function of C at x is given by $S(x|C) := max\,\{x^T y : y \in C\}$. This function being convex and finite has subdifferential*

$$\partial S(x|C) := \{z \in C : x^T z = S(x|C)\}.$$

Definition 9. *If $D \subseteq \mathbb{R}^n$ is convex, then the* ***normal cone*** *at a point z in D is defined as:*

$$N_D(z) := \{y \in \mathbb{R}^n : y^T(x - z) \leqq 0, \forall\, x \in D\}$$

If D is compact convex set, then taking into consideration both the definitions, one can infer that $y \in N_D(z) \Leftrightarrow z^T y = S(y|D)$ or say $z \in \partial S(y|D)$.

Consider $F(\cdot, \cdot)$ to be continuously differentiable such that $F(x, y) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, then
- $\nabla_x F, \nabla_y F$ denote gradient vectors with respect to x, y, respectively.
- $\nabla_{xx} F, \nabla_{yy} F$ are $n_1 \times n_1$ and $n_2 \times n_2$ symmetric Hessian matrices respectively.

3. HIGHER ORDER SYMMETRIC PROGRAMS

In this section we introduce multiobjective fractional symmetric primal dual pair. Let us suppose that S_1 and S_2 are two non-empty open sets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Further, suppose that A_1 and A_2 are closed and convex cones in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively, such that $A_1 \times A_2 \subset S_1 \times S_2$. Consider the following non-differentiable multiobjective fractional symmetric programs (MFNSP) and (MFNSD):

Primal (MFNSP)

$$K - \text{Min} \quad L(x, y, p) = (L_1(x, y, p), L_2(x, y, p), ..., L_k(x, y, p))$$

$$\textit{where } L_i(x, y, p) = \frac{f_i(x, y) + S(x|B_i) - y^T z_i + F_i(x, y, p_i) - p_i^T \nabla_{p_i} F_i(x, y, p_i)}{g_i(x, y) - S(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_{p_i} G_i(x, y, p_i)}$$

subject to

$$-\sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x, y) - z_i + \nabla_{p_i} F_i(x, y, p_i)$$
$$-L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))) \in A_2^*,$$

$$y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x, y) - z_i + \nabla_{p_i} F_i(x, y, p_i)$$
$$-L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))) \geqq 0,$$

$$\lambda \in \text{int}K^*, \; x \in A_1, \; z_i \in D_i, \; r_i \in F_i, \quad i = 1, 2, ..., k, \; \lambda^T e = 1.$$

Dual (MFNSD)

$$K - \text{Max} \quad M(u, v, q) = (M_1(u, v, q), M_2(u, v, q), ..., M_k(u, v, q))$$

$$\textit{where } M_i(u, v, q) = \frac{f_i(u, v) - S(v|D_i) + u^T w_i + \bar{F}_i(u, v, q_i) - q_i^T \nabla_{q_i} \bar{F}_i(u, v, q_i)}{g_i(u, v) + S(v|F_i) - u^T t_i + \bar{G}_i(u, v, q_i) - q_i^T \nabla_{q_i} \bar{G}_i(u, v, q_i)}$$

subject to

$$\sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + w_i + \nabla_{q_i} \bar{F}_i(u,v,q_i) \\ -M_i(u,v,q_i)(\nabla_x g_i(u,v) - t_i + \nabla_{q_i} \bar{G}_i(u,v,q_i))) \in A_1^*,$$

$$u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + w_i + \nabla_{q_i} \bar{F}_i(u,v,q_i) \\ -M_i(u,v,q_i)(\nabla_x g_i(u,v) - t_i + \nabla_{q_i} \bar{G}_i(u,v,q_i))) \leqq 0,$$

$$\lambda \in \text{int}K^*,\ v \in A_2, w_i \in B_i,\ t_i \in E_i, \quad i = 1,2,...,k,\ \lambda^T e = 1.$$

For $i = 1, 2, \ldots, k$ the following assumptions have been used while constructing the above programs pair:

(1.) $f_i, g_i : S_1 \times S_2 \to \mathbb{R}$ are differentiable functions,

(2.) The differentiable functions $F_i, G_i, \bar{F}_i, \bar{G}_i$ are such that $F_i, G_i : S_1 \times S_2 \times \mathbb{R}^{n_2} \to \mathbb{R}$, are higher-order approximation functions of f_i, g_i, respectively, with respect to second variable. $\bar{F}_i, \bar{G}_i : S_1 \times S_2 \times \mathbb{R}^{n_1} \to \mathbb{R}$ are higher-order approximation functions of f_i, g_i, respectively, with respect to first variable.

(3.) B_i, E_i are compact convex sets in $\mathbb{R}^{n_1}$ and D_i, F_i are compact convex sets in $\mathbb{R}^{n_2}$,

(4.) $p_i \in \mathbb{R}^{n_2}, q_i \in \mathbb{R}^{n_1}, \lambda \in \mathbb{R}^k$,

(5.) A_1^*, A_2^* are positive polar cones of A_1, A_2, respectively,

(6.) in the feasible region, the numerators and denominators are assumed to be nonnegative and positive, respectively.

Special Cases:.

1. If $K = \mathbb{R}_+^k$, $F_i = H_i$, $\phi_i = \bar{F}_i$, $\xi_i = \bar{G}_i$ and $C_1 = A_1$, $C_2 = A_2$, then this above discussed model becomes the one discussed by Dubey and Gupta [8].
2. If $k = 1, A_1 = \mathbb{R}_+^n$, $A_2 = \mathbb{R}_+^m$, with all the higher-order approximations are taken to be zero, or $p_i, q_i = 0$ and the sets $B_i = D_i = E_i = F_i = \{0\}$. Then the symmetric programs (MFNSP) and (MFNSD) reduce to the symmetric fractional program discussed by Chandra et al. [4].

To make the theorems easier, the following parametric program has been formulated. We take two parameters $l = (l_1, l_2, ..., l_k)$, and $m = (m_1, m_2, ..., m_k)$ and express the programs (MFNSP) and (MFNSD) equivalently as multiobjective non-differentiable symmetric programs (EMNSP) and (EMNSD), respectively.

Primal (EMNSP)

$K-Min \quad l=(l_1,l_2,...,l_k)$

subject to

$$f_i(x,y)+S(x|B_i)-y^Tz_i+F_i(x,y,p_i)-p_i^T\nabla_{p_i}F_i(x,y,p_i)$$
$$-l_i(g_i(x,y)-S(x|E_i)+y^Tr_i+G_i(x,y,p_i)-p_i^T\nabla_{p_i}G_i(x,y,p_i))=0, \quad (1)$$

$$-\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)-z_i+\nabla_{p_i}F_i(x,y,p_i)$$
$$-l_i(\nabla_y g_i(x,y)+r_i+\nabla_{p_i}G_i(x,y,p_i)))\in A_2^*, \quad (2)$$

$$y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)-z_i+\nabla_{p_i}F_i(x,y,p_i)$$
$$-l_i(\nabla_y g_i(x,y)+r_i+\nabla_{p_i}G_i(x,y,p_i)))\geqq 0, \quad (3)$$

$$\lambda\in \text{int}K^*,\ x\in A_1,\ z_i\in D_i,\ r_i\in F_i,\ i=1,2,...,k,\ \lambda^Te=1. \quad (4)$$

Dual (EMNSD)

$K-Max \quad m=(m_1,m_2,...,m_k)$

subject to

$$f_i(u,v)-S(v|D_i)+u^Tw_i+\bar{F}_i(u,v,q_i)-q_i^T\nabla_{q_i}\bar{F}_i(u,v,q_i)$$
$$-m_i(g_i(u,v)+S(v|F_i)-u^Tt_i+\bar{G}_i(u,v,q_i)-q_i^T\nabla_{q_i}\bar{G}_i(u,v,q_i)), \quad (5)$$

$$\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\bar{F}_i(u,v,q_i)$$
$$-m_i(\nabla_x g_i(u,v)-t_i+\nabla_{q_i}\bar{G}_i(u,v,q_i)))\in A_1^*, \quad (6)$$

$$u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\bar{F}_i(u,v,q_i)$$
$$-m_i(\nabla_x g_i(u,v)-t_i\nabla_{q_i}\bar{G}_i(u,v,q_i)))\leqq 0, \quad (7)$$

$$\lambda\in \text{int}K^*,\ v\in A_2, w_i\in B_i,\ t_i\in E_i,\ i=1,2,...,k,\ \lambda^Te=1. \quad (8)$$

4. DUALITY RESULTS

In this Section, we validate the duality relations between the equivalent symmetric programs (EMNSP) and (EMNSD) under generalized convexity assumptions.

Theorem 10 (Weak Duality Theorem). *Let (x,y,l,z,r,λ,p) be a feasible solution of (EMNSP) and (u,v,m,w,t,λ,q) be a feasible of (EMNSD). Assume that:*

(i) $(f(\cdot,v)+(\cdot)^Tw-m(g(\cdot,v)-(\cdot)^Tt))$ is higher-order $K-(\Phi^1,\rho^1)$ convex, in first variable, at u for fixed v, with respect to $\bar{F}-m\bar{G}$ and q where $\Phi^1:\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_1+1}\to\mathbb{R}$ and $\rho^1=(\rho_1^1,...,\rho_k^1)$,

(ii) $(-(f(x,\cdot)+(\cdot)^T z)+l(g(x,\cdot)-(\cdot)^T r))$ *is higher-order* $K-(\Phi^2,\rho^2)$ *convex, in second variable, at* y *for fixed* x*, with respect to* $-H+lG$ *and* p *where* $\Phi^2:\mathbb{R}^{n_2}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_2+1}\to\mathbb{R}$ *and* $\rho^2=(\rho_1^2,...,\rho_k^2)$ *,*

(iii) $\Phi^1(x,u,(a,\bar{\rho}))+a^T u\geqq 0,\ \forall\, a\in A_1^*,\ \rho^1\geqq 0$ *and* $\Phi^2(v,y,(b,\tilde{\rho}))+b^T y\geqq 0,\ \forall\, b\in A_2^*$ *and* $\rho^2\geqq 0$.

(iv) $\mathbb{R}_+^k\subseteq K,\ g_i(x,v)+v^T r_i-x^T t_i>0,\ \ i=1,2,...,k.$ *and for some* $\omega\in$ $int\mathbb{R}_+^k,\kappa\in K\setminus\{0\},\ \omega\kappa\in K\setminus\{0\}$.

Then $(m-l)\notin K\setminus\{0\}$.

Proof. We validate the theorem by contradiction. Suppose that the weak duality theorem does not hold, which means that $(m-l)\in K\setminus\{0\}$. Now for some $\lambda\in\text{int}K^*$ and using (iv) we have

$$\sum_{i=1}^{k}\lambda_i(l_i-m_i)(g_i(x,v)+v^T r_i-x^T t_i)<0. \tag{9}$$

(i) gives that

$$\begin{pmatrix} f_1(x,v)+x^T w_1-m_1(g_1(x,v)-x^T t_1)-(f_1(u,v)+u^T w_1-m_1(g_1(u,v)-u^T t_1)) \\ -(\bar{F}_1(u,v,q_1)-m_1\bar{G}_1(u,v,q_1))+q_1^T\nabla_{q_1}(\bar{F}_1(u,v,q_1)-m_1\bar{G}_1(u,v,q_1)) \\ -\Phi^1(x,u,(\nabla_x f_1(u,v)+w_1-m_1(\nabla_x g_1(u,v)-t_1)+\nabla_{q_1}(\bar{F}_1(u,v,q_1)-m_1\bar{G}_1(u,v,q_1))),\rho_1^1)) \\ -- \\ -- \\ f_k(x,v)+x^T w_k-m_k(g_k(x,v)-x^T t_k)-(f_k(u,v)+u^T w_k-m_k(g_k(u,v)-u^T t_k)) \\ -(\bar{F}_k(u,v,q_k)-m_k\bar{G}_k(u,v,q_k))+q_k^T\nabla_{q_k}(\bar{F}_k(u,v,q_k)-m_k\bar{G}_k(u,v,q_k)) \\ -\Phi^1(x,u,(\nabla_x f_k(u,v)+w_k-m_k(\nabla_x g_k(u,v)-t_k)+\nabla_{q_k}(\bar{F}_k(u,v,q_k)-m_k\bar{G}_k(u,v,q_k))),\rho_k^1)) \end{pmatrix}\in K$$

Since $\lambda\in\ \text{int}K^*$ we get the following

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i\big[&f_i(x,v)+x^T w_i-m_i(g_i(x,v)-x^T t_i)-(f_i(u,v)+u^T w_i-m_i(g_i(x,v)-u^T t_i)) \\ &-(\bar{F}_i(u,v,q_i)-m_i\bar{G}_i(u,v,q_i))+q_i^T\nabla_{q_i}(\bar{F}_i(u,v,q_i)-m_i\bar{G}_i(u,v,q_i)) \\ &\quad-\Phi^1(x,u;(\nabla_x f_i(u,v)+w_i-m_i(\nabla_x g_i(u,v)-t_i) \\ &\qquad\qquad+\nabla_{q_i}(\bar{F}_i(u,v,q_i)-m_i\bar{G}_i(u,v,q_i)),\rho_i^1))\big]\geqq 0.\end{aligned} \tag{10}$$

Divide this equation by $\sum_{i=1}^{k}\lambda_i=\tau$ (it is clear that $\sum_{i=1}^{k}\frac{\lambda_i}{\tau}=1$) and using convexity of Φ^1, the following is deduced

$$\sum_{i=1}^{k} \frac{\lambda_i}{\tau} \big[f_i(x,v) + x^T w_i - m_i(g_i(x,v) - x^T t_i) - (f_i(u,v) + u^T w_i - m_i(g_i(u,v) - u^T t_i)) \\ -(\bar{F}_i(u,v,q_i) - m_i \bar{G}_i(u,v,q_i)) + q_i^T \nabla_{q_i}(\bar{F}_i(u,v,q_i) - m_i \bar{G}_i(u,v,q_i)) \big] \\ \geqq \Phi^1(x,u,\frac{1}{\tau}\sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + w_i - m_i(\nabla_x g_i(u,v) - t_i) \\ + \nabla_{q_i}(\bar{F}_i(u,v,q_i) - m_i \bar{G}_i(u,v,q_i)), \rho_i^1)). \tag{11}$$

As $a1 = \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + w_i - m_i(\nabla_x g_i(u,v) - t_i) + \nabla_{q_i}(\bar{F}_i(u,v,q_i) - m_i \bar{G}_i(u,v,q_i))) \in A_1^*$ due to (6), multiply this with $\frac{1}{\tau} > 0$ to get $a = \frac{a1}{\tau}$ and since A_1^* is a convex cone, we have $a \in A_1^*$. Use this and $\bar{\rho} = \sum_{i=1}^{k} \frac{\lambda_i}{\tau} \rho_i^1 \geq 0$ (by hypothesis (iii)) to get the following inequality

$$\frac{1}{\tau}\sum_{i=1}^{k} \lambda_i [f_i(x,v) + x^T w_i - m_i(g_i(x,v) - x^T t_i) - (f_i(u,v) + u^T w_i - m_i(g_i(u,v) - u^T t_i)) \\ - (\bar{F}_i(u,v,q_i) - m_i \bar{G}_i(u,v,q_i)) + q_i^T \nabla_{q_i}(\bar{F}_i(u,v,q_i) - m_i \bar{G}_i(u,v,q_i))] \\ \geqq \Phi^1(x,u,\frac{1}{\tau}\sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + w_i - m_i(\nabla_x g_i(u,v) - t_i) \\ + \nabla_{q_i}(\bar{F}_i(u,v,q_i) - m_i \bar{G}_i(u,v,q_i)), \rho_i^1)) \\ = \Phi^1(x,u;(a,\bar{\rho})) \\ \geqq -a^T u \\ \geqq 0, \qquad \text{(due to (6))}. \tag{12}$$

On rearranging the terms in above equation and adding (5), then using feasibility conditions (4) and (8), we obtain the following

$$\sum_{i=1}^{k} \lambda_i (f_i(x,v) + x^T w_i - S(v|D_i) - m_i(g_i(x,v) - x^T t_i + v^T r_i) \geqq 0. \tag{13}$$

On the same lines, using (ii) the following can be obtained

$$\sum_{i=1}^{k} \lambda_i (-f_i(x,v) + v^T z_i - S(x|B_i) + l_i(g_i(x,v) + v^T r_i - x^T t_i) \geqq 0. \tag{14}$$

Adding equations (13) and (14) we get

$$\sum_{i=1}^{k} \lambda_i (l_i - m_i)(g_i(x,v) + v^T r_i - x^T t_i) \geqq 0.$$

This is a contradiction to equation (9). So it is concluded that the supposition was wrong and hence, weak duality holds under the given set of assumptions. □

Theorem 11 (Strong Duality Theorem). *Let $(\bar{x}, \bar{y}, \bar{l}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ be an efficient solution of (EMNSP) and fix $\lambda = \bar{\lambda}$ in (EMNSD). Further, if the following assumptions hold:*

(i) *For $i = 1, 2, ..., k$,*

$$F_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} F_i(\bar{x}, \bar{y}, 0) = \nabla_x F_i(\bar{x}, \bar{y}, 0) = \nabla_y F_i(\bar{x}, \bar{y}, 0) = 0,$$

$$G_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} G_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = \nabla_y G_i(\bar{x}, \bar{y}, 0) = 0,$$

$$\bar{F}_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \bar{F}_i(\bar{x}, \bar{y}, 0) = \bar{G}_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \bar{G}_i(\bar{x}, \bar{y}, 0) = 0$$

(ii) *for any $i = 1, 2, ..., k$, the Hessian matrix $\nabla_{p_i p_i}(F_i(\bar{x}, \bar{y}, \bar{p_i}) - l_i G_i(\bar{x}, \bar{y}, \bar{p_i}))$is positive or negative definite,*

(iii) *The set of vectors $\{\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_y F_i(\bar{x}, \bar{y}, \bar{p_i}) - l_i \nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p_i}), \nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p_i}) - l_i \nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p_i}),\ i = 1, 2, ..., k\}$ is linearly independent.*

(iv) *if for $\bar{p}_i \in \mathbb{R}^{n_2}$ such that $\bar{p}_i \neq 0$ implies $\sum_{i=1}^{k} \bar{p}_i(\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_y F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))) \neq 0$*

(v) *$\bar{l}_i > 0,\ i = 1, 2, ..., k$*

then the point $(\bar{x}, \bar{y}, \bar{l}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of the dual (EMNSD). Furthermore, if hypotheses of weak duality theorem hold for every feasible solution of dual, then $(\bar{x}, \bar{y}, \bar{l}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is an efficient solution of (EMNSD) and objective function values of (EMNSP) and (EMNSD) are equal.

Proof. Since $(\bar{x}, \bar{y}, \bar{l}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ is given to be an efficient solution of the primal (EMNSP), then by necessary optimality conditions [21] , there exist $\alpha \in K^*$, $\beta \in \mathbb{R}^k_+$, $\gamma \in A_2$, $\delta \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $\bar{w_i}$ and $\bar{t_i} \in \mathbb{R}^{n_1}$ such that the following hold

$$\Big[\sum_{i=1}^{k} \beta_i(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w_i} + \nabla_x F_i(\bar{x}, \bar{y}, \bar{p_i}) - \bar{l_i}(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t_i} + \nabla_x G_i(\bar{x}, \bar{y}, \bar{p_i})))$$
$$+ (\gamma - \delta\bar{y})^T \sum_{i=1}^{k} \lambda_i(\nabla_{yx} f_i(\bar{x}, \bar{y}) - l_i \nabla_{yx} g_i(\bar{x}, \bar{y}))$$
$$+ \sum_{i=1}^{k} \nabla_{p_i x}(F_i(\bar{x}, \bar{y}, \bar{p_i}) - l_i G_i(\bar{x}, \bar{y}, \bar{p_i})^T((\gamma - \delta\bar{y})\lambda_i - \beta_i\bar{p_i})\Big]^T (x - \bar{x}) \geqq 0,\ \forall\, x \in A_1. \tag{15}$$

$$\sum_{i=1}^{k}\beta_i(\nabla_y f_i(\bar{x},\bar{y})-\bar{z_i}+\nabla_y F_i(\bar{x},\bar{y},\bar{p_i})-\bar{l_i}(\nabla_y g_i(\bar{x},\bar{y})+\bar{r_i}+\nabla_y G_i(\bar{x},\bar{y},\bar{p_i})))$$
$$+(\gamma-\delta\bar{y})^T\sum_{i=1}^{k}\lambda_i\nabla_{yy}(f_i(\bar{x},\bar{y})-\bar{l_i}g_i(\bar{x},\bar{y}))+\sum_{i=1}^{k}(\nabla_{p_i y}F_i(\bar{x},\bar{y},\bar{p_i})$$
$$-\bar{l_i}\nabla_{p_i y}G_i(\bar{x},\bar{y},\bar{p_i}))((\gamma-\delta\bar{y})^T\lambda_i-\beta_i p_i)-\sum_{i=1}^{k}\delta\lambda_i(\nabla_y f_i(\bar{x},\bar{y})-\bar{z_i}+\nabla_{p_i}F_i(\bar{x},\bar{y},\bar{p_i})$$
$$-\bar{l_i}(\nabla_y g_i(\bar{x},\bar{y})+\bar{r_i}+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p_i}))))=0. \tag{16}$$

$$\alpha_i-\beta_i((g_i(\bar{x},\bar{y})-S(\bar{x}|E_i)+\bar{y}^T\bar{r_i}+G_i(\bar{x},\bar{y},\bar{p_i}))-\bar{p_i}^T\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p_i}))$$
$$-(\gamma-\delta\bar{y})^T(\nabla_y g_i(\bar{x},\bar{y})+r_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p_i}))=0,\ i=1,2,...,k. \tag{17}$$
$$((\gamma-\delta\bar{y})\lambda_i-\beta_i\bar{p_i})^T\nabla_{p_i p_i}(F_i(\bar{x},\bar{y},\bar{p_i})-l_iG_i(\bar{x},\bar{y},\bar{p_i}))=0, \tag{18}$$
$$\beta_i\bar{y}+(\gamma-\delta\bar{y})\lambda_i\in N_{D_i}(\bar{z_i}), \tag{19}$$
$$\beta_i\bar{l_i}\bar{y}+(\gamma-\delta\bar{y})\bar{l_i}\lambda_i\in N_{F_i}(\bar{r_i}), \tag{20}$$
$$(\gamma-\delta\bar{y})^T(\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i}F_i(\bar{x},\bar{y},\bar{p_i})$$
$$-l_i(\nabla_y g_i(\bar{x},\bar{y})+r_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p_i})))-\xi_i+\eta=0, \tag{21}$$
$$\gamma^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i}F_i(\bar{x},\bar{y},\bar{p_i})$$
$$-l_i(\nabla_y g_i(\bar{x},\bar{y})+r_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p_i})))=0, \tag{22}$$
$$\delta y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i}F_i(\bar{x},\bar{y},\bar{p_i})$$
$$-l_i(\nabla_y g_i(\bar{x},\bar{y})+r_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p_i})))=0, \tag{23}$$
$$\xi^T\bar{\lambda}=0,\ \eta(\bar{\lambda}e-1)=0, \tag{24}$$
$$\bar{w_i}\in B_i,\ \bar{t_i}\in E_i,\ \bar{x}^T\bar{t_i}=S(\bar{x}|E_i),\ \bar{x}^T\bar{w_i}=S(\bar{x}|B_i), \tag{25}$$
$$(\alpha,\beta,\gamma,\delta,\xi,\eta)\neq 0. \tag{26}$$

In (24) we have $\xi^T\bar{\lambda}=0$. As $\mathbb{R}_k^+\subseteq K\ \Rightarrow K^*\subseteq\mathbb{R}_k^+$ we have $\text{int}K^*\subseteq\text{int}\mathbb{R}_k^+$ implies $\bar{\lambda}>0$. So we have $\xi=0$. From hypothesis (ii) and equation (18) we get

$$(\gamma-\delta\bar{y})\bar{\lambda}_i=\beta_i\bar{p_i},\ i=1,2,...,k. \tag{27}$$

CLAIM: $\beta_i\neq 0,$ for any $i=1,2,...,k.$
Because if $\beta_{i_0}=0$, for some $i_0\in 1,2,...,k$, then we have

$$(\gamma-\delta\bar{y})\bar{\lambda}_{i_0}=0\overset{\bar{\lambda}>0}{\Longrightarrow}\gamma=\delta\bar{y} \tag{28}$$

Put this in equation (27), we get $\beta_i\bar{p_i}=0$ for each $i\in 1,...,k$. Using these values in (16) to get

$$\sum_{i=1}^{k}\beta_i(\nabla_y f_i(\bar{x},\bar{y}) - z_i + \nabla_y F_i(\bar{x},\bar{y},\bar{p_i}) - l_i(\nabla_y g_i(\bar{x},\bar{y}) + r_i + \nabla_y G_i(\bar{x},\bar{y},\bar{p_i})))$$
$$-\sum_{i=1}^{k}\delta\bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x},\bar{y},\bar{p_i}) - l_i(\nabla_y g_i(\bar{x},\bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p_i})))) = 0. \tag{29}$$

This due to (iii) gives $\beta_i = 0$ and $\delta\bar{\lambda}_i = 0$, for all $i = 1, 2, ..., k \Rightarrow \delta = 0, (\because\ \bar{\lambda} > 0)$. Now, due to (17) $\alpha_i = 0,\ \forall\, i = 1, 2, ..., k$. Equation (28) gives $\gamma = 0$ and equation (21) $\Rightarrow \eta = 0$ respectively. ξ is already 0. $\Rightarrow (\alpha, \beta, \gamma, \delta, \xi, \eta) = 0$, which is a contradiction to necessary optimality conditions constructed above. So we have proved our claim that $\beta_i \neq 0$ for any $i = 1, 2, ..., k$.
Now multiply equation (21) with $\bar{\lambda}_i$ and take its sum over the range of i to get

$$\begin{aligned}(\gamma - \delta\bar{y})^T \sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x},\bar{y},\bar{p_i})\\ - l_i(\nabla_y g_i(\bar{x},\bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p_i}))) + \eta^T(\bar{\lambda}e) = 0\end{aligned} \tag{30}$$

and (22)-(23) give

$$(\gamma-\delta\bar{y})^T \sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i} F_i(\bar{x},\bar{y},\bar{p_i})-L_i(\nabla_y g_i(\bar{x},\bar{y})+r_i+\nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p_i})))) = 0. \tag{31}$$

Now (31)-(30) give

$$\begin{aligned}&\eta^T\bar{\lambda}e = 0\\ &\Rightarrow \eta = 0 \qquad (\because \bar{\lambda} \neq 0,\ e \neq 0).\end{aligned} \tag{32}$$

Putting this in (21), we get

$$(\gamma-\delta\bar{y})^T(\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i} F_i(\bar{x},\bar{y},\bar{p_i})-l_i(\nabla_y g_i(\bar{x},\bar{y})+r_i+\nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p_i})))) = 0. \tag{33}$$

So due to (27)

$$\beta_i\bar{p}_i(\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i} F_i(\bar{x},\bar{y},\bar{p_i})-l_i(\nabla_y g_i(\bar{x},\bar{y})+r_i+\nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p_i})))) = 0. \tag{34}$$

As $\beta_i \neq 0$,

$$\bar{p}_i(\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i} F_i(\bar{x},\bar{y},\bar{p_i})-l_i(\nabla_y g_i(\bar{x},\bar{y})+r_i+\nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p_i})))) = 0. \tag{35}$$

Or $\sum_{i=1}^{k} \bar{p}_i(\nabla_y f_i(\bar{x},\bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x},\bar{y},\bar{p}_i) - l_i(\nabla_y g_i(\bar{x},\bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))) = 0,$

$$(36)$$

and by (iv), we have each of $\bar{p}_i = 0$. So from (27), we get

$$\gamma = \delta\bar{y}. \tag{37}$$

By putting the obtained values in (15) and (16), we get

$$\Big[\sum_{i=1}^{k} \beta_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))\Big]^T (x - \bar{x}) \geqq 0, \ \forall x \in A_1, \tag{38}$$

and

$$\sum_{i=1}^{k} (\beta_i - \delta\lambda_i)(\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i - \bar{l}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i))]^T = 0. \tag{39}$$

Again, using (iii), we have $\beta_i - \delta\bar{\lambda}_i = 0 \Rightarrow \beta_i = \delta\bar{\lambda}_i$. As $\bar{\lambda}_i > 0$, $\beta_i \in \mathbb{R}_+^k$ and $\beta \neq 0$, we have $\delta > 0$. Use this in (38) to get

$$[\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))]^T (x - \bar{x}) \geqq 0, \ \forall x \in A_1. \tag{40}$$

Put $x = 0$ (as (40) holds for any $x \in A_1$) we have

$$-\Big[\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))\Big]^T \bar{x} \geqq 0. \tag{41}$$

$$\Rightarrow \Big[\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))\Big]^T \bar{x} \leqq 0. \tag{42}$$

For convex cone A_1, $\bar{x} \in A_1 \Rightarrow x + \bar{x} \in A_1$, $\forall\, x \in A_1$. So (40) becomes

$$\Big[\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))\Big]^T x \geqq 0, \ \forall\, x \in A_1. \tag{43}$$

The above equation holds for every $x \in A_1$ so we can say that

$$\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)) \in A_1^*, \tag{44}$$

which is dual feasibility condition (6). Since (43) holds for every $x \in A_1$ and $\bar{x}$ is a feasible solution of primal, so at $x = \bar{x}$ the following is true. By putting $x = \bar{x}$ in (43), we have

$$\Big[\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))\Big]^T \bar{x} \geqq 0. \tag{45}$$

As $\bar{x} \in A_1 \ \Rightarrow$ (41) and (45) give

$$\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))]^T \bar{x} = 0. \tag{46}$$

Hence, the third feasibility condition of the dual, given by equation (7), is obtained. Now in (19), as $\gamma = \delta\bar{y}$, $\beta_i > 0 \Rightarrow \bar{y} \in N_{D_i}(\bar{z_i})$ which means, $\bar{y}^T\bar{z_i} = S(\bar{y}|D_i)$. Similarly, due to (20), (25) and hypothesis (v) we have $\bar{y}^T\bar{r_i} = S(\bar{y}|F_i)$. Moreover $p_i = 0$ as obtained above, use them in equation (1) to get

$$f_i(\bar{x},\bar{y}) + \bar{x}^T\bar{w}_i - S(\bar{y}|D_i) - \bar{l}_i(g_i(\bar{x},\bar{y}) - \bar{x}^T\bar{t}_i + S(\bar{y}|F_i)) = 0,\ i = 1,2,..,k. \tag{47}$$

Equation (37) gives that $\bar{y} = \frac{\gamma}{\delta} \in A_2$ From equations (44), (46), and (47) we can conclude that $(\bar{x},\bar{y},\bar{l},\bar{w},\bar{t},\bar{\lambda},\bar{q}=0)$ is a feasible solution of dual (EMNSD) and the objective values of (EMNSP) and (EMNSD) are equal, i.e., $\bar{l}_i = m_i$. Now, for the second part of the theorem, if $(\bar{x},\bar{y},\bar{l},\bar{w},\bar{t},\bar{\lambda},\bar{q}=0)$ is not an efficient solution for dual (EMNSD), then there exists another feasible solution $(\tilde{u},\tilde{v},\tilde{m},\tilde{w},\tilde{t},\tilde{\lambda},\tilde{q})$ of dual such that $\tilde{m} - \bar{l} \in K \setminus \{0\}$. But this contradicts weak duality theorem. Hence, $(\bar{x},\bar{y},\bar{l},\bar{w},\bar{t},\bar{\lambda},\bar{q}=0)$ is efficient for (EMNSD). Hence, the result. □

Theorem 12 (Converse Duality Theorem). *Let $(\bar{u},\bar{v},\bar{m},\bar{w},\bar{t},\bar{\lambda},\bar{q})$ be an efficient solution of (EMNSD) and fix $\lambda = \bar{\lambda}$ in (EMNSP). Then under the following set of assumptions:*

(i) *For $i = 1,2,...,k$,*

$$F_i(\bar{u},\bar{v},0) = \nabla_{p_i}F_i(\bar{u},\bar{v},0) = G_i(\bar{u},\bar{v},0) = \nabla_{p_i}G_i(\bar{u},\bar{v},0) = 0,$$
$$\bar{F}_i(\bar{x},\bar{y},0) = \nabla_{q_i}\bar{F}_i(\bar{u},\bar{v},0) = \nabla_x\bar{F}_i(\bar{u},\bar{v},0) = \nabla_y\bar{F}_i(\bar{u},\bar{v},0) = 0,$$
$$\bar{G}_i(\bar{x},\bar{y},0) = \nabla_{q_i}\bar{G}_i(\bar{u},\bar{v},0) = \nabla_x\bar{G}_i(\bar{u},\bar{v},0) = \nabla_y\bar{G}_i(\bar{u},\bar{v},0) = 0$$

(ii) *for any $i = 1,2,...,k$, the Hessian matrix $\nabla_{q_iq_i}(\bar{F}_i(\bar{u},\bar{v},\bar{q}_i) - m_i\bar{G}_i(\bar{u},\bar{v},\bar{q}_i))$is positive or negative definite,*

(iii) *the set of vectors $\{\nabla_x f_i(\bar{u},\bar{v}) + w_i + \nabla_x\bar{F}_i(\bar{u},\bar{v},\bar{q}_i) - m_i\nabla_x g_i(\bar{u},\bar{v}) - t_i + \nabla_x\bar{G}_i(\bar{u},\bar{v},\bar{q}_i)$,*
$\nabla_x f_i(\bar{u},\bar{v}) + w_i + \nabla_{q_i}\bar{F}_i(\bar{u},\bar{v},\bar{q}_i) - m_i\nabla_x g_i(\bar{u},\bar{v}) - t_i + \nabla_{q_i}\bar{G}_i(\bar{u},\bar{v},\bar{q}_i),\ i = 1,2,...,k\}$ are linearly independent.

(iv) *if for $\bar{q}_i \in \mathbb{R}^{n_1}$ such that $\bar{q}_i \neq 0$, implies $\sum_{i=1}^{k}\bar{q}_i(\nabla_x f_i(\bar{u},\bar{v}) + w_i + \nabla_x\bar{F}_i(\bar{u},\bar{v},\bar{q}_i) - m_i\nabla_x g_i(\bar{u},\bar{v}) - t_i + \nabla_x\bar{G}_i(\bar{u},\bar{v},\bar{q}_i)) \neq 0$*

(v) *$\bar{m}_i > 0,\ i = 1,2,...,k$.*

the point $(\bar{u},\bar{v},\bar{m},\bar{w},\bar{t},\bar{\lambda},\bar{q})$ is a feasible solution of (EMFNSP). Furthermore if hypotheses of weak duality theorem hold for every feasible solution of (EMFNSP), then $(\bar{u},\bar{v},\bar{m},\bar{w},\bar{t},\bar{\lambda},\bar{q})$ is an efficient solution of (EMFNSP).

Proof. The proof follows on the lines of Theorem 11. □

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