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RELIABILITY OF AN ENGINEERING SYSTEM CHARACTERIZED BY HOT AND COLD STANDBY: A COMPOUND BOUNDARY VALUE PROBLEM

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Abstract: We analyse the reliability (survival function) of a duplex system characterized by hot standby and sustained by an auxiliary unit in cold standby. The entire system is attended by two heterogeneous repairmen. Our methodology is based on the theory of sectionally holomorphic functions combined with the notion of dual transforms. Finally, we also study the total occupational time of the repairman responsible for the repair of the failed priority unit during the survival time of the system.

Keywords: Duplex System, Priority Rule, Stopping Time, Survival Function.

MSC: 60K10.

1. INTRODUCTION

A recent survey [14] on reliability of technical systems in industry citing 138 references observes that "systems reliability becomes a crucial aspect of intelligent manufacturing". Standby provides a powerful tool to increase the reliability and quality of operational systems, e.g. [2], [6], [9], [17]. A frequently employed standby mode is the so-called "cold" standby. The notion of cold standby signifies that a backup unit is kept in reserve with a zero failure rate, until the online unit fails. The involvement of cold standby redundancy in satellite systems has been cited by [10]. A crucial variant of cold standby is the so-called "hot" standby or active standby redundancy. The notion of hot standby signifies that the backup unit has the same failure rate in standby as in the operative state. Note that the hot standby mode is often indispensable to implement a fast automatic replacement of the failed online unit by the underlying backup unit. An example of a system with hot standby is the light plant of a tunnel connected with a single operative generator sustained by a generator in hot standby. Figure 1.1 shows a functional block-diagram of the operational system. Note that systems with hot standby are rather scarce in the Literature e.g. [22].

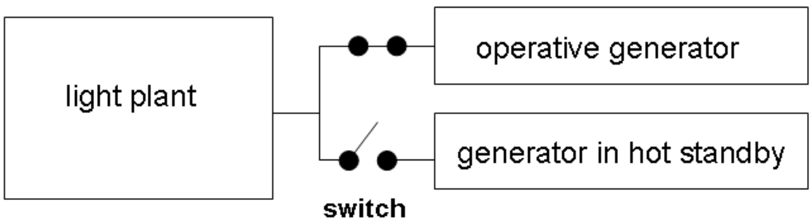

Figure 1: Operational light plant

Standby systems are frequently endowed with priority rules. For instance, an external power supply station of a technical plant has usually overall (break-in) priority (called pre-emptive priority) in operation with regard to an internal (local) power generator in standby, i.e the local generator is only deployed if the external station is down (or eventually shut off for periodic maintenance). In particular, a general repairable duplex system characterized by pre-emptive priority in operation with regard to a cold standby unit, subjected to arbitrary distributions for failure and repair, has been introduced by [20]. As a preliminary modification, we first consider a duplex system composed of a priority unit (called the p-unit) sustained by a different unit in hot standby (called the h-unit).The system (henceforth called the **H**-system) is attended by two heterogeneous repairmen R_p and R_h. A failure of the p-unit is always directed to repairman R_p , whereas a failure of the h-unit is always allocated to repairman R_h. Moreover, unit p has (pre-emptive) priority in operation with regard to the h-unit. Thus the **H**-system acts as a closed network evolving in time, i.e. each failed unit goes immediately into repair and conversely, the repaired p-unit becomes immediately operative, whereas the repaired h-unit lines up in hot standby if the p -unit is operative or becomes instantaneously operative if unit p is still under repair. Note that the priority rule implies that the event "The h-unit is operative and the p-unit is waiting in standby" is a P-null set. The **H**- system is down if both units are under repair, partially down if only one unit is under repair and completely up is both units are up(available). Next, we introduce the **S**-system. The **S**-system is composed of the

H-system sustained by an auxiliary unit (called the s-unit) in cold standby. Both the p-unit and the h-unit have break-in priority in operation with regard to the s-unit. Thus the s-unit is only deployed if the **H**-system is down. The **S**-system is up if at least one unit is up. Otherwise, the **S**-system is down. Finally, we assume that the s-unit has its own repair facility R_s. A practical example of an s-unit is the so-called ram air turbine (RAT). The device consists of a small propeller that, upon request, drops out of the bottom of an aircraft (cf. the landing gear) converting kinetic energy, induced by the airstream, into electrical power. Thus the RAT is actually a small wind turbine! Note that this auxiliary power device can provide almost all vital components with the required amount of power needed to monitor the plane's flight control in case of emergency. So, the RAT increases the reliability of the aircraft. However, note that the device is only deployed if the global (internal) power generator system (usually a parallel system) is down. Therefore, the RAT is a non-priority unit designed to operate in the exceptional case of emergency.

In order to derive the survival function of the **S**-system, we employ a stochastic process describing the various states of the **S**-system and endowed with time-dependent transition measures satisfying coupled partial differential equations. The solution procedure of the equations is based on a refined application of the theory of sectionally holomorphic functions, e.g. [8], [11] combined with the notion of dual transforms, [18].The main problem is to convert a functional equation into a boundary value problem on the real line.

Furthermore, we introduce a security interval $[0, \tau)$ related to a security level $0 < \delta < 1$ and satisfying a suitable risk criterion. The security interval ensures a survival of the **S**-system up to time τ with probability δ.

As an example, we consider the case of a Coxian repair time distributions. Some graphs are displaying the survival function jointly with the security interval corresponding to a security level of 90%.

Finally, we study the *total* occupational time of repairman R_p during the survival time of the **S**-system.Note that our **S**-system is a statistical variant of the duplex system introduced by [21].

2. STOCHASTIC PROCESS, STOPPING TIME, SURVIVAL TIME

We now focus on the survival time of the **S**-system. In order to introduce a precise definition of the survival time we employ a stochastic process $\{N_t, t \geq 0\}$ with (discrete) state space $\{A, B, C, C_s, D\}$, where D is an absorbing state. The process $\{N_t, t \geq 0\}$ is characterized by the following exhaustive set of mutually independent events:

$\{N_t = A\}$: All units of the **S**-system are up at time t,

$\{N_t = B\}$: The **H**-system is up, repairman R_p is busy and the s-unit is in cold standby at time t,

$\{N_t = C\}$: The **H**-system is up, repairman R_h is busy and the s-unit is in cold standby at time t,

$\{N_t = C_s\}$: The **H**-system is down and the s-unit is operative at time t,

$\{N_t = D\}$: The **S**-system is down at time t,

Note that the absorbing state D signifies that the process $\{N_t\}$, once entered state D at some random time θ, cannot escape state D. Therefore, taking our priority rule into account, we may assume that a failure of the **s**-unit is catastrophic, i.e. terminates the lifetime of the **S**-system. The inclusion of state D into the state space of the process $\{N_t\}$ invokes the introduction of a so-called stopping time, e.g. [3, 5]. Consequently, we first define the non-Markovian process $\{N_t\}$ on a filtered probability space $\{\Omega, \mathcal{A}, P, \mathfrak{F}\}$ where the history $\mathfrak{F} := \{\mathfrak{F}_t, t \geq 0\}$ satisfies the Dellacherie-conditions

- $\mathfrak{F}_0$ contains the P-null sets of $\mathcal{A}$,
- $\forall t \geq 0,\ \mathfrak{F}_t = \bigcap_{u<t} \mathfrak{F}_u$ i.e.the family $\mathfrak{F}$ is right-continuous.

Consider the $\mathfrak{F}$-stopping time (Markov time)

$$\theta := \inf \{t > 0 :\ N_t = D | N_0 = A\}.$$

We assume that the **S**-system starts functioning at some time origin $t = 0$ in state A, i.e. let $N_0 = A$ with probability one. Thus, from $t = 0$ onwards, θ is the survival time (lifetime) of the **S**-system. The corresponding survival function is denoted by $\mathfrak{R}(t)$. Clearly,

$$\mathfrak{R}(t) = Pr\{\theta > t\},\ t \geq 0.$$

It should be noted that θ does not depend on the repair time of the s-unit. Therefore, the state space of the process $\{N_t\}$ is sufficient (exhaustive) to describe the random behaviour of the **S**-system during the survival time θ. Figure 2.0 displays the transitions of N_t related to failures and repairs. An upward (downward) arrow corresponds to a repair (failure) of a unit. Along with the survival function of the **S**-system, we now introduce a security interval $[0, \tau)$, where

$$\tau := \sup \left\{t \geq 0\ :\ \mathfrak{R}(t^-) \geq \delta\right\}$$

for some $0 < \delta < 1$, called the security level. In practice, δ is usually large. For instance, $\delta = 0.9$. Therefore, we require that the **S**-system satisfies the risk criterion $\lim_{t\uparrow\tau} \mathfrak{R}(t) \geq \delta \gg 0$. Note that the security interval, corresponding to the security level δ, ensures a continuous operation (survival) of the **S**-system up to time τ with probability δ. The various states of the **S**-system are described by functional block-diagrams in figures 2.1-2.5.

We recall that the p-unit has overall (pre-emptive) priority in operaron with regard to both the h-unit and the s-unit, whereas the h-unit has only (pre-emptive) priority in operation with regard to the s-unit. Consequently, the s-unit is only deployed whenever the **H**-system is down. See figure 2.4.

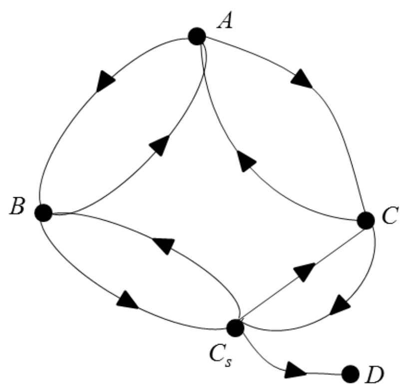

Figure 2: Transition diagram related to failures and repairs.

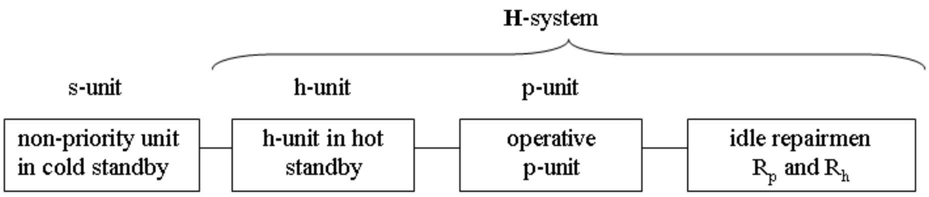

Figure 3: Functional block-diagram of the **S**-system operating in state A.

3. ASSUMPTIONS, DEFINITIONS, PROPERTIES

3.1. Assumptions

Consider the **S**-system satisfying the following assumptions. The operative p-unit has a failure-free time f with distribution $F(\cdot), F(0) = 0$ and a constant repair rate μ. A repair of the failed p-unit is always carried out by repairman R_p The h-unit has a constant failure rate λ and a repair time r with distribution $R(\cdot), R(0) = 0$. A repair of the failed h-unit is always carried out by repairman R_h. The s-unit has a zero failure rate in standby (cold standby) and a constant failure rate λ_s in the operative state. We recall that the s-unit is only deployed if the **H**-system is down. Therefore, θ is independent of the repair time r_s of the s-unit. Consequently, the repair time distribution of r_s needs no specification. All underlying random variables are supposed to be independent and a repaired unit functions as good as new.

3.2. Definitions and properties

- Characteristic functions (and their duals) are formulated in terms of a *complex* transform variable.

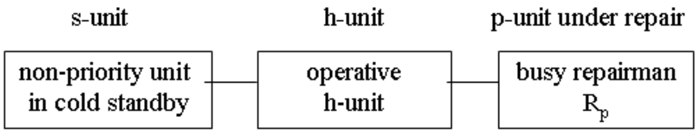

Figure 4: Functional block-diagram of the **S**-system operating in state B.

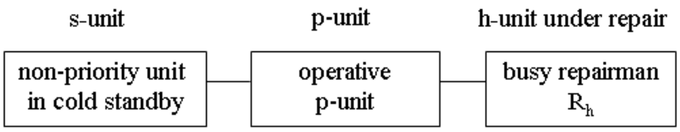

Figure 5: Functional block-diagram of the **S**-system operating in state C.

For instance,

$$\mathbf{E}e^{i\omega r} = \int_0^\infty e^{i\omega x}\, dR(x),\ \operatorname{Im}\omega \geq 0.$$

Note that

$$\mathbf{E}e^{-i\omega r} = \int_{-\infty}^0 e^{i\omega x}\, d\{(1 - R((-x)^-)\},\ \operatorname{Im}\omega \leq 0.$$

The corresponding Fourier-Stieljes transforms are called *dual* transforms. Without loss of generality (see Remarks 6.1), we may assume that F and R have density functions of bounded variation on $[0, \infty)$ with finite mean. In addition, let $\mathbf{E}r^2 < \infty$.

- A (vector) Markov characterization of the non-Markovian process $\{N_t, t \geq 0\}$, with absorbing state D, is piecewise and conditionally defined by:

  $\{(N_t, X_t)\}$ if $N_t = A$, where X_t denotes the remaining failure-free time of the p-unit being operative at time t,

  $\{N_t\}$ if $N_t = B$ or D,

  $\{(N_t, Y_t)\}$ if $N_t = C_s$, where Y_t denotes the remaining repair time of failed h-unit under progressive repair at time t,

  $\{(N_t,\, X_t,\, Y_t)\}$ if $N_t = C$.

  The state space of the underlying Markov process is given by

  $\{(A, x)\} \bigcup \{B\} \bigcup \{(C_s, y)\} \bigcup \{(C, x, y)\} \bigcup \{D\}, x \geq 0,\ y \geq 0.$

  For $K = A, B, C, C_s, D$ let $p_K(t) := Pr\{N_t = K\},\ t \geq 0$ where $\sum_K p_K(t) = 1.$

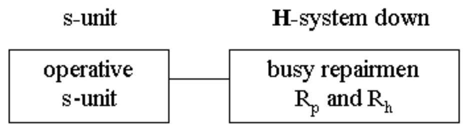

Figure 6: Functional block-diagram of the **S**-system operating in state C_s.

Figure 7: The **S**-system's down state D.

Finally, we introduce the measures

$$\begin{aligned} p_A(t,\, x)dx &:= Pr\left\{N_t = A,\, x - \Delta x < X_t \leq x\right\}, \\ p_{C_s}(t,\, y)dy &:= Pr\left\{N_t = C_s,\, y - \Delta y < Y_t \leq y\right\}, \\ p_C(t,\, x,\, y)dxdy &:= Pr\left\{N_t = C,\, x - \Delta x < X_t \leq x, y - \Delta y < Y_t \leq y\right\}. \end{aligned}$$

Note that, for instance,

$$p_C(t) = \int_0^\infty \int_0^\infty p_C(t,x,y)dxdy.$$

- The indicator (function) of an event $\{N_t = K\} \in \mathcal{A}$ is denoted by $\mathbf{1}\{N_t = K\}$.
- The complex plane and the real line are respectively denoted by $\mathbf{C}$ and $\mathbf{R}$ with obvious superscript notations such as $\mathbf{C}^+$ and $\mathbf{C}^-$. For instance,

$$\mathbf{C}^+ := \{\omega \in \mathbf{C} : \text{Im}\ \omega > 0\}.$$

- The Laplace transform of any locally integrable and bounded function on $[0,\infty)$ is denoted by the corresponding character marked with an asterisk. For instance,

$$p^*(z) := \int_0^\infty e^{-zt}p(t)dt, \ \text{Re}\ z > 0.$$

Moreover, if $p(t)$ is of bounded variation and right-continuous on $[0,\infty)$, we have

$$zp^*(z) = p(0) + \int_0^\infty e^{-zt}dp(t), \ \text{Re}\ z > 0.$$

- Let $\alpha(\tau),\ \tau \in \mathbf{R}$ be a bounded and continuous function.
$\alpha(\cdot)$ is called $\Gamma-$integrable if

$$\lim_{\substack{T\to\infty \\ \epsilon\downarrow 0}} \int_{\Gamma_{T,\varepsilon}} \alpha(\tau)\frac{d\tau}{\tau - u}, \ u \in \mathbf{R}$$

exists, where $\Gamma_{T,\varepsilon} := (-T,\, u-\varepsilon]\bigcup[u+\varepsilon, T)$. The corresponding integral, denoted by

$$\frac{1}{2\pi i}\int_{\Gamma}\alpha(\tau)\frac{d\tau}{\tau-u}$$

is called a Cauchy principal value in double sense.

- A function $\alpha(\tau),\ \tau\ \in \mathbf{R}$ is Lipschitz-continuous (**L**-continuous) on **R** if $\forall\tau_1,\, \tau_2 \in \mathbf{R}$ there exists a constant c such that

$$|\alpha(\tau_2)-\alpha(\tau_1)| \le c|\tau_2-\tau_1|.$$

The function $\alpha(\tau),\ \tau \in \mathbf{R}$ is called **L**-continuous at infinity if

$$|\alpha(\tau)| = O\left(\frac{1}{|\tau|}\right),\ |\tau| \to \infty.$$

- Note that the **L**-continuity of $\alpha(\cdot)$ on **R** and at infinity is sufficient for the existence of the Cauchy-type integral

$$\frac{1}{2\pi i}\int_{\Gamma}\alpha(\tau)\frac{d\tau}{\tau-\omega},\ \omega\ \in\ \mathbf{C}.$$

See [8] for further details.

- Let $F(t)$ be any probability distribution on $[0,\infty)$. The n-fold convolution of F is denoted by $F^{n\star}$. For $n=0$, $F^{n\star}$ represents the Heaviside step function with the unit-jump at $t=0$, i.e.

$$F^{0\star} := \begin{cases} 1, \text{if}\ \ t \ge\ 0, \\ 0, \text{if}\ \ t < 0. \end{cases}$$

- We frequently use the characteristic function

$$\mathbf{E}e^{i\tau\nu} = \begin{cases} \dfrac{\mathbf{E}e^{i\tau r}-1}{i\tau\mathbf{E}r}, & \text{if } \tau \neq 0, \\ 1, & \text{if } \tau = 0. \end{cases} \tag{1}$$

Note that

$$\mathbf{E}e^{i\omega\nu} = (\mathbf{E}r)^{-1}\int_0^{\infty} e^{i\omega x}(1-R(x))dx, \operatorname{Im}\omega \ge 0. \tag{2}$$

Property 3.1
The function $\gamma^{+}(\omega) := 1+\lambda\mathbf{E}r\mathbf{E}e^{i\omega\nu},\ \operatorname{Im} w \ge 0$, has no zeros in $\mathbf{C}^{+}\bigcup\mathbf{R}$.
Proof Consider an alternating renewal process, e.g. Birolini[2,452-456] with an up and down state. Let $1-e^{-\lambda t}$ (respectively $R(t)$) be the sojourn time distribution of the process in the up state (respectively in the down state). Furthermore, let

$p_R(t)$ be the probability that the process is up at time t given that it was up at time $t = 0$. hlBy renewal theory, e.g.[15],

$$p_R(t) = \int_{0-}^{t} e^{-\lambda(t-u)} d \sum_{n=0}^{\infty} \varphi^{n\star}(u),$$

where

$$\varphi(u) := \int_{0}^{u} (1 - e^{-\lambda(u-v)}) dR(v)$$

is the distribution function of a cycle.

The following properties are valid for an arbitrary R.

- $p_R(0) = 1,\ 0 < p_R(t) \leq 1,\ p_R(\infty) = (1 + \lambda_s \mathbf{E} r)^{-1}$.
- $p_R(t)$ is Lebesque-absolutely continuous on $(0, \infty)$ and of bounded variation and right-continuous on $[0, \infty)$.
- $p_R^*(z)$ is given by

$$p_R^*(z) = \frac{1}{z + \lambda_s(1 - \mathbf{E} e^{-zr})}, \mathrm{Re}\, z > 0. \tag{3}$$

Applying the product rule to Eq. (3.3), reveals that

$$(1 + \lambda_s \mathbf{ErE} e^{i\omega\nu})^{-1} = \int_{0-}^{\infty} e^{iwt} dp_R(t).$$

On the other hand

$$\left| \int_{0-}^{\infty} e^{iwt} dp_R(t) \right| \leq \int_{0-}^{\infty} |dp_R(t)| = \mathrm{var}_{[0,\infty]} p_R(t),$$

where the notation $\mathrm{var}_{[0,\infty)} p_R(t)$ stands for the *total variation* of $p_R(\cdot)$ on $[0, \infty)$. See [1])(page 128) for an appropriate definition. Note that the bounded variation property of $p_R(\cdot)$ on $[0, \infty)$ implies that $\mathrm{var}_{[0,\infty)} p_R(t) < \infty$. Hence, the function $1 + \lambda_s \mathbf{ErE} e^{i\omega\nu}$ is free from zeros in $\mathbf{C}^+ \bigcup \mathbf{R}$.

- **Property 3.2**
  The function $\gamma_z^-(\omega) := z + i\omega + \lambda(1 - \mathbf{E} e^{-i(\omega - i(z+\lambda_s))r_h})$, $\mathrm{Im}\, \omega \leq 0$ has no zeros in $\mathbf{C}^- \bigcup \mathbf{R}$.

  A straightforward proof based on Rouché's theorem, e.g. [12] of property 3.2 is similar to the proof of a property in Vanderperre and Makhanov [20, Lemma 7.1] and therefore omitted.

- **Corollary 3.1**
  The function $1/\gamma^+(\omega)$ (respectively $1/\gamma_z^-(\omega)$) is bounded on $\mathbf{C}^+ \bigcup \mathbf{R}$ (respectively on $\mathbf{C}^- \bigcup \mathbf{R}$).

- **Property 3.3**

  The function

$$\frac{\mathbf{E}e^{i\tau r}}{\gamma^+(\tau)\gamma_z^-(\tau)},\ \tau \in \mathbf{R}$$

  is **L**-continuous on **R** and at infinity.

  Proof

  First note that

$$\left|\frac{\partial}{\partial\tau}\mathbf{E}e^{i\tau r}\right| \leq \mathbf{Er}$$

  whereas Eq.(3.2) entails that

$$\left|\frac{\partial}{\partial\tau}\gamma^+(\tau)\right| \leq (\mathbf{Er})^{-1}\int_0^\infty x(1-R(x))dx = \frac{1}{2}\frac{\mathbf{Er}^2}{\mathbf{Er}}.$$

  Moreover,

$$\left|\frac{\partial}{\partial\tau}\gamma_z^-(\tau)\right| \leq 1+\mu\mathbf{Ef}.$$

  The mean value theorem for derivatives, e.g. [1](page 110), implies that each of the functions $\mathbf{E}e^{i\tau r}$,$1/\gamma^+(\tau)$, $1/\gamma_z^-(\tau)$ is a bounded **L**-continuous function. Consequently, the function, $\mathbf{E}e^{i\tau r}/(\gamma^+(\tau)\gamma_z^-(\tau))$ being a product of bounded **L**-continuous functions is also **L**- continuous on **R**.

  Finally, the maximun modulus theorem, e.g. [1](page 454), applied to $\gamma_z^-(\tau)$ entails that

$$\gamma_z^-(\tau) = O(\frac{1}{|\tau|}),\ \tau \to \infty$$

  whereas

$$\lim_{|\tau|\to\infty}\gamma^+(\tau) = 1.$$

  Hence, the function $\mathbf{E}e^{i\tau r}/(\gamma^+(\tau)\gamma_z^-(\tau))$ is **L**-continuous on **R** and at infinity.

- A probability distribution $R(\cdot), R(0) = 0$ of a random variable r is called a Coxian distribution if

$$\mathbf{E}e^{i\tau r} = \frac{A_m(\tau)}{B_n(\tau)},\ m < n,\ \tau \in \mathbf{R},$$

  where $A_m(\tau), B_n(\tau)$ are polynomials of degree m, n.[4] has shown that this exclusive family of distributions is surprisingly large.

4. DIFFERENTIAL EQUATIONS

In order to derive a set of differential equations, we observe the behaviour of the **S**-system in some time interval $[t, t+\Delta]$, $\Delta \downarrow 0$. Applying a general birth and death technique, e.g. [19] and taking the absorbing state D into account, yields the balance equations

$$p_A(t+\Delta,\, x-\Delta) = p_A(t,x)(1-\lambda\Delta) + \mu p_B(t)\frac{dF}{dx}\Delta + p_C(t,x,0)\Delta + o(\Delta),$$

$$p_B(t+\Delta) = p_B(t)(1-(\lambda+\mu)\Delta) + p_A(t,0)\Delta + p_{C_s}(t,0)\Delta + o(\Delta),$$

$$p_C(t+\Delta,\, x-\Delta,\, y-\Delta) = p_C(t,x,y) + \mu p_{C_s}(t,y)\frac{dF}{dx}\Delta + \lambda p_A(t,x)\frac{dR}{dy}\Delta + o(\Delta),$$

$$p_{C_s}(t+\Delta,\, y-\Delta) = p_{C_s}(t,y)(1-(\mu+\lambda_s)\Delta) + \lambda p_B(t)\frac{dR}{dy}\Delta + p_C(t,0,y)\Delta + o(\Delta),$$

$$p_D(t+\Delta) = p_D(t) + \lambda_s p_{C_s}(t)\Delta + o(\Delta),$$

where the notation $o(\Delta)$, $\Delta \downarrow 0$ stands for any function $\mathcal{K}(\cdot)$ such that

$$\lim_{\Delta\downarrow 0} \frac{\mathcal{K}(\Delta)}{\Delta} = 0.$$

Taking the definition of *directional* derivative into account, for instance,

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) p_C(t,x,y) := \lim_{\Delta\downarrow 0} \frac{p_C(t+\Delta,\, x-\Delta,\, y-\Delta) - p_C(t,\, x,\, y)}{\Delta}$$

entails that for $t > 0$, $x > 0$, $y > 0$,

$$\left(\lambda + \frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) p_A(t,x) = \mu p_B(t)\frac{dF}{dx} + p_C(t,x,0), \tag{4}$$

$$\left(\mu + \lambda + \frac{d}{dt}\right) p_B(t) = p_A(t,0) + p_{C_s}(t,0), \tag{5}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) p_C(t,x,y) = \mu p_{C_s}(t,y)\frac{dF}{dx} + \lambda p_A(t,x)\frac{dR}{dy}, \tag{6}$$

$$\left(\lambda_s + \mu + \frac{\partial}{\partial t} - \frac{\partial}{\partial y}\right) p_{C_s}(t,x) = p_C(t,0,y) + \lambda p_B(t)\frac{dR}{dy}, \tag{7}$$

$$\frac{d}{dt} p_D(t) = \lambda_s p_{C_s}(t). \tag{8}$$

Note that the initial condition $N_0 = A, X_0 = f$ with propability one, entails that $p_A(0,x) = \frac{dF}{dx}$. Moreover, $Pr\{\theta \le t\} = p_D(t)$. Finally, observe that the equations (4.1)-(4.5) are consistent with the probability law $\sum_K p_K(t) = 1$ and that $p_A(0) = 1$.

5. FUNCTIONAL EQUATION

First we remark that our set of differential equations is well-adapted to a transformation by means of Laplace-Fourier transforms of the underlying transition functions. As a matter of fact, the transition functions are bounded on their appropriate regions and locally integrable with respect to t. Consequently, each Laplace transform exists for $\operatorname{Re} z > 0$. Moreover, the obvious integrability of the density functions and the transition functions with regard to x, y also implies the integrability of the corresponding partial derivatives. The advantage of the proposed Laplace-Fourier methodology is the possibility to represent the Laplace transform of the survival function explicitly in terms of a particular Cauchy integral. Applying a Laplace-Fourier transform technique to equations (4.1)-(4.4) and taking the initial condition into account, yields the equations

$$(z+\lambda+i\omega)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t=A\})dt + p_A^*(z,0) =$$

$$\mu p_B^*(z)\mathbf{E}e^{i\omega f} + \int_0^\infty e^{i\omega x}p_C^*(z,x,0)dx + \mathbf{E}e^{i\omega f}, \tag{9}$$

$$(z+\lambda+\mu)p_B^*(z) = p_A^*(z,0) + p_{C_s}^*(z,0) \tag{10}$$

$$(z+i\omega+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}e^{i\eta Y_t}\mathbf{1}\{N_t=C\})dt + \int_0^\infty e^{i\omega x}p_C^*(z,x,0)dx +$$

$$\int_0^\infty e^{i\eta y}p_C^*(z,0,y)dy = \mu\,\mathbf{E}e^{i\omega f}\int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}\mathbf{1}\{N_t=C_s\})dt$$

$$+\lambda\mathbf{E}e^{i\eta r}\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t=A\})dt, \tag{11}$$

$$(z+\lambda_s+\mu+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}\mathbf{1}\{N_t=C_s\})dt + p_{C_s}^*(z,0) =$$

$$\int_0^\infty e^{i\eta y}p_C^*(z,0,y)dy + \lambda p_B^*(z)\mathbf{E}e^{i\eta r}, \tag{12}$$

$$zp_D^*(z) = \lambda_s p_{C_s}^*(z). \tag{13}$$

Adding Eqs. (5.1)−(5.4) yields the functional equation

$$(z+\lambda(1-\mathbf{E}e^{i\eta r})+i\omega)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t=A\})dt +$$

$$(z+\lambda(1-\mathbf{E}e^{i\eta r})+\mu(1-\mathbf{E}e^{i\omega f}))p_B^*(z)+$$
$$(z+i\omega+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}e^{i\eta Y_t}\mathbf{1}\{N_t=C\})dt+$$

$$(z+\lambda_s+\mu(1-\mathbf{E}e^{i\omega f})+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}\mathbf{1}\{N_t=C_s\})dt=\mathbf{E}e^{i\omega f}, \quad (14)$$

valid for $\operatorname{Re} z>0,\ \operatorname{Im}\omega\geq 0,\ \operatorname{Im}\eta\geq 0.$

6. A COMPOUND SOKHOTSKI-PLEMELJ PROBLEM, SURVIVAL FUNCTION

Inserting $\omega=iz,\eta=0$ in Eq.(5.6) reveals that

$$(z+\mu(1-\mathbf{E}e^{-zf})p_B^*(z)+(z+\lambda_s+\mu(1-\mathbf{E}e^{-zf}))p_{C_s}^*(z)=\mathbf{E}e^{-zf}. \quad (15)$$

On the other hand, applying the product rule for Lebesgue-Stieltjes transforms to Eq.(5.5) shows that

$$\mathbf{E}e^{-z\theta}=\lambda_s p_{C_s}^*(z). \quad (16)$$

Eliminating $p_{C_s}^*(z)$ in Eq.(6.1) by means of Eq.(6.2) entails that

$$\frac{1-\mathbf{E}e^{-z\theta}}{z}=\frac{\left(1+\mu\frac{1-\mathbf{E}e^{-zf}}{z}\right)(1+\lambda_s p_B^*(z))+\lambda_s\frac{1-\mathbf{E}e^{-zf}}{z}}{z+\lambda_s+\mu(1-\mathbf{E}e^{-zf})}. \quad (17)$$

Note that $\mathcal{R}^*(z)$ is uniquely determined by the relationship

$$\frac{1-\mathbf{E}e^{-z\theta}}{z}=\int_0^\infty e^{-zt}\mathcal{R}(t)dt=\mathcal{R}^*(z),\operatorname{Re} z>0.$$

Hence, in order to derive $\mathcal{R}^*(z)$, we first have to derive the unknown $p_B^*(z)$ appearing in Eq.(6.3). Substituting $\eta=\tau,\omega=-\tau+iz,\tau\in\mathbf{R},\operatorname{Re} z>0$ into Eq. (5.6) and noting that $z+i\omega+i\eta=0$, yields the boundary value equation

$$\gamma_z^-(\tau)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\tau Y_t}\mathbf{1}\{N_t=C\})dt-i\tau\gamma^+(\tau)\int_0^\infty e^{-zt}\mathbf{E}(e^{i(-\tau+iz)X_t}\mathbf{1}\{N_t=A\})dt=$$

$$\mathbf{E}e^{i(-\tau+iz)f}-[z+\lambda(1-\mathbf{E}e^{i\tau r})+\mu(1-\mathbf{E}e^{i(-\tau+iz)f}]p_B^*(z). \quad (18)$$

Next, dividing Eq.(6.4) by the factor $\gamma^+(\tau)\gamma_z^-(\tau)$ (an operatiion justified by Properties 3.1 and 3.2) reveals that

$$\psi^+(z,\tau)-\psi^-(z,\tau)=\varphi(z,\tau),\ \tau\in\mathbf{R} \quad (19)$$

where

$$\psi^{+}(z,\tau)=\frac{\int_0^\infty\ e^{-zt}\mathbf{E}(e^{i\tau Y_t}\mathbf{1}\{N_t=C\})dt}{\gamma^{+}(\tau)},$$

$$\psi^{-}(z,\tau)=\frac{\int_0^\infty\ e^{-zt}\mathbf{E}(e^{i(-\tau+iz)X_t}\mathbf{1}\{N_t=A\})dt}{\gamma_z^{-}(\tau)},$$

$$\varphi(z,\tau):=\frac{\mathbf{E}e^{i(-\tau+iz)f}}{\gamma^{+}(\tau)\gamma_z^{-}(\tau)}-\frac{z+\lambda(1-\mathbf{E}e^{i\tau r})+\mu(1-\mathbf{E}e^{i(-\tau+iz)f})}{\gamma^{+}(\tau)\gamma_z^{-}(\tau)}p_B^{*}(z).$$

Note that the extended function $\psi^{+}(z,\omega),\operatorname{Im}\omega\geq 0$ (respectively $\psi^{-}(z,\omega)$, $\operatorname{Im}\omega\leq 0$) is analytic in $\mathbf{C}^{+}$ (respectively in $\mathbf{C}^{-}$) whereas the compound "neutral" term $\varphi(z,\tau)$ only exists on the real line. In addition, Property 3.3 implies that the component functions of $\varphi(z,\tau)$ are all $\mathbf{L}$-continuous on $\mathbf{R}$ and at infinity.

Consequently, Eq.(6.5) constitutes a (compound) Sokhotski-Plemelj boundary value problem on the real line solvable by the theory of sectionally holomorphic functions. For direct reference we have compiled some basic definitions and properties located in the Appendix. We obtain

Property 6.1
The unique solution of Eq.(6.5) is given by

$$\frac{1}{2\pi i}\int_\Gamma\ \varphi(z,\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}.$$

In addition

$$\psi^{+}(z,\omega)=\frac{1}{2\pi i}\int_\Gamma\ \varphi(z,\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}^{+}. \tag{20}$$

Observe that Eq.(6.6) is not valid for $\omega=0$. Indeed, on the one hand, we have by continuity

$$\lim_{\omega\to 0}\psi^{+}(z,\omega)=\psi^{+}(z,0)=\frac{p_{C_s}^{*}(z)}{1+\lambda\mathbf{Er}}. \tag{21}$$

On the other hand, an application of Sokhotski-Plemelj formulas reveals that

$$\lim_{\substack{\omega\to 0\\ \omega\in\mathbf{C}^{+}}}\frac{1}{2\pi i}\int_\Gamma\ \varphi(z,\tau)\frac{d\tau}{\tau-\omega}=\frac{1}{2}\varphi(z,0)+\frac{1}{2\pi i}\int_\Gamma\ \varphi(z,\tau)\frac{d\tau}{\tau}. \tag{22}$$

However, evaluating the Cauchy-type integrals in Eq.(6.6) by means of the algorithm elaborated in Vanderperre and Makhanov [20] and taking Eq.(6.7) into account, generates an additional equation in terms of $p_B^{*}(z)$, $p_{C_s}^{*}(z)$ independent of Eq.(6.1). Hence, the required $p_B^{*}(z)$ follows from the solution of a simultaneous pair of linear equations. Consequently, having determined $p_B^{*}(z)$, we obtain $\mathcal{R}^{*}(z)$ from Eq.(6.3). Note that Properties 3.1.3.2 and 3.3 are also valid for general distributed variables f,r,ν with finite mean. Consequently, the bounded variation assumptions imposed on F and R in Section 3.2 are totally superfluous for the existence of $\mathcal{R}^*(z)$. Unfortunately, the explicit form of $\mathcal{R}^*(z)$ is surprisingly complicated, even if one of the distributions F or R represents a Coxian distribution. The following example shows the details.

7. NUMERICAL EXAMPLE, COXIAN DISTRIBUTION

Recall that a probability distribution is called Coxian if its Laplace-Stieltjes transform is a quotient of two polynomials, [4].
Further, Sokhotski-Plemelj formulas are rather theoretical. In applications, we first try to elaborate the Cauchy-type integrals and perform the limit procedure thereafter. Hence, we are faced (see Eq.(6.7)-(6-8)) with elaborating the equation

$$p^*_{C_s}(z)(1+\lambda\mathbf{E}\mathbf{r})^{-1} = \lim_{\substack{\omega\to 0\\ \omega\in\mathbf{C}^+}} \frac{1}{2\pi i}\int_\Gamma \varphi(z,\tau)\frac{d\tau}{\tau-\omega}. \tag{23}$$

In order to obtain computational results, we assume that $R(\cdot)$ is a Coxian distribution. The simplest case is a constant repair rate ρ, i.e.

$$\mathbf{E}e^{i\omega r} = \rho/(\rho - i\omega),\ \omega \neq -i\rho.$$

Applying the algorithm elaborated in Vanderperre and Makhanov [20] to Eq.(7.1) yields the additional equation

$$\lambda p^*_B(z)T(z) - \rho p^*_{C_s}(z)(T(z)+\lambda_s) = \lambda\mathbf{E}e^{-(z+\rho+\lambda)f}, \tag{24}$$

where

$$T(z) := z+\rho+\lambda+\mu(1-\mathbf{E}e^{-(z+\rho+\lambda)f}),\ \mathrm{Re}\, z \geq 0$$

From Eqs.(6.1) and (7.2) we finally obtain

$$p^*_B(z) = \frac{\begin{vmatrix} \mathbf{E}e^{-zf} & z+\lambda_s+\mu(1-\mathbf{E}e^{-zf}) \\ \lambda\mathbf{E}e^{-(z+\rho+\lambda)f} & -\rho(T(z)+\lambda_s) \end{vmatrix}}{\begin{vmatrix} z+\mu(1-\mathbf{E}e^{-zf}) & z+\lambda_s+\mu(1-\mathbf{E}e^{-zf}) \\ \lambda T(z) & -\rho(T(z)+\lambda_s) \end{vmatrix}} \tag{25}$$

The value $p^*_B(0)$ has an interesting probabilistic interpretation, i.e. let $\mathcal{B}_\theta$ be the *total* occupational time of repairman R_p during the survival time θ of the **S**-system. Clearly $\mathcal{B}_\theta$ has the same distribution as the *total* sojourn time of $\{N_t\}$ in state B during θ. Hence,

$$\mathcal{B}_\theta = \int_0^\theta \mathbf{1}\{N_t = B\}dt.$$

Applying Fubini-Tonelli's theorem, e.g. [7](page 239), entails that

$$\mathbf{E}\mathcal{B}_\theta = \mathbf{E}\int_0^\theta \mathbf{1}\{N_t = B\}dt = \int_0^\infty p_B(t)dt = p^*_B(0).$$

From Eqs. (6.3) and (7.3) we obtain
Property 7.1

$$\mathbf{E}\mathcal{B}_\theta = \frac{1}{\lambda}\left(\frac{\rho}{\lambda_s} + \frac{\rho + \lambda \mathbf{E} e^{-(\rho+\lambda)f}}{\rho + \lambda + \mu(1 - \mathbf{E}e^{-(\rho+\lambda)f})}\right),$$

$$\mathbf{E}\theta = (1 + \mu\mathbf{Ef})\left(\frac{1}{\lambda_s} + \mathbf{E}\mathcal{B}_\theta\right) + \mathbf{Ef}.$$

We recall that the **S**-system fails whenever the s-unit fails. Therefore it is of interest to study the impact of the failure rate λ_s of the s-unit on the survival function $\mathcal{R}^*(\cdot)$.
As a numerical example we consider the Erlang-2 distribution

$$F(t) = 1 - e^{-t}(1 + t),\ t \geq 0$$

to model the failure-free time **f** of the p-unit. Note that $\mathbf{E}e^{-z\mathbf{f}} = (1+z)^{-2}$ whereas $\mathbf{Ef} = 2$. The various numerical values of the parameters (Case 1-3) are shown in Table 7.1. The data is synthetic. It is selected in order to analyze numerically the sensitivity of the model to the parameter λ_s (failure rate of the s-unit in the operative state), which is one of the most important characteristics of the system.

Table 7.1 Values of the input parameters

Case	λ	ρ	μ	λ_s
1	1.0	0.5	1.5	0.5
2	1.0	0.5	1.5	1.5
3	1.0	0.5	1.5	2.0

A an illustration, we first deal with Case 1. The required laborious technical manipulations performed in an entirely *automatic* mode using the symbolic engine of Mathematica 12([16]) show that

$$\mathcal{R}^*(z) = \frac{N(z)}{D(z)},\ \mathrm{Re}\, z > 0,$$

where $N(z) = 43.0 + 516.719z + 1281.69z^2 + 1513.75z^3 + 1033.44z^4 + 433.5z^5 + 111.0z^6 + 16.0z^7 + 1.0z^8$ and
$D(z) = 2.875 + 74.625z + 569.219z^2 + 1317.94z^3 + 1525.5z^4 + 1034.94z^5 + 433.5z^6 + 111.0z^7 + 16.0z^8 + 1.0z^9$.
The equation $D(z) = 0$ has the roots $z_1 = -3.32225 - 0.940162i$, $z_2 = \bar{z}_1$, $z_3 = -1.94535 - 0.889027i$, $z_4 = \bar{z}_3$, $z_5 = -1.87055 - 0.918955i$, $z_6 = \bar{z}_5$, $z_7 = -1.54241$, $z_8 = -0.109295$, $z_9 = -0.0719968$.
Clearly, $\mathcal{R}(t)$ is continuous on $(0, \infty)$ and of bounded variation on $[0, \infty)$. Hence, by the inversion theorem, e.g. [1])(page 342)

$$\mathcal{R}(t) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{-iT}^{iT} e^{zt}\frac{N(z)}{\prod_{k=1}^{9}(z - z_k)})dz.$$

Finally, applying the residue theorem, e.g. [1](page 468, Th. 16.39), omitting numerically negligible terms and using the identity

$$(u+iv)e^{(a+ib)t}+(u-iv)e^{(a-ib)t}=2e^{at}(u\cos bt-v\sin bt)$$

reveals that

$$\mathcal{R}(t)=2e^{-3.32225t}(0.0215172\cos 0.940162t{+}0.01939\sin 0.94016t)-$$

$$2e^{-1.87055t}(0.03627\cos 0.91896t+0.01667\sin 0.91896t)-0.05140e^{-1.54241t}$$

$$+1.08089e^{-0.072t}.$$

In a similar way we obtain
Case 2

$$\mathcal{R}(t)=2e^{-3.52217t}(0.0923\cos 0.73108t{+}0.0964\sin 0.731078t){+}$$

$$2e^{-2.0727t}(-0.11317\cos 0.7277t+0.0041\sin 0.7277t)-0.15027e^{-1.64204t}+$$

$$1.192e^{-0.16823t}.$$

Case 3

$$\mathcal{R}(t)=2e^{-3.6307t}(0.1705\cos 0.3787t+0.3631\sin 0.3787t)+$$

$$2e^{-2.1604t}(-0.17\cos 0.5692t+0.069\sin 0.5692t)-0.23238e^{-1.7163t}+$$

$$1.23096e^{-0.20148t}.$$

Figure 7.1 shows the graph of $\mathcal{R}(t)$, case 1-3 with the security interval $[0,t_{sec,i}]\,, i = 1,2,3$ corresponding to the security level $\delta = 0.9$. The graph illustrates that the security interval is a nonlinear function of the failure rate of the s-unit. Figure 7.2 displays the security interval as a function of λ, λ_s on $[0.1]\times[0.1]$. The security interval decreases faster in the λ_s-direction. In particular,

$$||\frac{\partial t_{sec}}{\partial \lambda_s}||_2 \approx 6.79,\ \ ||\frac{\partial t_{sec}}{\partial \lambda}||_2 \approx 5.12.$$

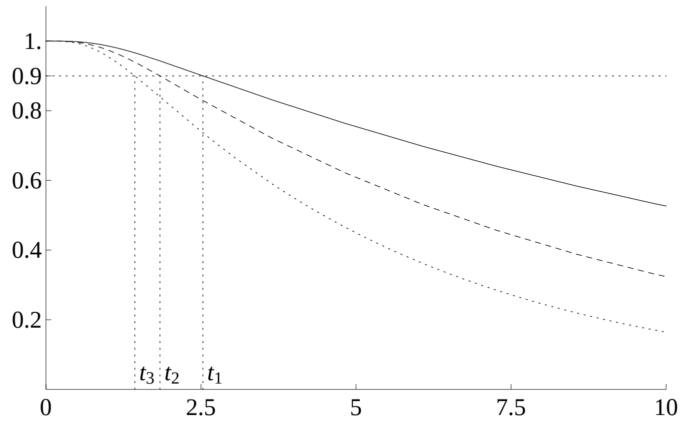

Figure 8: Graph of $\mathcal{R}(t)$, case 1-solid line, case 2-dashed line, case 3- dotted line, $[0, t_i]$ is the security interval for $\sigma = 0.9$, $t_1 = 2.53131$, $t_2 = 1.83898$, $t_3 = 1.43351$.

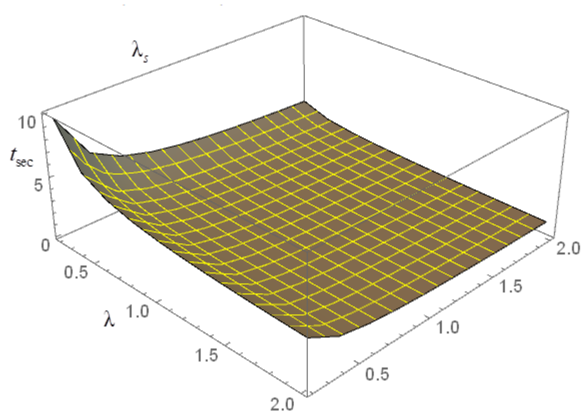

Figure 9: Security interval t_{sec} as a function of λ and λ_s

8. CONCLUSIONS

The Laplace transform of the survival function related to the **S**-system can be derived by solving a set of coupled partial differential equations corresponding to a stochastic process with an absorbing barrier. The important case of Coxian distributions shows how to obtain computational results for the survival function by a numerical analysis based on the inversion formula for Laplace transforms. Therefore, the proposed methodology provides a tangible contribution to statistical reliability engineering and its ramifications. Further, there exists a variety of extensions and generalizations of the proposed model. In particular, finding the solution for arbitrary distributions such as Weibull distribution requires a numerical method. Further, the model is closely related to the modified Gnedenko-Ushakov system[9] characterized by a preemptive priority rule and sustained by an auxiliary unit in cold standby. The system is attended by two heterogeneous repairmen. We conjecture that the proposed methodology combined with numerical inversion of the Laplace transform is applicable to the Gnedenko-Ushakov model in case of the Coxian and Erlang repair.

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9. APPENDICES

For direct reference, we propose to state some particular properties of sectionally holomorphic functions and their ramifications for the solution of some boundary value problems on the real line. See [8] (pp. 1-360), [11] (pp.1-73), [13] (pp. 118-242) for proofs and details.
Let $\varphi(\tau)$ be a function satisfying the Hölder (Lipschitz) condition on $\mathbf{R}$ and at infinity. In addition, let

$$\mathcal{L}^+(u) := \lim_{\substack{\omega\to u\\ \omega\in\mathbf{C}^+}} \frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau-\omega},\ u\in\mathbf{R},$$

$$\mathcal{L}^-(u) := \lim_{\substack{\omega\to u\\ \omega\in\mathbf{C}^-}} \frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau-\omega},\ u\in\mathbf{R}.$$

We have

$$\mathcal{L}^+(u) = \frac{1}{2}\varphi(u) + \frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau-u}. \tag{26}$$

$$\mathcal{L}^-(u) = -\frac{1}{2}\varphi(u) + \frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau-u}, \tag{27}$$

Hence, for $u\in\mathbf{R}$,

$$\mathcal{L}^+(u) - \mathcal{L}^-(u) = \varphi(u), \tag{28}$$

$$\frac{\mathcal{L}^+(u)+\mathcal{L}^-(u)}{2} = \frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\,\frac{d\tau}{\tau-u}. \tag{29}$$

The relations (A.1)−(A.4) are called the Sokhotski-Plemelj formulas on the real line. The functions $\mathcal{L}^+(u)$, $\mathcal{L}^-(u)$ are continuous on $\mathbf{R}$ and infinity. The function $\varphi(\tau)$ has a unique decomposition and the resulting boundary value Eq.(A.3) has a unique regular solution

$$\frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau-\omega}, \tag{30}$$

valid for all $\omega\in\mathbf{C}$ and the Cauchy-type integral generates a regular sectionally holomorphic function in $\mathbf{C}$ cut along the real line. Furthermore

$$\mathcal{L}^+(\omega) = \int_\Gamma \varphi(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}^+, \tag{31}$$

$$\mathcal{L}^-(\omega) = \int_\Gamma \varphi(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}^-. \tag{32}$$