

# OPTIMAL CONTROL POLICY TO PRODUCTION AND INVENTORY SYSTEM WITH PROMOTION EFFORT DEPENDENT DEMAND IN SEGMENTED MARKET

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Received: February 2020 / Accepted: May 2020

**Abstract:** Increased competition in market leads to the interaction between marketing and production of a firm in segmented market. This paper considers the problem of finding an optimal promotion and production strategy, where the consumer demand rate depends on differentiated and mass promotion efforts. Differentiated promotions can reach each segment independently and mass promotion reaches it with a fixed segment-spectrum. Under reasonable conditions, two optimal control problems are formulated. The first one considers a single source inventory and multi-segmented demand problem. In the second, a multi-segmented inventory and demand problem is considered, where it is assumed that the firm chooses an inventory directed to each segment. The solution to both problems is obtained by using Pontryagin's maximum principle. Numerical examples are provided to illustrate the applicability of proposed models. The discretized version of the problem is formulated and solved on some numeric data.

**Keywords:** Inventory-Production, Segmentation, Optimal Control Problem, Promotional Efforts.

**MSC:** 49J15, 90B60, 49N90, 97M40.

## 1. INTRODUCTION

With technological development, the advent of newer goods and services has been on a rise. The basic model given by Bass [6] has proved to be of great

importance while analysing the diffusion of newly produced products in the market. The market size has also shifted to one with dynamic potential. Goods produced at a particular source may be kept at various warehouses before reaching the users. Production companies have to focus on formulation of strategies for production and storage of goods in order to meet the required market demand. These policies of production and inventory planning, on one hand, focus on meeting the demands and on the other, aim at maximising profits for the firm.

Further, the market is partitioned into segments with customers in different segments having varied demands for the product. This has necessitated the decision makers to formulate different production and inventory properties for different sections. In order to increase the sales of the product, promotional policies have to be employed effectively. Since the population is segmented into groups, it becomes essential to have customised promotional strategies. In addition to segmented promotion, a uniform level of promotion may also be applied to the entire population. This results in differentiated and mass promotional strategies. Few authors like Duran *et al.* [23], and Chen and Li [28] have given mathematical models on inventory and production planning in a segmented market.

For designing policies to optimise profits, applicability of optimal control theory has resulted in meaningful outcomes. The dynamic nature of the said theory has made it suitable to be used in formulating plans for inventory and production. The techniques of optimal control have been used to solve problems in other areas as well. Davis and Elzinga [3] have solved an investment problem in the field of finance with the use of optimal control model. Elton and Gruber [5] have also used similar models to devise plans for investment.

In the field of economics, optimal control models have been employed by Arrow and Kurz [12] to decide the allocation of resources in various sectors in order to achieve required growth. Seierstad and Sydsaeter [2] and Feichtinger [8] have also employed optimal controls in the field of economics. Feichtinger *et al.* [9] have utilised this concept to analyse marketing strategies and determine advertising expenditures. Optimal advertising strategies using the concept have been discussed by Hartl [20]. Sethi [21] has discussed advertising models designed to maximise profits, incorporating the concepts of free and fixed end points. Sethi [22] has further summarised various optimal control models, which have been studied in the field of advertising and marketing. Maintenance models have been considered by Pierskalla and Voelker [27]. Amit [18] has made use of optimal control models in the process of petroleum recovery. Optimising the use of limited natural resources using such dynamic models have been discussed by Derzko and Sethi [15] and Heaps [25]. Optimal control models to solve marketing and production planning problems have been given by Abad [17]. The author has employed a reverse-time parametric technique to solve such problems.

Yi *et al.* [7] have applied the concept of optimal control to production and inventory control systems where the demand is assumed to vary stochastically. They used dynamic programming principle to solve the model. Authors Benkherouf *et al.* [13] have established the optimal control of an inventory system where there is a possibility that the customer returns back the product. A policy for the handling

of these returned products is discussed. The case of the products being returned is also studied by Gayon *et al.*[11]. The behaviour of return is stochastic in the production-inventory system under consideration. Pooya and Pakdaman [1] have applied optimal control theory to a production-inventory system whose solution is given by the use of neural networks. In the paper given by Ahmadi *et al.*[24], it is assumed that the consumer follows Poisson distribution and an optimal control policy is developed for the inventory system with a constant commitment lead time.

In the paper given by Li and Wang [14], the authors consider a production inventory setup in which the inventory records are not maintained accurately. It is assumed that these error rates follow normal distribution. The problem is aimed to provide an optimal control policy for replenishment of inventory and production. The paper given by Duan *et al.*[29] discusses the problem of finding an optimal production and inventory policy where the products under consideration deteriorate with time. In this paper the demand is assumed to dependent on price and the potential demand is explained by a stochastic process. The paper given by Bhatnagar and Bing [19] considers joint policies for transshipment and production control for the case when the products are stored at different locations to meet the demands occurring there. An optimization problem in a dynamic inventory system is considered by Shah *et al.*[16]. The model studied assumes that the demand for the items is dependent on price, service, and time. The products that are stocked are assumed to deteriorate at a constant rate. A decision model given by Kumar *et al.*[26] discuss the pricing and warranty policies for a new product incorporating free replacement during warranty period and re-working throughout production process. An optimal solution of the production-Inventory set-up has been discussed by Giri and Sharma [4]. In this article, there is a single manufacturer and a single retailer and the production-inventory model is studied under the assumption that the demand depends on price. The manufacturer is assumed to give cash discounts to the retailers.

In our paper, a model in the marketing-production system is considered, where we assume that the population under consideration is divided into segments with customers in each segment having common requirements and preferences. This leads to different levels of efforts made to popularise the product in different sections. The models considered in early studies have taken the demand to be either some fixed value or varying with price. But here we assume that demand depends on the level of mass and differentiated promotions and varies continuously with time. The model is formulated in two different scenarios. One considers the situation in which the product is produced at a single production unit and stored in a single warehouse before reaching the customers in various segments. In the second scenario, the storage is also done segment-wise. The main aim in both the cases is to maximize the profits while manufacturing products to meet the demand. The calculation of profits takes into account the expenditure incurred in promotion.

The paper is organized in the following manner. The model notations, which have been used in the paper, are mentioned in Section 2. In section 3, we have discussed the Model Development. We have further considered the two cases

in next section. In section 4, we have studied the situation when production takes place at a single source, the inventory is also maintained at single source but due to segmentation of market, the demand occurs at multiple destinations. Further, we have discussed the optimal control problem in which the inventory is also maintained at different segments in section 5. Section 6 gives numerical illustrations for the two models discussed in previous section. Section 7 is the concluding section and provides some possible areas of research in future.

## 2. MODEL NOTATIONS

The following notations have been used throughout the paper.

$P(t)$	: Production rate at time t,
$I_i(t)$	: Inventory level at time t in $i^{th}$ segment,
$N_i(t)$	: Cumulative sales by time t in $i^{th}$ segment,
$\bar{N}_i$	: Total potential buyers in the $i^{th}$ segment,
$h_i(I_i(t))$	: Holding cost rate for $i^{th}$ segment, (multi destination)
$K(P(t))$	: Cost rate corresponding to the production rate,
$u_i(t)$	: Differentiated promotional effort rate for $i^{th}$ segment at time t
$u(t)$	: Mass market promotional effort rate at time t
$d_i(t)$	: Demand rate at time t in $i^{th}$ segment,
$r_i$	: The revenue rate per unit sale in $i^{th}$ segment,
$\rho$	: Constant non-negative discount rate.

## 3. MODEL DEVELOPMENT

Let the whole market be partitioned into  $n$  segments and we consider a case in which a manufacturer is producing only one product and sells it in a partitioned market. To fulfil the consumer demand for products, the production house plans a production-inventory setup. Once the goods are manufactured, they are stored in warehouses before they reach the customers. Promotion plays an important role to generate the awareness and purchase for a product. The demand of product is assumed to be generated through differentiated and mass promotions in segment specific market. The demand rate is assumed to be dependent on the level of promotional efforts and follows the Bass Model. Thus, the consumer demand rate function is given by

$$d_i(t) = \frac{dN_i}{dt} = (p_i + q_i \frac{N_i(t)}{\bar{N}_i})(u_i(t) + \alpha u(t))(\bar{N}_i - N_i(t)), N_i(0) = N_0 \quad (1)$$

Where,  $(\bar{N}_i - N_i(t))$  represents the potential customers who have not purchased the product.

We consider two different situations. In the first situation, we assume that demand from all segments depends on one inventory source. The second one, we assume that there are  $n$  inventory points corresponding to each market segment and demand from each segment can selectively reach each target inventory source.

Then evolution of inventory rate in the segments for the two situations is given by the following differential equations respectively:

$$\frac{d}{dt}I(t) = P(t) - \sum_{i=1}^n d_i(t), I(0) = I_0 \tag{2}$$

$$\frac{d}{dt}I_i(t) = \gamma_i P(t) - d_i(t), I_i(0) = I_{i0} \tag{3}$$

When a new product is introduced in market, the firm needs to set up its capacity and decide on its inventory policy to maximize its revenue over the new product life cycle. The firm aims to achieve maximum total profit during the time period  $T$  under consideration in segmented market. Hence, the total profit is given by

$$\max_{P(t) \geq \sum_{i=1}^n d_i(t)} J = \int_0^T e^{-\rho t} \left( \sum_{i=1}^n \left[ r_i N_i(t) - \frac{\epsilon_i}{2} u_i^2(t) \right] - K(P(t)) - \frac{\epsilon}{2} u^2(t) - h(I(t)) \right) dt \tag{4}$$

where, holding cost rate for the  $i^{th}$  segment depends on the level of inventory and production cost rate on the production rate. For the illustration purpose, let us assume that functions  $h_i(I_i(t)) = h_i I_i(t)$  and  $K(P) = \frac{k}{2} P^2$ . All functions considered here are assumed to be non-negative, continuous, and differentiable. Now, there are two types of the optimization problems corresponding to equation (2) and equation (3): (P1) Single source inventory, and multi-segmented demand problem; (P2) Multi Segmented Inventory and Demand Problem.

#### 4. SINGLE SOURCE INVENTORY and MULTI-SEGMENTED DEMAND OPTIMAL CONTROL PROBLEM

This case considers a situation when the production and storage occur at a single source. The goods are then sold in various segments depending on respective demands. The objective is to obtain the maximum total profit during the planning period in segmented market. Hence, the optimal control problem is given as follows:

$$\begin{cases} \text{Max } J_{P(t) \geq \sum_{i=1}^n d_i(t)} = \int_0^T e^{-\rho t} \left( \sum_{i=1}^n \left[ r_i N_i(t) - \frac{\epsilon_i}{2} u_i^2(t) \right] - \frac{k}{2} P(t) - \frac{\epsilon}{2} u^2(t) - hI(t) \right) dt, \\ \text{subject to} \\ \frac{d}{dt}I(t) = P(t) - \sum_{i=1}^n d_i(t), I(0) = I_0 \\ d_i(t) = \frac{dN_i}{dt} = (p_i + q_i \frac{N_i(t)}{N_i})(u_i(t) + \alpha u(t))(\bar{N}_i - N_i(t)), N_i(0) = N_0 \end{cases} \tag{5}$$

The Hamiltonian and Lagrangian functions [2] for the above optimal control problem are respectively given by:

$$H = \left( \sum_{i=1}^n \left[ r_i N_i \dot{t} - \frac{\epsilon_i}{2} u_i^2(t) \right] - \frac{k}{2} P(t) - hI(t) - \frac{\epsilon}{2} u^2(t) \right) + \lambda(t) \left( P(t) - \sum_{i=1}^n d_i(t) \right) \quad (6)$$

$$L = \left( \sum_{i=1}^n \left[ r_i N_i \dot{t} - \frac{\epsilon_i}{2} u_i^2(t) \right] - \frac{k}{2} P(t) - hI(t) - \frac{\epsilon}{2} u^2(t) \right) + (\lambda(t) + \mu(t)) \left( P(t) - \sum_{i=1}^n d_i(t) \right) \quad (7)$$

The Hamiltonian presented here represents the total profit obtained as a result of the various policy decisions, while taking into account their immediate and future effects. Hamiltonian  $H$  is strictly concave for all segments. Following the maximum principle[2], the necessary conditions for optimality are given as:

$$H(t, N_i^*, I^* u_i^*, u^*, \lambda, \mu) \geq H(t, N_i^*, I^* u_i, u, \lambda, \mu), \quad (8)$$

$$\frac{\partial L^*}{\partial u_i} = 0; \quad \frac{\partial L^*}{\partial u} = 0; \quad \frac{\partial L^*}{\partial P} = 0 \quad (9)$$

$$\frac{d}{dt} \lambda(t) = \rho \lambda - \frac{\partial L}{\partial I}, \quad \lambda(T) = 0 \quad (10)$$

$$\left( P(t) - \sum_{i=1}^n d_i(t) \right) \geq 0, \quad \mu(t) \geq 0, \quad \mu(t) \left( P(t) - \sum_{i=1}^n d_i(t) \right) = 0 \quad (11)$$

After solving equation (10), we have:

$$\lambda(t) = \frac{h}{\rho} (e^{-\rho(T-t)} - 1) \quad (12)$$

Using expressions in equation (9), we have

$$u_i^*(t) = \frac{1}{\epsilon_i} (r_i - \lambda_i - \mu_i) \frac{\partial \dot{N}_i}{\partial u_i} \quad (13)$$

$$u^*(t) = \frac{1}{\epsilon} \left[ \sum_{i=1}^n (r_i - \lambda_i - \mu_i) \frac{\partial \dot{N}_i}{\partial u} \right] \quad (14)$$

$$P^*(t) = \frac{1}{k} (\lambda(t) + \mu(t)) \quad (15)$$

From the expressions (13) and (14), we observe that initially both the optimal promotional effort rate are at maximum promotion effort rate level and as untapped market potential level decreases, which in turn reduces the potential adopters, the decision maker should lower the promotional effort level at the end of the planning period. From equation (11), there are two conditions either  $P(t) - \sum_{i=1}^n d_i(t) = 0$  or  $P(t) - \sum_{i=1}^n d_i(t) > 0$

#### 4.1. Case 1:

Let  $S$  be the subset of planning period  $[0, T]$ , corresponding to the condition  $P(t) - \sum_{i=1}^n d_i(t) = 0$ . Then,  $\frac{d}{dt}I(t) = 0$  on  $S$ . Hence  $I^*(t)$  is constant on  $S$ . The optimal production rate is given by the following equation:

$$P^*(t) = \sum_{i=1}^n \left[ \left( p_i + q_i \frac{N_i(t)}{\bar{N}_i} \right) (u_i^* + \alpha_i u^*(t)) (\bar{N}_i - N_i(t)) \right] \quad (16)$$

and the values of adjoint variables  $\lambda(t)$  and  $\mu(t)$  are given by:

$$\lambda(t) = \frac{h}{\rho} (e^{-\rho(T-t)} - 1) \quad (17)$$

$$\mu(t) = kP^*(t) - \frac{h}{\rho} (e^{-\rho(T-t)} - 1) \quad (18)$$

#### 4.2. Case 2:

When  $P(t) - \sum_{i=1}^n d_i(t) > 0$  for  $t \in [0, T] \setminus S$ , then  $\mu_i(t) = 0$  on  $t \in [0, T] \setminus S$ . Then the optimal value of the production and inventory level at time  $t$  can be obtained by

$$\frac{dP(t)}{dt} = \rho P(t) + \frac{h}{k}, \text{ and} \quad (19)$$

$$\frac{d^2}{dt^2}I(t) - \rho \frac{dI}{dt} = \frac{h}{K} + \sum_{i=1}^n (\rho \dot{N}_i(t) - \ddot{N}_i(t)), I(0) = I_0, P(T) = 0 \quad (20)$$

The above problem is a two point boundary value problem. The problem is solved in the next proposition.

**Proposition 1.** *The optimal value of inventory can be described as*

$$I^*(t) = c_1 e^{-\sqrt{\rho}t} + c_2 e^{\sqrt{\rho}t} + Q(t), \quad (21)$$

and the corresponding production is

$$P^*(t) = -c_1 \sqrt{\rho} e^{-\sqrt{\rho}t} + c_2 \sqrt{\rho} e^{\sqrt{\rho}t} + \frac{d}{dt}Q(t) + \sum_{i=1}^n d_i(t) \quad (22)$$

where,  $Q(t)$  is a particular solution, and values of the constant  $c_1, c_2$  are given in the proof.

**Proof:** The characteristic equation of (20) is  $m^2 - \rho = 0$ . It has two real roots,  $m_1 = -\sqrt{\rho}$ ;  $m_2 = \sqrt{\rho}$ . Therefore, the solution of differential equation (20) is given by:

$$I(t) = c_1 e^{-\sqrt{\rho}t} + c_2 e^{\sqrt{\rho}t} + Q(t),$$

where  $Q(t)$  is the particular solution of equation. By using the initial and terminal conditions (20), the value of constants  $c_1$  and  $c_2$  can be determined as follows:

$$c_1 + c_2 + Q(0) = I_0,$$

$$-c_1\sqrt{\rho}e^{-\sqrt{\rho}T} + c_2\sqrt{\rho}e^{\sqrt{\rho}T} + \frac{d}{dt}Q(T) + \sum_{i=1}^n d_i(T) = 0.$$

By putting  $a_1 = I_0 - Q(0)$  and  $a_2 = -(\frac{d}{dt}Q(T) + \sum_{i=1}^n d_i(T))$  in above equations, we obtain a system of two linear equations with two variables given as follows:

$$c_1 + c_2 = a_1$$

$$-c_1\sqrt{\rho}e^{-\sqrt{\rho}T} + c_2\sqrt{\rho}e^{\sqrt{\rho}T} = a_2$$

which has the unique solution as follows:

$$c_1 = \frac{a_2 - \sqrt{\rho}e^{\sqrt{\rho}T}a_1}{-\sqrt{\rho}e^{-\sqrt{\rho}T} - \sqrt{\rho}e^{\sqrt{\rho}T}}; \quad c_2 = \frac{-a_1\sqrt{\rho}e^{-\sqrt{\rho}T} - a_2}{-\sqrt{\rho}e^{-\sqrt{\rho}T} - \sqrt{\rho}e^{\sqrt{\rho}T}}$$

The value of  $P^*$  can be determined by using the values of  $I^*(t)$  and the state equation (2). As  $(P(t) - \sum_{i=1}^n d_i(t)) \geq 0$ , optimal value  $P^*$  can be given by:

$$P^*(t) = \max\left(P(t), \sum_{i=1}^n d_i(t)\right)$$

### 5. MULTI SEGMENTED INVENTORY and DEMAND OPTIMAL CONTROL MODEL

In this section, we assume a single source production and a multi destination demand-inventory system. Accordingly, we define the profit maximization objective function as follows:

$$\begin{cases} \text{Max} J_{\gamma_i P(t) > d_i(t)} = \int_0^T e^{-\rho t} \left( \sum_{i=1}^n [r_i \dot{N}_i(t) - h_i I_i(t) - \frac{\epsilon_i}{2} u_i^2(t)] - \frac{k}{2} P^2 - \frac{\epsilon}{2} u^2(t) \right) dt \\ \text{subject to} \\ \frac{d}{dt} I_i(t) = \gamma_i P(t) - d_i(t), I_i(0) = I_{i0}, \\ \gamma_i P(t) - d_i(t) \geq 0 \end{cases} \tag{23}$$

Here  $\gamma_i > 0, \sum_{i=1}^n \gamma_i = 1, \gamma_i P(t)$  defines the relative segment production rate towards  $i^{th}$  segment and  $h_i(I_i(t))$  is the inventory cost for  $i^{th}$  segment. The Hamiltonian and Lagrangian functions for the above optimal control problem are respectively given by:

$$H = \left( \sum_{i=1}^n [r_i \dot{N}_i(t) - h_i I_i(t) - \frac{\epsilon_i}{2} u_i^2(t)] - \frac{k}{2} P^2 - \frac{\epsilon}{2} u^2(t) \right) + \sum_{i=1}^n \lambda_i(t) [\gamma_i P(t) - d_i(t)]$$



(24)

$$L = \left( \sum_{i=1}^n \left[ r_i \dot{N}_i(t) - h_i I_i(t) - \frac{\epsilon_i}{2} u_i^2(t) \right] - \frac{k}{2} P^2 - \frac{\epsilon}{2} u^2(t) \right) + \sum_{i=1}^n (\lambda_i(t) + \mu_i(t)) (\gamma_i P(t) - d_i(t)) \quad (25)$$

. By using Maximum Principle [2], we have the following values of optimal control variable and adjoint variables, respectively:

$$u_i^*(t) = \frac{1}{\epsilon_i} (r_i - \lambda_i - \mu_i) \frac{\partial \dot{N}_i(t)}{\partial u_i} \quad (26)$$

$$u^*(t) = \frac{1}{\epsilon} \left[ \sum_{i=1}^n (r_i - \lambda_i - \mu_i) \frac{\partial \dot{N}_i(t)}{\partial u} \right] \quad (27)$$

$$\sum_{i=1}^n (\lambda_i + \mu_i) \gamma_i = kP(t) \quad (28)$$

$$\frac{d}{dt} \lambda_i(t) = \rho \lambda_i + h_i, \lambda_i(T) = 0 \quad (29)$$

Similarly, from equation (23), we either have  $\gamma_i P(t) - d_i(t) = 0$  or  $\gamma_i P(t) - d_i(t) > 0$

### 5.1. Case 1:

Let S be a subset of planning horizon  $[0, T]$ . When  $\gamma_i P(t) - \dot{N}_i(t) = 0$ , then  $\frac{dI_i}{dt} = 0$  on S. In this case,  $I(t)$  obtains a constant value on S and the rate of production is given by:

$$P^*(t) = \frac{1}{\gamma_i} \left( p_i + q_i \frac{N_i(t)}{\bar{N}_i} \right) (u_i^* + \alpha_i u^*(t)) (\bar{N}_i - N_i(t)) \quad (30)$$

### 5.2. Case 2:

In this case,  $\gamma_i P(t) - d_i(t) > 0$ . Then  $\mu_i(t) = 0 \forall i, t \in [0, T] \setminus S$ . From equation (28), the optimal production rate is given by the following differential equation:

$$\frac{dP}{dt} = \rho P(t) + \frac{1}{k} \sum_{i=1}^n r_i h_i, P(T) = 0 \quad (31)$$

The value of the adjoint variable is:

$$\lambda_i(t) = \frac{h_i}{\rho} (e^{-\rho(T-t)} - 1) \quad (32)$$

The optimal promotional effort is given by:

$$u_i^*(t) = \frac{1}{\epsilon_i} \left[ r_i - \frac{h_i}{\rho} (e^{-\rho(T-t)} - 1) (p_i + q_i \frac{N_i(t)}{\bar{N}_i}) (\bar{N}_i - N_i(t)) \right] \quad (33)$$

$$u^*(t) = \frac{1}{\epsilon} \left[ r_i - \frac{h_i}{\rho} (e^{-\rho(T-t)} - 1) \alpha_i (p_i + q_i \frac{N_i(t)}{\bar{N}_i}) (\bar{N}_i - N_i(t)) \right] \quad (34)$$

The necessary conditions for  $I_i^*(t)$  to be an optimal solution to problem is given by:

$$\frac{d^2 I_i(t)}{dt^2} - \rho \frac{dI_i(t)}{dt} = \eta_i(t), I_i(0) = I_i^0, P(T) = 0 \quad (35)$$

where,  $\eta_i(t) = -\rho d_i(t) + \frac{\gamma_i}{k} (\sum_{i=1}^n \gamma_i h_i) + \frac{d}{dt} d_i(t)$ . The above problem is a two point boundary value problem. It is solved by the same method, which was used in previous section.

## 6. NUMERICAL ILLUSTRATION

In this section, the numerical examples are presented to illustrate the applicability of the formulated model. The discounted optimal control problem(5) is transformed into the equivalent discrete form of the model as given by Rosen [10]. The corresponding discrete version of the optimal control problem as in section 4 is given as:

$$\max J_{P(k) \geq \sum_{i=1}^n d_i(k)} = \sum_{k=1}^T \left[ \left( \sum_{i=1}^n \left[ r_i (N_i(k+1) - N_i(k)) - \frac{\epsilon_i}{2} u_i^2(k) \right] - K(P(k)) - \frac{\epsilon}{2} u^2(k) - h(I(k)) \right) \frac{1}{(1+\rho)^{k-1}} \right] \quad (36)$$

$$I(k+1) = I(k) + P(k) - \sum_{i=1}^n [N_i(k+1) - N_i(k)] \quad (37)$$

$$N_i(k+1) = N_i(k) + (p_i + q_i \frac{N_i(k)}{\bar{N}_i}) (u_i(k) + \alpha u(k)) (\bar{N}_i - N_i(k)) \quad (38)$$

Similarly, the optimal control problem corresponding to equation (23) can be converted to discrete form. The discrete optimal control models are solved by using Lingo11. We assume that the duration of all the time periods are equal. The number of market segments is 3. The following values of parameters are used:

Segments	$\bar{N}_i$	$p_i$	$q_i$	$\alpha_i$	$r_i$	$\epsilon_i$	$\gamma_i$	Initial Sales
1	169107	0.000766	0.137604	0.3	4000	22750	0.05	11760
2	152460	0.001161	0.480575	0.19	4400	22750	0.065	11231
3	97581	0.00138	0.540395	0.189	4200	22750	0.0878	8260

Table 1: Values of Parameters

In addition, we assume the unit costs  $h_i = .01$  and  $k = 5$ . The optimal value of the profit for first optimal control problem is 639570900. The following table gives the optimal production rates and differentiated and mass promotional efforts in the segmented market in case of a single source inventory.

Time Periods	P	$u_1(t)$	$u_2(t)$	$u_3(t)$	$u(t)$
1	10612	.3791	1.0000	1.0000	.2750
2	14788	.3573	1.0000	1.0000	.2630
3	19820	.3362	1.0000	.9809	.2491
4	24789	.3158	1.0000	.9220	.2330
5	28552	.2962	1.0000	.8460	.2140
6	29816	.2774	1.0000	.7522	.1926
7	26391	.2594	.9117	.6468	.1698
8	20026	.2423	.76711	.5410	.1459
9	14076	.2260	.6261	.4467	.1233
10	9762	.2104	.50744	.3694	.1043
11	6977	.1957	.4152	.3082	.1000
12	5175	.1817	.346921	.2601	.1000
13	3973	.1686	.2901	.2218	.1000
14	3144	.1563	.2473	.1908	.1000
15	2554	.1447	.2131	.1655	.1000
16	2121	.1339	.1853	.1446	.1000
17	1794	.1238	.1624	.1271	.1000
18	1542	.1143	.1433	.1123	.1000
19	1344	.1055	.1273	.1000	.1000
20	1224	.1000	.1137	.1000	.1000
21	1140	.1000	.1020	.1000	.1000
22	1096	.1000	.1000	.1000	.1000
23	1061	.1000	.1000	.1000	.1000
24	1029	.1000	.1000	.1000	.1000
25	0	.1000	.1000	.1000	.1000

Table 2: The Optimal production and inventory rate in segmented market

The optimal value of profit is 370693700 for second optimal control problem considering single source production and multi destination demand-inventory system. Table 3 gives the optimal Production rates and promotional efforts in the segmented market.

Time Periods	P	$u_1(t)$	$u_2(t)$	$u_3(t)$	$u(t)$
1	5471	1.0000	0.3031	0.3827	0.2449
2	5934	1.0000	0.2877	0.3472	.0.2353
3	6637	1.0000	0.2859	0.3338	0.2266
4	7398	1.0000	0.2848	0.3244	0.2177
5	8213	1.0000	0.2840	0.3183	0.2087
6	9080	1.0000	0.2837	0.3149	0.1997
7	9991	1.0000	0.2838	0.3139	0.1907
8	10551	0.9630	0.2731	0.3031	0.1811
9	10841	0.9093	0.2578	0.2888	0.1712
10	10983	0.8546	0.2424	0.2755	0.1613
11	10975	0.7996	0.2271	0.2632	0.1514
12	10825	0.7450	0.2120	0.2516	0.1417
13	10549	0.6913	0.2120	0.2406	0.1321
14	10165	0.6391	0.1829	0.2302	0.1228
15	9698	0.5888	0.1690	0.2203	0.1139
16	9170	0.5408	0.1558	0.2107	0.1054
17	8616	0.4954	0.1430	0.2015	.1000
18	8072	0.4528	0.1304	0.1924	.1000
19	7523	0.4131	0.1187	0.1837	.1000
20	6980	0.3760	0.1077	0.1755	.1000
21	6593	0.3498	0.1000	0.1718	.1000
22	6618	0.3483	0.1000	0.1806	.1000
23	6633	0.3467	0.1000	0.1910	.1000
24	0	0.1174	0.1000	0.2079	.1000
25	0	0.1174	0.1000	0.2079	.1000

Table 3: The Optimal production and inventory rate in segmented market

## 7. CONCLUSION

This paper presents concept of the interdependence between marketing and production in segmented market using optimal control model of marketing-production system. We have formulated two optimal control problems with the assumption that consumer demand in each segment is generated by mass and differentiated promotions. We derive dynamic production and promotional effort policies by using Pontryagin's maximum principle to achieve the maximum total profit associated with inventory and production rate over the planning period. The first problem considered is a single source inventory and multi-segmented demand problem. The second problem is a multi segmented inventory and demand problem. The two problems are solved and analyzed by using Pontryagin's maximum principle. After discretizing, the proposed optimal control problems for promotion dependent demand were solved using Lingo11. Numerical examples are provided to illustrate the effectiveness of the proposed method and solution procedure. For further studies, the proposed optimal control models can be extended in a competitive environment. Another direction for future research is to incorporate factors such as price, quality, and cost along with differentiated and mass market promotional effort expenditure.

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