Yugoslav Journal of Operations Research
32 (2022), Number 2, 153–165
DOI: https://doi.org/10.2298/YJOR200715027G

THE INVERSE MINISUM CIRCLE LOCATION PROBLEM

Mehraneh GHOLAMI
Faculty of Mathematical Sciences, Shahrood University of Technology, University Blvd., Shahrood, Iran
mehraneh.gholami@gmail.com

Jafar FATHALI
Faculty of Mathematical Sciences, Shahrood University of Technology, University Blvd., Shahrood, Iran
fathali@shahroodut.ac.ir

Received: July 2020 / Accepted: December 2021

Abstract: Let n weighted points be given in the plane. The inverse version of the minisum circle location problem deals with modifying the weights of points with minimum cost, such that the sum of the weighted distances from the circumference of a given circle C with radius r, to the given points is minimized. The classical model of this problem contains infinite constraints. In this paper, a mathematical model with finite constraints is presented. Then an efficient method is developed for solving this problem.

Keywords: Minisum Circle Location, Inverse Facility Location, Variable Weight.

MSC: 90B85, 90B06.

1. INTRODUCTION

Location theory is an essential branch of operations research. It has been applied to solve many problems in the real world, such as finding the best location for facility servers, the best position to build a railroad and industrial buildings, etc. In a location problem, a set of points, that are supposed to be the location of clients, is given and the aim is to find the location of facility servers such that these facilities serve clients in an optimal way. The circle location problem is one of the location models that today has found a lot of interest and has many applications in real life. In this problem, we should find a circle in the plane such that the weighted distances from the given points to the circumference of the circle are minimized. Three kinds of objective functions have been considered for the circle location problem: 1-minimizing the sum of weighted squares of distances from clients to the circumference of the circle (the least squares model), 2-minimizing the maximum of these weighted distances (the minimax model), 3- minimizing the sum of these weighted distances (the minisum model). Also, the circle may be with a fixed or variable radius.

Drezner et al. [11] considered the squares and minimax circle location problems in the plane and suggested them as models for the out-of-roundness problems. The minisum case of the circle location problem has been investigated by Brimberg et al. [6]. If the radius of the circle is a variable they show that there exists an optimal circle passing through two of the existing facilities. The discrete case of minisum circle location problem is studied by Labbé et al. [16]. Some applications of circle location models such as the problem of locating circular facilities, e.g., a circular irrigation pipe, circular conveyor belts, or ring roads, and out-of-roundness problem are mentioned in Drezner et al. [11]. Recently, Gholami and Fathali [13] considered the semi-obnoxious minisum circle location problem. They presented some properties on this problem and developed a meta-heuristic algorithm to solve it.

Most of the classical location models deal with finding the optimal locations of the facilities concerning different criteria such as time, cost, and distances between the clients and facilities. Among them, the Fermat-Weber, p-median, and p-center problems are three important classic facility location models. The Fermat-Weber problem asks to find the location of a new facility in the plane such that the sum of weighted distances from the given points to the new facility is minimized. The p-median and p-center problems ask to choose the location of p facilities from a set of candidate points such that respectively the sum and maximum weighted distances from the clients to the closest facility are minimized. However, in some real applications, the facilities may already exist and the problem is to change the parameters with the minimum cost such that the given locations are optimal. These kinds of location models are called inverse location problems. These problems have been considered by many authors. Among them, Cai et al. [10] showed that the inverse center problem is NP-hard. The inverse median problem has been investigated by Burkard et al. [8], [7] and Baroughi-Bonab et al. [5]. Burkard et al. [8] proposed an $O(nlogn)$ algorithm for the inverse 1-median problem on a tree. The time complexity of this problem is improved to linear time by Galavii [12]. The inverse 1-median problem on a cycle has been considered by Burkard et al. [9]. They developed an $O(n^2)$ algorithm for this problem. Alizadeh et al. [2] presented an $O(nlogn)$ time algorithm for the inverse 1-center location problem with edge length augmentation on trees. Later, Alizadeh and Burkard [3] showed that the inverse 1-center problem can be solved in $O(n^2)$ time. The inverse 1-median and 1-center problems on trees with Chebyshev and Hamming norms have been considered by Guan and Zhang [15] and Nguyen and Sepasian [20], respectively. Later, Sepasian and Rahbarnia [23] solved the inverse 1-median problem with varying vertex and edge length on trees. Nguyan [19] presented a solution algorithm for the inverse 1-median problem with variable vertex weight on block graphs. Nazari et al. [17, 18] considered the inverse of the backup 2-median problem with variable edge length and vertex weight. Alizadeh et al. [1] and Alizadeh and Etemad [4] presented some combinatorial algorithms for inverse obnoxious median and center problems, respectively. Omidi et al. [22] and [21] developed algorithms for solving some cases of the inverse balanced facility location models. Recently, Gholami and Fathali [14] proposed mathematical models for the inverse minimax circle location problem with variable weights.

In this paper, we consider the inverse minisum circle location problem with varying weights of vertices. This problem asks to modify the weight of points with minimum cost such that a given circle becomes optimal. Using some properties on minisum circle location problem we presented a mathematical model for solving the inverse problem. Then an efficient approach is proposed for solving the presented model.

In what follows, we define the minisum circle location problem and some previously presented properties on this problem in Section 2. In Section 3, a mathematical model for the variable radius case of the inverse model is presented. Then a method is developed to solve this problem. Section 4 contains the summary and conclusion on this paper.

2. THE MINISUM CIRCLE LOCATION PROBLEM

Let n points $\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n$, where $\mathbf{p}_j = (x_j, y_j)$ for $j = 1, ..., n$ are the location of clients, be given in the plane. For every point $\mathbf{p}_j$, $j = 1, ..., n$ a nonnegative weight w_j is associated. The minisum circle location problem is to find the circle $C = C(\mathbf{x}, r)$ with center $\mathbf{x}$ and radius r such that the sum of weighted distances from all customers to the circumference of the circle C is minimized, i.e.

$$\min_{\mathbf{x},r} F(C) = \sum_{j=1}^{n} w_j |d(\mathbf{x}, \mathbf{p}_j) - r|,$$

where $d(\mathbf{x}, \mathbf{p}_j)$ represents the distance between point $\mathbf{p}_j$ and the center of circle C.

Let

$$J_+(C) = \{j : d(\mathbf{x}, \mathbf{p}_j) > r\},$$

$$J_0(C) = \{j : d(\mathbf{x}, \mathbf{p}_j) = r\},$$

and

$$J_-(C) = \{j : d(\mathbf{x}, \mathbf{p}_j) < r\},$$

be the points inside, on and outside a given circle $C(\mathbf{x}, r)$. Then

$$|d(\mathbf{x}, \mathbf{p}_j) - r| = \begin{cases} d(\mathbf{x}, \mathbf{p}_j) - r & j \in J_+(C) \\ 0 & j \in J_0(C) \\ r - d(\mathbf{x}, \mathbf{p}_j) & j \in J_-(C) \end{cases}$$

and the minisum circle location problem can be written as follows;

$$\min_{\mathbf{x},r} F(C) = \sum_{j\in J_-(C)} w_j(r - d(\mathbf{x},\mathbf{p}_j)) + \sum_{j\in J_+(C)} w_j(d(\mathbf{x},\mathbf{p}_j) - r). \tag{1}$$

The proof of the following two important properties can be found in [6].

Theorem 1. *An optimal solution of problem (1) passes through at least two existing points.*

Theorem 2. *Let $C(\mathbf{x}, r)$ be an optimal solution of (1). Then*

$$\left| \sum_{j\in J_-(C)} w_j - \sum_{j\in J_+(C)} w_j \right| \le \sum_{j\in J_0(C)} w_j$$

Using these two theorems, to find the optimal solution of the Minisum Circle Location Problem (MCLP), we need to find the bisector of any two existing points. Brimberg et al. [6] presented a method for finding the optimal solution of MCLP based on finding bisectors of existing points. According to their method, first, the set of all intersection points of any three bisectors is found. These intersection points can be candidates for the center of the optimal solution. Then by solving some nonlinear programming problems, some new candidate points on the line segments and half lines of bisectors are obtained. Among all candidate points those that satisfy in Theorem 2, are chosen as the center of candidate circles. Then the circle with the best objective function is selected for the optimal solution of MCLP. In the next section, we use this method to present an approach for solving the Inverse Minisum Circle Location Problem (IMCLP).

3. INVERSE MINISUM CIRCLE LOCATION PROBLEM

Let the circle $C^* = C(\mathbf{p}^*, r^*)$ be given and we want to modify the weights of existing points w_j, for $j = 1, ..., n$, to $\hat{w}_j = w_j + q_j^+ - q_j^-$ with minimum cost, such that the circle C^* is optimal compared to any other circle with a positive radius. Here $q_j^+ \ge 0$ and $q_j^- \ge 0$ are the values of augmenting and reduction of the weight of point $\mathbf{p}_j$ and bounded from above by u_j^+ and u_j^-, respectively. Suppose that c_j^+ and c_j^- denote the cost of augmenting and reduction of per unit of w_j. Then IMCLP can be modeled as follow.

$$\min \sum_{j=1}^{n} (c_j^+ q_j^+ + c_j^- q_j^-) \tag{2}$$

$$\sum_{j=1}^{n} \hat{w}_j |d(\mathbf{p}^*, \mathbf{p}_j) - r^*| \le \sum_{j=1}^{n} \hat{w}_j |d(\mathbf{x}, \mathbf{p}_j) - r| \quad \text{for all } \mathbf{x} \in \mathbb{R}^2, r \in \mathbb{R}^+ \tag{3}$$

$$\hat{w}_j = w_j + q_j^+ - q_j^- \quad j = 1, \cdots, n \tag{4}$$

$$0 \le q_j^+ \le u_j^+ \qquad j = 1, \cdots, n \tag{5}$$

$$0 \le q_j^- \le u_j^- \qquad j = 1, \cdots, n. \tag{6}$$

This is a nonlinear model with an infinite number of constraints. Therefore, we are going to find a simpler model and its optimal solution by considering the Brimberg et al. [6] method.

For any circle $C = C(\mathbf{x}, r)$, let

$$F_1(C) = \sum_{j=1}^{n}(w_j + q_j^+ - q_j^-)|d(\mathbf{x}, \mathbf{p}_j) - r|,$$

Note that, by Theorem 1 since the center of an optimal circle should be on at least one bisector of existing points, therefore, if $\mathbf{p}^*$ does not lie on any of these bisectors then the inverse model doesn't have any solution. Otherwise, we should change the weight of vertices such that the center of the given circle became better than any point on the line segments of bisectors which are between the intersection of any two bisectors.

First, let B be the set of intersections of any three bisectors. For any $\mathbf{b}_i \in B$ let r_i^B be the distance between $\mathbf{b}_i$ and existing points corresponding to the bisectors. Consider the circle $C_i^B = C(\mathbf{b}_i, r_i^B)$ and find $J_+(C_i^B)$ and $J_-(C_i^B)$ for any $\mathbf{b}_i \in B$. Then for C_i^B and the given values of q_j^+ and q_j^- the following two cases may occur.

1. The condition in Theorem 2 holds, i.e.

$$|\sum_{j\in J_-(C_i^B)}(w_j + q_j^+ - q_j^-) - \sum_{j\in J_+(C_i^B)}(w_j + q_j^+ - q_j^-)| \leq \sum_{j\in J_0(C_i^B)}(w_j + q_j^+ - q_j^-).$$

   In this case C_i^B is a candidate for the optimal solution of MCLP with current weights, therefore, in order to $C^* = C(\mathbf{p}^*, r^*)$ become optimal the constraint $F_1(C^*) \leq F_1(C_i^B)$ should be held.
2. The condition in Theorem 2 does not hold. In this case the circle C_i^B is not optimal and should be ignored.

To enforce these cases to the model, we define the following binary variable.

$$y_i^B = \begin{cases} 1 & \text{if the condition in Theorem 2 does not hold for circle } C_i^B \\ 0 & \text{Otherwise} \end{cases}$$

Then for any $i = 1, ..., |B|$ minimizing My_i^B with the following constraints should be considered in the model.

$$|\sum_{j\in J_-(C_i^B)}\hat{w}_j - \sum_{j\in J_+(C_i^B)}\hat{w}_j| \leq \sum_{j\in J_0(C_i^B)}\hat{w}_j + My_i^B \tag{7}$$

$$F_1(C^*) \leq F_1(C_i^B) + My_i^B \tag{8}$$

$$y_i^{\mathcal{B}} \in \{0, 1\}. \tag{9}$$

Where $\hat{w_j} = w_j + q_j^+ - q_j^-$ and M is a big enough value.

Second, let A be the set of all intersection of any two bisectors and L be the set of all line segments of bisectors that are between any two points of A. Suppose

$l_i \in L$ is a line segment on the bisector of existing points $\mathbf{p}_r$ and $\mathbf{p}_s$ which lies between two points $\mathbf{a}_m, \mathbf{a}_t \in A$. For any point $\mathbf{a} \in l_i$, let $C_a^{l_i}$ be the circle with center in $\mathbf{a}$ and radius equal to $d(\mathbf{a}, \mathbf{p}_r)$. Again we consider the two following cases for $C_a^{l_i}$.

1. The condition in Theorem 2 holds, then all points in l_i satisfy in this theorem (see [6]) and the points with minimum objective function can be candidates for the optimal solution of MCLP with current weights. These points can be found by solving the following problem.

$$\begin{aligned}
\min\ f_i(\lambda_i^L) = & \sum_{j \in J_+(C_a^{l_i})} \hat{w}_j (d(\lambda_i^L \mathbf{a}_m + (1-\lambda_i^L)\mathbf{a}_t, \mathbf{p}_j) - d(\lambda_i^L \mathbf{a}_m + (1-\lambda_i^L)\mathbf{a}_t, \mathbf{p}_r)) \\
& + \sum_{j \in J_-(C_a^{l_i})} \hat{w}_j (d(\lambda_i^L \mathbf{a}_m + (1-\lambda_i^L)\mathbf{a}_t, \mathbf{p}_r) - d(\lambda_i^L \mathbf{a}_m + (1-\lambda_i^L)\mathbf{a}_t, \mathbf{p}_j)) \\
& \text{subject to } \ 0 \le \lambda_i^L \le 1.
\end{aligned}$$

   Let λ_i^{L*} be the optimal solution of this problem. Then in order to $C^* = C(\mathbf{p}^*, r^*)$ be optimal, the constraint $F_1(C^*) \le f_i(\lambda_i^{L*})$ should be considered.
2. The condition in Theorem 2 does not hold. In this case no one of the circle $C_a^{l_i}$ is optimal and ignored.

We define again the following binary variable.

$$y_i^L = \begin{cases} 1 & \text{if the condition in Theorem 2 does not hold for the circles } C_a^{l_i} \\ 0 & \text{Otherwise} \end{cases}$$

Then the following constraints with minimizing My_i^L should be considered in the model.

$$|\sum_{j \in J_-(C_i^{l_i})} \hat{w}_j - \sum_{j \in J_+(C_i^{l_i})} \hat{w}_j| \le \sum_{j \in J_0(C_i^{l_i})} \hat{w}_j + My_i^L \tag{10}$$

$$F_1(C^*) \le f_i(\lambda_i^L) + My_i^L \ \ for\ all\ 0 \le \lambda_i^L \le 1 \tag{11}$$

$$y_i^L \in \{0, 1\}. \tag{12}$$

The constraint (11) can be replaced by

$$F_1(C^*) \le \min_{0 \le \lambda_i^L \le 1} (f_i(\lambda_i^L) + My_i^L) \tag{13}$$

Third, we should consider the half lines on the bisectors. Let H be the set of all line segments of bisectors that one end of them are in A. Suppose $h_i \in H$ be the half line on the bisector B_{rs} with end point $\mathbf{a}_m \in A$. For any point $\mathbf{b} \in h_i$, let $C_b^{h_i}$ be the circle with center in $\mathbf{b}$ and radius $d(\mathbf{b}, \mathbf{p}_r)$. Again, the following two cases will be considered.

1. The condition in Theorem 2 holds, then all points in h_i satisfy in this theorem and the points with minimum objective function will be optimal solution candidates of MCLP with current weights. These points can be obtained by solving the following problem.

$$\begin{aligned}
\min\ g_i(\lambda_i^H) = & \sum_{j\in J_+(C_b^{h_i})} \hat{w}_j(d(\lambda_i^H\mathbf{b}+(1-\lambda_i^H)\mathbf{a}_m,\mathbf{p}_j)-d(\lambda_i^H\mathbf{b}+(1-\lambda_i^H)\mathbf{a}_m,\mathbf{p}_r))\\
& +\sum_{j\in J_-(C_b^{h_i})} \hat{w}_j(d(\lambda_i^H\mathbf{b}+(1-\lambda_i^H)\mathbf{a}_m,\mathbf{p}_r)-d(\lambda_i^H\mathbf{b}+(1-\lambda_i^H)\mathbf{a}_m,\mathbf{p}_j))\\
& \text{subject to}\ \ 0\le\lambda_i^H\le 1.
\end{aligned}$$

   Suppose λ_i^{H*} be the optimal solution of this problem. Then the constraint $F_1(C^*)\le g_i(\lambda_i^{H*})$ should be considered.
2. The condition in Theorem 2 does not hold, then no one of the circles $C_b^{h_i}$ is optimal and ignored.

The following binary variables are considered.

$$y_i^H = \begin{cases} 1 & \text{if the condition in Theorem 2 does not hold for the circles } C_b^{h_i}\\ 0 & \text{Otherwise}\end{cases}$$

Then minimizing My_i^H with the following constraints should be considered.

$$|\sum_{j\in J_-(C_i^{h_i})}\hat{w}_j-\sum_{j\in J_+(C_i^{h_i})}\hat{w}_j|\le\sum_{j\in J_0(C_i^{h_i})}\hat{w}_j+My_i^H \tag{14}$$

$$F_1(C^*)\le g_i(\lambda_i^H)+My_i^H\ \ for\ all\ 0\le\lambda_i^H\le 1 \tag{15}$$

$$y_i^H\in\{0,1\}. \tag{16}$$

The constraint (15) can be replaced by

$$F_1(C^*)\le\min_{0\le\lambda_i^H\le 1}(g_i(\lambda_i^H)+My_i^H) \tag{17}$$

Finally, using the equations (7 to 16) we obtain the following model for IMCLP.

$$\min\sum_{j=1}^{n}(c_j^+q_j^+ + c_j^-q_j^-) + M\sum_{b_i\in B} y_i^B + M\sum_{l_i\in L} y_i^L + M\sum_{h_i\in H} y_i^H \tag{18}$$

$$\textit{subject to} \tag{19}$$

$$|\sum_{j\in J_-(C_i^B)}\hat{w}_j - \sum_{j\in J_+(C_i^B)}\hat{w}_j| \le \sum_{j\in J_0(C_i^B)}\hat{w}_j + My_i^B,\quad b_i\in B \tag{20}$$

$$|\sum_{j\in J_-(C_i^{l_i})}\hat{w}_j - \sum_{j\in J_+(C_i^{l_i})}\hat{w}_j| \le \sum_{j\in J_0(C_i^{l_i})}\hat{w}_j + My_i^L\quad l_i\in L \tag{21}$$

$$|\sum_{j\in J_-(C_i^{h_i})}\hat{w}_j - \sum_{j\in J_+(C_i^{h_i})}\hat{w}_j| \le \sum_{j\in J_0(C_i^{h_i})}\hat{w}_j + My_i^H\quad h_i\in H \tag{22}$$

$$F_1(C^*) \le F_1(C_i^B) + My_i^B\quad b_i\in B \tag{23}$$

$$F_1(C^*) \le \min_{0\le\lambda_i^L\le 1}(f_i(\lambda_i^L) + My_i^L)\quad l_i\in L \tag{24}$$

$$F_1(C^*) \le \min_{0\le\lambda_i^H\le 1}(g_i(\lambda_i^H) + My_i^H)\quad h_i\in H \tag{25}$$

$$\hat{w}_j = w_j + q_j^+ - q_j^-\quad j=1,\cdots,n \tag{26}$$

$$y_i^H\in\{0,1\}\quad h_i\in H \tag{27}$$

$$y_i^L\in\{0,1\}\quad l_i\in L \tag{28}$$

$$y_i^B\in\{0,1\}\quad b_i\in B \tag{29}$$

$$0\le q_j^+\le u_j^+\qquad j=1,\cdots,n \tag{30}$$

$$0\le q_j^-\le u_j^-\qquad j=1,\cdots,n. \tag{31}$$

To solve this model, by starting with initial values of q_j^+ and q_j^-, we first solve the following two models for each $l_i\in L$ and $h_i\in H$, respectively.

$$\begin{aligned} f_i(\lambda_i^{L*}) = {} & \min(f_i(\lambda_i^L) + My_i^L) \\ & |\sum_{j\in J_-(C_i^{l_i})}\hat{w}_j - \sum_{j\in J_+(C_i^{l_i})}\hat{w}_j| \le \sum_{j\in J_0(C_i^{l_i})}\hat{w}_j + My_i^L \\ & y_i^L\in\{0,1\} \\ & 0\le\lambda_i^L\le 1 \end{aligned}$$

and

$$
\begin{aligned}
g_i(\lambda_i^{H*}) = \min(& g_i(\lambda_i^H) + My_i^H) \\
& |\sum_{j\in J_-(C_i^{h_i})} \hat{w}_j - \sum_{j\in J_+(C_i^{h_i})} \hat{w}_j| \leq \sum_{j\in J_0(C_i^{h_i})} \hat{w}_j + My_i^H \\
& y_i^H \in \{0,1\} \\
& 0 \leq \lambda_i^H \leq 1.
\end{aligned}
$$

Then let

$$
G^* = \min_{\substack{i=1,\ldots,|L| \\ k=1,\ldots,|H|}} (f_i(\lambda_i^{L*}), g_i(\lambda_i^{H*})). \tag{32}
$$

We replace the constraints (24) and (25) with the following constraint.

$$
F_1(C^*) \leq G^*. \tag{33}
$$

Then the obtained model is solved. In the optimal solution if the new values of q_j^+ and q_j^- are obtained then we find again G^* with the new values of q_j^+ and q_j^- and solve the main model. This procedure is repeated until we don't have any change in the values of q_j^+ and q_j^-.

Note that in some cases G^* may be zero or a very small value, for example consider the case that existing points lie on (or very near) the circumference of a circle $C \neq C^*$. In these cases the constraint (33) is infeasible and C^* can not be optimal by any changes on the weights of points. Therefore the problem is infeasible. This infeasibility can be checked before the start of the algorithm.

We should also mention that the novelty of this paper is converting a model with infinite constraints to a finite one and proposing an efficient heuristic algorithm for solving it. However, the presented method may not find the optimal solution for all instances.

Example 3. *consider the inverse circle location problem with five existing points. The coordinates, weights, upper and lower bounds of weights and the cost of modifying per unit of weights are given in Table 1. The optimal circle with the current weights is $\bar{C} = (0, 6)$ with $F_1(\bar{C}) = 2$. Suppose that we want to modify the weights of vertices with minimum cost such that the circle $C^* = C(P^*, r^*)$ become optimal. Where $P^* = (0, \frac{11}{12})$ and $r^* = \frac{61}{12}$ (see Figure 1). With respect to the current weights we obtain $F_1(C^*) = \frac{550}{3}$.*

We start with initial values $q_j^+ = 0$ and $q_j^- = 0$. The bisectors of existing points are depicted in Figure 2. Since $P^* \in B_{14} \cap B_{34}$, then the problem has a solution. We observe that $J_-(C^*) = \emptyset$, $J_+(C^*) = \{2\}$ and $J_0(C^*) = \{1, 3, 4\}$, thus $F_1(C^*) = (\frac{11}{6})(100 + q_2^+ - q_2^-)$.

The members of B, i.e. the intersections of any three bisectors, and their corresponding radius are given in Table 2. Then we should find all line segments

i	$\mathbf{p}_i = (x_i, y_i)$	w_i	u_i^-	u_i^+	c_i^-	c_i^+
1	$\mathbf{p}_1 = (0, 6)$	100	99	900	1	1
2	$\mathbf{p}_2 = (0, -6)$	100	99	900	1	1
3	$\mathbf{p}_3 = (5, 0)$	1	0.5	999	1	1
4	$\mathbf{p}_4 = (-5, 0)$	1	0.5	999	1	1

Table 1: The information of existing points

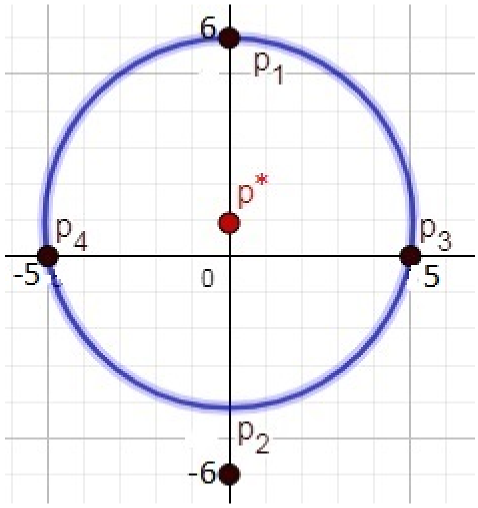

Figure 1: The given circle and existing points

and half lines on the bisectors, i.e. the members of L and H, respectively. Then the best solutions on these line segments and half lines should be calculated. The members of L and H and the best solutions for each member of them are presented in tables 3 and 4. Using these tables a mixed integer linear programming should be solved.

Table 2: The intersections of any three bisectors

i	$\mathbf{b}_i$	r_i^B	$J_+(C_i^B)$	$J_-(C_i^B)$	$J_0(C_i^B)$
1	$\mathbf{b}_1 = (0, 0.917)$	5.083	$\{2\}$	$\{\}$	$\{1, 3, 4\}$
2	$\mathbf{b}_2 = (0, -0.917)$	5.083	$\{1\}$	$\{\}$	$\{2, 3, 4\}$
3	$\mathbf{b}_3 = (1.100, 0)$	6.100	$\{\}$	$\{3\}$	$\{1, 2, 4\}$
4	$\mathbf{b}_4 = (-1.100, 0)$	6.100	$\{\}$	$\{4\}$	$\{1, 2, 3\}$

The optimal solution of this model is $q_2^- = \frac{1088}{11}$ with the value of objective function $\frac{1088}{11}$. Then by solving all sub-problems with these new values of q_j^+ and q_j^- we find

$$\min_{\substack{i=1,\ldots,|L| \\ k=1,\ldots,|H|}} (f_i(\lambda_i^{L*}), g_i(\lambda_i^{H*})) = 2.$$

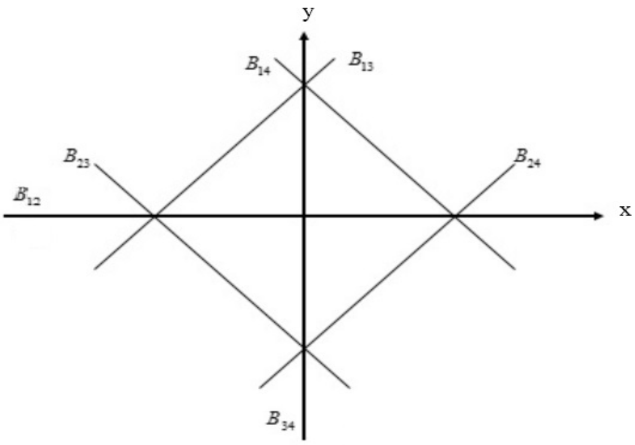

Figure 2: The bisectors of existing points

Table 3: The members of L and the best solutions on them

i	bisector	$\mathbf{a}_m$	$\mathbf{a}_t$	$\mathbf{p}_r$	$J_+(C_i^L)$	$J_-(C_i^L)$	$J_0(C_i^L)$	$f_i(\lambda_i^{L*})$
1	B_{12}	$(-1.100, 0.00)$	$(0.00, 0.00)$	$(0, 6)$	$\{\}$	$\{3, 4\}$	$\{1, 2\}$	2.00
2	B_{12}	$(0.00, 0.00)$	$(1.100, 0.00)$	$(0, 6)$	$\{\}$	$\{3, 4\}$	$\{1, 2\}$	2.00
3	B_{13}	$(-1.100, 0.00)$	$(0.00, 0.917)$	$(0, 6)$	$\{2\}$	$\{3, 4\}$	$\{1\}$	2.20
4	B_{14}	$(0.00, 0.917)$	$(1.100, 0.00)$	$(0, 6)$	$\{2\}$	$\{3, 4\}$	$\{1\}$	2.20
5	B_{23}	$(-1.100, 0.00)$	$(0.00, -0.917)$	$(0, -6)$	$\{1\}$	$\{3, 4\}$	$\{2\}$	2.20
6	B_{24}	$(0.00, -0.917)$	$(1.100, 0.00)$	$(0, -6)$	$\{1\}$	$\{3, 4\}$	$\{2\}$	2.20
7	B_{34}	$(0.00, -0.917)$	$(0.00, 0.00)$	$(5, 0)$	$\{1, 2\}$	$\{\}$	$\{3, 4\}$	1183.33
8	B_{34}	$(0.00, 0.00)$	$(0.00, 0.917)$	$(5, 0)$	$\{1, 2\}$	$\{\}$	$\{3, 4\}$	1183.33

Thus $G^* = 2$. Then in the new optimal solution the values of q_j^+ and q_j^-, do not change. Therefore, the algorithm is terminated with the values $\hat{w}_1 = 100$, $\hat{w}_2 = \frac{12}{11}$, $\hat{w}_3 = 1$ and $\hat{w}_4 = 1$. One can see that with these new weights $F_1(\bar{C}) = F_1(C^*) = 2$.

4. SUMMARY AND CONCLUSION

In this paper, we considered the inverse minisum circle location problem by modifying the weight of vertices. A mathematical model was proposed for the problem, then an efficient approach is presented for solving the model.

As the future works, the other kinds of inverse circle location problems such as inverse circle problem with a fixed radius, inverse minimax problem and inverse circle location on networks with variable edge lengths can be considered.

Table 4: The members of H and the best solutions on them

i	bisector	$\mathbf{a}_m$	$\mathbf{b}$	$\mathbf{p}_r$	$J_+(C_i^H)$	$J_-(C_i^H)$	$J_0(C_i^H)$	$g_i(\lambda_i^{H*})$
1	B_{12}	$(-1.100, 0.00)$	$(-2.100, 0.00)$	$(0, 6)$	$\{3\}$	$\{3\}$	$\{1, 2\}$	2.20
2	B_{12}	$(1.100, 0.00)$	$(2.100, 0.00)$	$(0, 6)$	$\{4\}$	$\{3\}$	$\{1, 2\}$	2.20
3	B_{13}	$(-1.100, 0.00)$	$(-2.100, -0.830)$	$(0, 6)$	$\{\}$	$\{2, 4\}$	$\{1, 3\}$	2.20
4	B_{13}	$(0.00, 0.917)$	$(6.00, 5.917)$	$(0, 6)$	$\{2, 4\}$	$\{\}$	$\{1, 3\}$	183.33
5	B_{14}	$(0.00, 0.917)$	$(-1.00, 1.750)$	$(0, 6)$	$\{2, 3\}$	$\{\}$	$\{1, 4\}$	183.33
6	B_{14}	$(1.100, 0.00)$	$(6.00, -4.083)$	$(0, 6)$	$\{\}$	$\{2, 3\}$	$\{1, 4\}$	2.20
7	B_{23}	$(-1.100, 0.00)$	$(-2.100, 0.830)$	$(0, -6)$	$\{\}$	$\{1, 4\}$	$\{2, 3\}$	2.20
8	B_{23}	$(0.00, -0.917)$	$(6.00, -5.917)$	$(0, -6)$	$\{1, 4\}$	$\{\}$	$\{2, 3\}$	183.33
9	B_{24}	$(0.00, -0.917)$	$(1.00, -1.750)$	$(0, -6)$	$\{1, 3\}$	$\{\}$	$\{2, 4\}$	183.33
10	B_{24}	$(1.100, 0.00)$	$(2.100, 0.830)$	$(0, -6)$	$\{\}$	$\{1, 3\}$	$\{2, 4\}$	2.20
11	B_{34}	$(0.00, -0.917)$	$(0.00, -1.917)$	$(5, 0)$	$\{1\}$	$\{2\}$	$\{3, 4\}$	183.33
12	B_{34}	$(0.00, -0.917)$	$(0.00, 1.917)$	$(5, 0)$	$\{2\}$	$\{1\}$	$\{3, 4\}$	183.33

REFERENCES

[1] Alizadeh, B., Afrashteh, E., and Baroughi-Bonab, F., "Combinatorial algorithms for some variants of inverse obnoxious median location problem on tree networks", *Journal of Optimization Theory and Applications*, 178 (3) (2018) 914-934.

[2] Alizadeh, B., Burkard, R.E., and Pferschy, U., "Inverse 1-center location problems with edge length augmentation on trees", *Computing*, 86 (4) (2009) 331-343.

[3] Alizadeh, B., and Burkard, R.E., "Combinatorial algorithms for inverse absolute and vertex 1-center location problems on trees", *Networks*, 58 (3) (2011) 190-200.

[4] Alizadeh, B., and Etemad, R., "Optimal algorithms for inverse vertex obnoxious center location problems on graphs", *Theoretical Computer Science*, 707 (2018) 36-45.

[5] Baroughi-Bonab, F., Burkard, R.E., and Gassner, E., "Inverse p-median problems with variable edge lengths", *Mathematical Methods of Operations Research*, 73 (2) (2011) 263-280.

[6] Brimberg, J., Juel, H., and Schóbel, A., "Locating a minisum circle in the plane", *Discrete Applied Mathematics*, 157 (5) (2009) 901–912.

[7] Burkard, R. E., Galavii, M., and Gassner, E., "The inverse Fermat-Weber problem", *European Journal of Operation Research*, 206 (1) (2010) 11-17.

[8] Burkard, R. E., Pleschiutschnig, C., and Zhang, J. Z., "Inverse median problems", *Discrete Optimization*, 1 (1) (2004) 23-39.

[9] Burkard, R. E., Pleschiutschnig, C., and Zhang, J. Z., "The inverse 1-median problem on a cycle", *Discrete Optimization*, 5 (2) (2008) 242-253.

[10] Cai, M.C., Yang, X.G., and Zhang, J., "The complexity analysis of the inverse center location problem", *Journal of Global Optimization*, 15 (2) (1999) 213-218.

[11] Drezner, Z., Steiner, G., and Wesolowsky, G.O., "On the circle closest to a set of points", *Computers and Operations Research*, 29 (6) (2002) 637–650.

[12] Galavii, M., "The inverse 1-median problem on a tree and on a path", *Electronic Notes in Discrete Mathematics*, 36 (2010) 1241-1248.

[13] Gholami, M., and Fathali, J., "The semi-obnoxious minisum circle location problem with Euclidean norm", *International Journal of Nonlinear Analysis and Applications*, 12 (1) (2021) 669-678.

[14] Gholami, M. and Fathali, J., "Mathematical models for the variable weights version of the inverse minimax circle location problem", *Journal of Mathematical Modeling* 9 (1) (2021) 137-144.

[15] Guan, X. C., and Zhang, B. W., "Inverse 1-median problem on trees under weighted Hamming distance", *Journal of Global Optimization*, 54 (1) (2012) 75-82.

[16] Labbé, M., Laporte, G., Rodriguez-Martin, I., and Gonzalez, J.J.S., "Locating median cycles in networks", *European Journal of Operation Research* 160 (2) (2005) 457–470

[17] Nazari, M., Fathali, J., Nazari, M., and Varedi-Koulaei, S.M., "Inverse of backup 2-median

problems with variable edge lengths and vertex weight on trees and variable coordinates on the plane", *Production and Operations Management*, 9 (2) (2018) 115-137.

[18] Nazari, M., and Fathali, J., "Reverse backup 2-median problem with variable coordinate of vertices", *Journal of Operational Research and Its Applications*, 15 (2) (2018) 63-88.

[19] Nguyen, K.T., "Inverse 1-median problem on block graphs with variable vertex weights", *Journal of Optimization Theory and Applications*, 168 (3) (2016) 944-957.

[20] Nguyen, K.T., and Sepasian, A.R., "The inverse 1-center problem on trees with variable edge lengths under Chebyshev norm and Hamming distance", *Journal of Combinatorial Optimization*, 32 (3) (2016) 872-884.

[21] Omidi, S., and Fathali, J., "Inverse single facility location problem on a tree with balancing on the distance of server to clients", *Journal of Industrial and Management Optimization*, 18 (2) (2022) 1247-1259.

[22] Omidi, S., Fathali, J., and Nazari, M., "Inverse and reverse balanced facility location problems with variable edge lengths on trees", *OPSEARCH*, 57 (2) (2020) 261-273.

[23] Sepasian, A.R., and Rahbarnia, F., "An O(nlogn) algorithm for the inverse 1-median problem on trees with variable vertex weights and edge reductions", *Optimization*, 64 (3) (2015) 595-602.