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CONSTRUCTION AND ANALYSIS OF GRAPH MODELS FOR MULTIPROCESSOR INTERCONNECTION NETWORKS

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Abstract: A graph G can serve as a model for the Multiprocessor Interconnection Networks (MINs) in which the vertices represent the processors, while the edges represent connections between processors. This paper presents several graphs that could qualify as models for efficient MINs based on the small values of the graph tightness previously introduced by Cvetković and Davidović in 2008. These graphs are constructed using some well-known and widely used graph operations. The tightness values of these graphs range from $O(\sqrt[4]{N})$ to $O(\sqrt{N})$, where N is the order of the graph under consideration. Also, two new graph tightness values, namely *Third type mixed tightness* $t_3(G)$ and *Second type of Structural tightness* $t_4(G)$ are defined in this paper. It has been shown that these tightness types are easier to calculate than the others for the considered graphs. Moreover, their values are significantly smaller.

Keywords: Interconnection Networks, Graph Tightness, Line Graph, Chromatic Number, Chromatic Index.

MSC: 05C50, 68M10.

1. INTRODUCTION

Throughout this paper, we consider only finite, undirected, and simple graphs. Let G be a simple graph with the adjacency matrix $A = A(G)$. Here $\lambda_1(G), \lambda_2(G), ...$ $, \lambda_n(G)$ are the eigenvalues of A in non-increasing order and they form the spectrum of G. The largest eigenvalue $\lambda_1(G)$ is called index of G; $\mathsf{m} = \mathsf{m}(G)$ denotes the number of distinct eigenvalues of G. Connected and undirected graphs are considered as models for MINs. The nodes (vertices) represent the processors, and the edges represent the connection links between the processors. The time taken to exchange data between different processing units is one of the main communication overheads in multiprocessor systems. Interconnection networks with shorter paths between processors, along with the average number of connections per processor, are preferred. In order to minimize communication time within multiprocessor networks, they must comply with two contradictory characteristics: reduce the number

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of wires (diameter D) and maximize the rate of exchange of data (maximum vertex degree Δ). The diameter D and maximum vertex degree Δ are two main factors of the graph that play an essential role in the design of multiprocessor topologies.

In 2003, Elsässer et al. [10] established a link between the graph spectra and the design of multiprocessor topologies. The key conclusion was as follows: if $\mathsf{m}\Delta$ is small for a given graph G, it was anticipated that the corresponding multiprocessor topology would have excellent communication properties and was called well-suited; also, it was noted that there exists an optimal load balancing algorithm that completed load balancing within $\mathsf{m} - 1$ computational steps. Graphs with large $\mathsf{m}\Delta$ have been named ill-suited and found to be unsuitable for multiprocessor network design.

Let δ and Δ be the minimum and maximum degree, respectively, $\overline{d}$ be the average vertex degree, λ_1 be the largest eigenvalue, D be the diameter, and m be the number of distinct eigenvalues. From [9], considering the inequalities, $\delta \leq \overline{d} \leq \lambda_1 \leq \Delta$ and $D \leq \mathsf{m} - 1$, in 2008, Cvetkoviĉ and Davidoviĉ [7], defined four types of graph tightness values, namely $t_1(G), stt(G), spt(G)$, and $t_2(G)$. Here the use of largest eigenvalue $\lambda_1(G)$ and diameter D, instead of Δ and m, was considered more suitable. Dragoš Cvetkoviĉ [6] proved that the index of the graph $\lambda_1(G)$ is equal to the dynamic mean value of the vertex degrees. Since the dynamical mean value of the vertices takes into account not only the immediate vertex neighbors but also the neighbors of the neighbors, Cvetkoviĉ and Davidoviĉ [7] suggested that it was appropriate to use the largest eigenvalue (index). Furthermore, they showed that the four tightness values are partially ordered by the relation '$\leq$' as follows:

$$t_2(G) \leq stt(G) \leq t_1(G)$$
$$t_2(G) \leq spt(G) \leq t_1(G)$$

Later, they concluded that the graphs with small tightness values of $t_2(G)$ are more suitable for the design of multiprocessor interconnection networks.

In this paper, a few interesting graphs are considered, and it is demonstrated that they could be suitable models for MINs. It is noted that determining the chromatic number for these and similar graphs is easy. This allows the introduction of two additional tightness values, $t_3(G)$ and $t_4(G)$, which could be calculated efficiently. The relation '$\leq$' can also be used to partially arrange these new tightness values. Keeping in mind the emphasis on λ_1, from [19] we consider the inequality $\chi(G) \leq 1 + \lambda_1(G)$ and define $t_3(G)$ and $t_4(G)$ based on the chromatic number of the graph. Also, we show that graphs with small values of $t_3(G)$ and $t_4(G)$ are well suited for the design of multiprocessor interconnection networks.

The rest of the paper is organized as follows: in Section 2, we present some known definitions and theorems which will be used later; in Section 3, based on the chromatic number of the graph, the two new tightness values $t_3(G)$ and $t_4(G)$ are defined, showing that graphs with small values of $t_3(G)$ and $t_4(G)$ are well suited for the design of multiprocessor interconnection networks; Section 4 describes examples of several new families of well-suited graphs whose tightness values range from $O(\sqrt[4]{N})$ to $O(\sqrt{N})$, where N is the number of vertices of the graph under consideration. The examples shown under Section 4 are the results of some well-known graph operations.

2. PRELIMINARIES

Several definitions and theorems that will be used in the remainder of this paper are given in this section. Harary and Norman [12] used the term line graph for the very first time in 1960. However, these concepts were studied by Whitney [18] in 1932 and Krausz [14] in 1943.

Definition 2.1. [12] The line graph $L(G)$ of a graph G has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if they are adjacent as edges in G.

A proper vertex (edge) coloring of a graph G is an assignment of colors to the vertices (edges) of G, so that adjacent vertices (edges) are uniquely colored. A proper vertex (edge) coloring that uses colors from a set of k colors is a k-vertex (edge) coloring.

Definition 2.2. [5] The minimum positive integer k for which G is k-vertex colorable is called the **chromatic number** of G and is denoted by $\chi(G)$. The **chromatic index** (or edge chromatic number) $\chi^{'}(G)$ of a graph G is the minimum positive integer k for which G is k-edge colorable. Furthermore, $\chi^{'}(G) = \chi(L(G))$ for every non empty graph G.

According to Vizing [17], the definition of Class one and Class two graphs are given below:

Definition 2.3. [17] Let $\Delta(G)$ be the maximum vertex degree of the graph G. Graphs that have $\chi^{'}(G) = \Delta(G)$ are called Class one graphs. Graphs with $\chi^{'}(G) = \Delta(G) + 1$ are called Class two graphs.

The concept of total graphs was introduced by Behzad [1] in 1970.

Definition 2.4. [1] The total graph $T(G)$ of a graph G is that graph whose vertex set is $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in G.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. For the construction of new well-suited graphs, the following graph operations are considered.

Definition 2.5. [11] The Cartesian product $G \Box H$ of graphs G and H has vertex set $V(G \Box H) = V(G) \times V(H)$, and edge set $E(G \Box H) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$.

Definition 2.6. [11] The Tensor product (direct product) $G \times H$ of graphs G and H has vertex set $V(G \times H) = V(G) \times V(H)$, and edge set $E(G \times H) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)\}$.

Definition 2.7. [3] Let X be a finite set. The Johnson graph of the e-sets in X has vertex set $\binom{X}{e}$, the collection of e-subsets of X. Two vertices γ, δ are adjacent whenever $\gamma \cap \delta$ has cardinality $e - 1$. When X is some unspecified n-set, the graph is denoted as $\binom{n}{e}$ or $J(n, e)$.

Definition 2.8. [3] The Rook's graph is defined as the Cartesian product of two complete graphs K_n and K_m, expressed as $K_n \Box K_m$. It is also called as $m \times n$ grid by Brouwer [3].

Definition 2.9. [3] The n-crown graph is defined as the complement of the $2 \times n$ grid, i.e., it is isomorphic to the complement of Rook's graph $\overline{K_2 \Box K_n}$.

The following are the definitions of four types of graph tightness introduced by Cvetkoviċ and Davidoviċ [7].

Definition 2.10. ***First type mixed tightness*** $t_1(G)$ of a graph G is defined as the product of the number of distinct eigenvalues m and the maximum vertex degree Δ of G, i.e., $t_1(G) = \mathsf{m}\Delta$.

Definition 2.11. ***Structural tightness*** $stt(G)$ is the product $(D+1)\Delta$, where D is diameter and Δ is the maximum vertex degree of a graph G, i.e., $stt(G) = (D+1)\Delta$.

Definition 2.12. ***Spectral tightness*** $spt(G)$ is the product of the number of distinct eigenvalues m and the largest eigenvalue λ_1 of a graph G, i.e., $spt(G) = \mathsf{m}\lambda_1$.

Definition 2.13. ***Second type mixed tightness*** $t_2(G)$ is defined as a product of the diameter D of G and the largest eigenvalue λ_1, i.e., $t_2(G) = (D+1)\lambda_1$.

In the analysis of a graph's tightness, the following theorem seems to be of fundamental importance [7].

Theorem 2.14. [7] For any kind of tightness, the number of connected graphs with a bounded tightness is finite.

The following theorem gives the eigenvalues of $G \times H$:

Theorem 2.15. [6] The eigenvalues of $G \times H$ are just the pairwise products of the eigenvalues of G and H.

The following theorem gives the eigenvalues of $G \Box H$:

Theorem 2.16. [6] The eigenvalues of $G \Box H$ are just the pairwise sums of the eigenvalues of G and H.

The following result gives an explicit formula for the eigenvalues of $L(G)$ in terms of the eigenvalues of a regular graph G.

Corollary 2.17. [4] If G is a regular graph of degree r, with n vertices and $m(=\frac{nr}{2})$ edges, and eigenvalues θ_i for $i = 1, 2, ..., n$, then line graph $L(G)$ is $(2r-2)$-regular with eigenvalues $(\theta_i + r - 2)$ for $i = 1, ..., n$, and -2 with the multiplicity $(m - n)$.

The following result gives an explicit formula for the eigenvalues of $L(G)$ in terms of the signless Laplace eigenvalues of a non-regular graph G.

Proposition 2.18. [4] Let G be a graph on n vertices, having m edges, and let $q_1 \geq q_2 \geq ... \geq q_n$ be the signless Laplace eigenvalues of G, then the eigenvalues of Line graph of G are $\theta_i = q_i - 2$ for $i = 1, 2, ..., n$, and $\theta_i = -2$ if $n < i \leq m$.

A graph $G = (V, E)$ is bipartite if V can be partitioned into two sets V_1 and V_2, such that every edge of G joins a vertex of V_1 and a vertex of V_2. If the degree of each vertex is r, then the graph is called as r-regular graph.

Theorem 2.19. [5] (Konig's Theorem) If G is a non empty bipartite graph, then $\chi'(G) = \Delta(G)$.

A factor of a graph refers to its spanning subgraph. A sequence of pairwise edge-disjoint subgraphs $G_1, G_2, \ldots, G_n$ whose union is G is called a decomposition of G, and is represented as $G = \bigcup_1^n G_i$. If G_i is r-regular spanning subgraph of G, then every G_i is called an r-factor, and G is called r-factorable graph. A graph M is a matching if each vertex has a degree of 0 or 1. Thus, the edge set of a 1-factor in a graph G is a perfect matching in G. So, a graph G has a 1-factor if and only if G has a perfect matching.

Theorem 2.20. [5] A regular graph G is of Class one if and only if G is 1-factorable.

Corollary 2.21. [5] Every regular graph of odd order is of Class two.

Theorem 2.22. [15] If $\chi'(G) = \Delta(G)$, then $\chi'(G \Box H) = \Delta(G+H) = \Delta(G)+\Delta(H)$

Theorem 2.23. [13] Let G and H be two graphs such that $\chi'(H) = \Delta(H)$. Then $\chi'(G \times H) = \Delta(G \times H) = \Delta(G)\Delta(H)$

3. NEW TIGHTNESS VALUES BASED ON THE CHROMATIC NUMBER

Cvetkoviċ and Davidoviċ [7] showed that the tightness values $t_1(G), stt(G), spt(G)$, and, $t_2(G)$ are partially ordered by the relation '$\leq$' as follows:

$$t_2(G) \leq stt(G) \leq t_1(G)$$
$$t_2(G) \leq spt(G) \leq t_1(G)$$

Later, they concluded that the graphs with small tightness values of $t_2(G)$ are more suitable for the design of multiprocessor interconnection networks.

All the graphs presented in this paper are line graphs of regular graphs, bipartite graphs, or products of these graphs. It is a well known fact that the index of r-regular graph is equal to the vertex degree r, while the complete bipartite graph $K_{p,q}$ has spectrum $\pm\ \sqrt{pq}$, and 0 whose multiplicity is $p+q-2$. As a result, we can determine the eigenvalues of the graphs based on whether the graph is regular or bipartite, as mentioned below.

To get the eigenvalues of a Line graph, one must first compute the spectrum of the original graph, knowing whether it is a regular or bipartite graph. The spectrum of the resulting Line graph is then computed using the Corollary 2.17 and the Proposition 2.18. The eigenvalues of the graphs generated from graph operations, such as Tensor product and Cartesian product are computed using the Theorem 2.15 and the Theorem 2.16. However, by determining whether the graph is a Class one or Class two graph, using the results from the preliminaries section, the chromatic index of these graphs is easily obtained without any complicated calculations. Since, it is known that $\chi^{'}(G) = \chi(\text{Line Graph}(G))$ [5] for any non-empty graph G, determining the edge chromatic number of the graphs presented here is all that is required.

Considering the Line graph of Tensor product $K_n \times K_p$, we show that computing the chromatic number is more straightforward than computing the largest eigenvalue for this graph. The largest eigenvalue λ_1 is computed as follows:
The complete graph K_n is an $(n-1)$ regular graph and the characteristic polynomial is $P(K_n, x) = (x-n+1)(x+1)^{n-1}$. From the polynomial, it is clear that the eigenvalues of K_n are $(n-1)$ and -1 with the multiplicities 1 and $n-1$. From Theorem 2.15, it is known that the eigenvalues of $K_n \times K_p$ are $(n-1)(p-1)$,$(n-1)(-1)^{p-1}$, $(-1)^{n-1}(p-1)$, and $(-1)^{n-1}(-1)^{p-1}$. From Corollary 2.17, the eigenvalues of the line graph of Tensor product of $K_n \times K_p$ are calculated as follows:

$$\begin{aligned}
&\lambda_1 = np - n - p + 1 + np - n - p + 1 - 2 = 2np - 2(n+p)\\
&\lambda_2 = (n-1)(-1) + np - n - p + 1 - 2 = np - 2n - p\\
&\lambda_3 = (-1)(p-1) + np - n - p + 1 - 2 = np - 2p - n\\
&\lambda_4 = (-1)(-1) + np - n - p + 1 - 2 = np - n - p\\
&\text{and } \lambda_5 = -2
\end{aligned}$$

Therefore $\lambda_1 = 2np - 2(n+p)$.

The chromatic number of the Line graph of $K_n \times K_p$ can be quickly computed as follows: the first step is to figure out whether the graph is a Class one or Class two graph; Theorem 2.20, Corollary 2.21, and Theorem 2.23 are then used to compute the chromatic index of the graph. Here, the chromatic index is $np-n-p+2$ when the number of vertices in the original graph is odd, and $np-n-p+1$ when the number of vertices in the original graph is even. The observation that one could quickly determine the chromatic number for the graphs presented as examples in this paper leads to the introduction of two additional tightness values, $t_3(G)$ and $t_4(G)$, which can be partially ordered by the relation '$\leq$'. The basis for the present investigation is the following result from [19] by Wilf.

Theorem 3.1. [19] If χ is the chromatic number and λ_1 is the largest eigenvalue, then

$$\chi \leq 1 + \lambda_1 \tag{3.1}$$

with equality if and only if G is a complete graph or an odd circuit.

The maximum and minimum vertex degree of graph G is denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The average vertex degree of G is represented as $\overline{d} = \overline{d}(G)$. From [9], we have

$$\delta \leq \overline{d} \leq \lambda_1 \leq \Delta \text{ and} \tag{3.2}$$

$$D \leq \mathsf{m} - 1, \text{ where D is the diameter.} \tag{3.3}$$

Rewrite (3.1) as

$$\chi - 1 \leq \lambda_1 \tag{3.4}$$

Recalling Definition 2.12, which states $spt(G) = \mathsf{m}\lambda_1$ and from (3.4), the new tightness value called the Third type mixed tightness $t_3(G)$ can be defined as follows:

Definition 3.2. ***Third type mixed tightness*** $t_3(G)$ is the product of the number of distinct eigenvalues m and $(\chi - 1)$, where χ is the chromatic number of a graph G, i.e., $t_3(G) = \mathsf{m}(\chi - 1)$.

Considering Definition 2.13, which states $t_2(G) = (D + 1)\lambda_1$ and equation(3.4), the new tightness value called the Second type of Structural tightness $t_4(G)$ can be defined as follows:

Definition 3.3. ***Second type of Structural tightness*** $t_4(G)$ is the product $(D + 1)(\chi - 1)$, where D is diameter and χ is the chromatic number of a graph G.

From Definition 3.2, Definition 3.3, equations (3.2), (3.3), and (3.4), the new tightness values can be partially ordered as follows:

$$\begin{aligned}
&t_3(G) \leq spt(G) \leq t_1(G)\\
&t_4(G) \leq t_2(G) \leq stt(G) \leq t_1(G)\\
&t_4(G) \leq t_2(G) \leq spt(G) \leq t_1(G), \text{ and}\\
&t_4(G) \leq t_3(G)
\end{aligned}$$

Hence, from the above inequalities, it is clear that the graphs with small values of $t_3(G)$ and $t_4(G)$ are well suited for the design of the multiprocessor interconnection topologies. In Theorem 2.14 [7], it has been proved that the number of connected graphs with bounded tightness is finite for the four types of tightness values defined. The following theorem proves that this criterion also applies to the two new tightness values defined in this paper.

Theorem 3.4. The number of connected graphs with the bounded Third type mixed tightness $t_3(G)$ and Second type of Structural tightness $t_4(G)$ is finite.

Proof. The following inequality holds for the number of vertices n in a graph G:

$$n \leq 1 + \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 \cdots + \Delta(\Delta - 1)^{D-1} \tag{3.5}$$

As in the proof of Theorem 2.14 [7], we assume that $t(G) \leq a$, for a given positive integer a, where $t(G)$ represents the two new tightness values $t_3(G)$ and $t_4(G)$. We now prove that for the new tightness values, both the diameter D and maximum vertex degree Δ are bounded by a number denoted as b. According to Brooks [2], it is known that $\chi(G) \leq 1 + \Delta(G)$, and from Cvetkovič et al. [9], we have $D \leq \mathsf{m} - 1$ for the diameter D. Here, m is the number of distinct eigenvalues, and $\chi(G)$ is the chromatic number of G. Note that $\Delta \leq a$ and $D \leq a - 1$, as shown in the proof of Theorem 2.14 [7]. Now for $t_3(G) = \mathsf{m}(\chi - 1)$, $t_3(G) \leq a$ implies

$$\begin{aligned} &\mathsf{m}(\chi - 1) \leq a \Rightarrow \ \mathsf{m} \leq a \text{ and } (\chi - 1) \leq a, \text{ which implies} \\ &D \leq a - 1, \ \Delta \leq a, \text{ and we assign } b = a; \end{aligned}$$

and for $t_4(G) = (D + 1)(\chi - 1)$, when $t_4(G) \leq a$, the following holds :

$$\begin{aligned} &(D + 1)(\chi - 1) \leq a \Rightarrow \ D + 1 \leq a \text{ and } (\chi - 1) \leq a, \\ &\text{which implies } D \leq a - 1, \Delta \leq a, \text{ and we assign } b = a; \end{aligned}$$

Based on the relationship in (3.5), and assuming that both D and Δ are bound by the number b, we have the following:

$$\begin{aligned} n &\leq 1 + \Delta + \Delta^2 + \Delta^3 \cdots + \Delta^D \leq 1 + \Delta + \Delta^2 + \Delta^3 \cdots + \Delta^b \\ &\leq 1 + b + b^2 + b^3 \cdots + b^b \end{aligned}$$

Hence, we prove that a connected graph with the given number of vertices n and a bounded tightness is also bounded. Therefore, we conclude that the number of connected graphs with the bounded tightness $t_3(G)$ and $t_4(G)$ is finite. □

4. GRAPHS SUITABLE FOR MINs

One can find examples of well-suited MINs resulting from some graph operations with tightness values as $O(\sqrt{N})$ or $O(N)$ in [8]. In this section, we present examples of graphs resulting from several graph operations. Graph operations include line graphs of graph products, such as the Cartesian product and the Tensor product of graphs. Also, we consider the line graphs of Johnson graphs, Rook graphs, and Crown graphs. The resulting graphs are considered as well-suited interconnection network models since their tightness values range from $O(\sqrt[4]{N})$ to $O(\sqrt{N})$, where N is the number of vertices of the graph that is considered.

Obtaining the chromatic number of an arbitrary graph is NP-Hard, but one can get the chromatic number for the well-known graphs using Sage[16]. The graphs presented here are line graphs of some regular or bipartite graphs or products of these graphs. For every non-empty graph G, $\chi'(G) = \chi(L(G))$, according to Definition2.2. As a result, obtaining the chromatic index of the graphs provided here is sufficient. We can determine the edge chromatic index of these graphs using the results in the Preliminaries section. The computations in this section are performed using Sage. [16].

Let G_c be the set of connected graphs with at least two vertices, and $t(G) \in \{t_1(G), stt(G), spt(G), t_2(G), t_3(G), t_4(G)\}$. Now consider the following notations:

$$\begin{aligned} S^{O(\sqrt{N})} &= \{G : G \in G_c, \ t(G) = O(\sqrt{N})\} \\ S^{O(\sqrt[3]{N})} &= \{G : G \in G_c, \ t(G) = O(\sqrt[3]{N})\} \\ S^{O(\sqrt[4]{N})} &= \{G : G \in G_c, \ t(G) = O(\sqrt[4]{N})\} \end{aligned}$$

Some of the notations used in the examples are given below in Table 1. Throughout the examples, we consider the order of the original graph and its regularity. The graph parameters such as D, Δ, m, λ_1, and N^{OG} are computed. Since $\chi^{'}(G) = \chi(L(G))$, the graph's chromatic number is derived from the edge chromatic index obtained for such graphs using the theorems stated in Section 2.

Table. 1 Notations

N	Number of vertices in the newly constructed graph
D	Diameter
m	Number of distinct eigenvalues of G.
Δ	Maximum degree
λ_1	Largest eigenvalue of G
N^{OG}	Number of vertices in the original graph

Example 4.1. The set $S^{O(\sqrt{N})}$ contains the following graphs:
4.1.1. Line graph of Tensor product $K_n \times K_p$
4.1.2. Line graph of Tensor product $K_n \times K_{p,p}$
4.1.3. Line graph of Cartesian product $K_{1,n-1} \square K_{1,p-1}$
4.1.4. Line graph of Complete graph K_n
4.1.5. Line graph of Complete Bipartite graph $K_{n,n}$
4.1.6. Line graph of Crown graph $K_{n,n} - I$
4.1.7. Line graph of Complete Tripartite graph $K_{n,n,n}$

4.1.1. Line graph of Tensor product $K_n \times K_p$: Consider $G_1 = L(K_n \times K_p)$ = Line graph of Tensor product $K_n \times K_p$, for $n > 2$ and $p > 2$. All relevant parameters of G_1 are summarized in Table 2.

Table. 2 Line graph of Tensor product $K_n \times K_p$, for $n > 2$ and $p > 2$.

N	D	m	Δ	λ_1
$\frac{n^2p^2-n^2p-np^2+np}{2}$	≤ 3	≤ 5	$2np-2(n+p)$	$2np-2(n+p)$

Table 3 presents some properties of the Tensor product $K_n \times K_p$, for $n > 2$:

Table. 3 Tensor product $K_n \times K_p$, for $n > 2$ and $p > 2$.

N^{OG}	Δ	Is Regular?
$n * p$	$np - n - p + 1$	Yes

The chromatic number of the Line graph of Tensor product $K_n \times K_p$ is calculated as follows: if the number of vertices N^{OG} is odd, then from Corollary 2.21 it is clear that the edge chromatic number (chromatic index) of $K_n \times K_p$ is $np - n - p + 2$; if the number of vertices N^{OG} is even, then from Theorem 2.20 the edge chromatic number (chromatic index) of $K_n \times K_p$ is $np - n - p + 1$. Also, from Definition 2.2, $\chi^{'}(K_n \times K_p) = \chi(L(K_n \times K_p))$. If $n = p$, then the tightness values are given as follows:

$$t_1(G_1) \leq 5(2n^2 - 2(n+n)) \leq 10n^2 - 10(2n) = O(\sqrt{N});$$
$$stt(G_1) \leq 4(2n^2 - 2(n+n)) \leq 8n^2 - 8(2n) = O(\sqrt{N});$$
$$spt(G_1) \leq 5(2n^2 - 2(n+n)) \leq 10n^2 - 10(2n) = O(\sqrt{N});$$
$$t_2(G_1) \leq 4(2n^2 - 2(n+n)) \leq 8n^2 - 8(2n) = O(\sqrt{N}).$$

If $n = p$, the new tightness values $t_3(G_1)$ and $t_4(G_1)$ are also given as follows:

$t_3(G_1) = \mathsf{m}(\chi - 1) \leq 5(n^2 - 2n) = O(\sqrt{N})$; if no. of vertices is even
$t_3(G_1) = \mathsf{m}(\chi - 1) \leq 5(n^2 - 2n + 1) = O(\sqrt{N})$; if no. of vertices is odd
$t_4(G_1) = (D+1)(\chi - 1) \leq 4(n^2 - 2n) = O(\sqrt{N})$; if no. of vertices is even
$t_4(G_1) = (D+1)(\chi - 1) \leq 4(n^2 - 2n + 1) = O(\sqrt{N})$; if no. of vertices is odd

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt{N})$, and hence G_1 can be used as a model for MINs. The Line graph of Tensor product $K_3 \times K_3$ is given in Fig. 1.

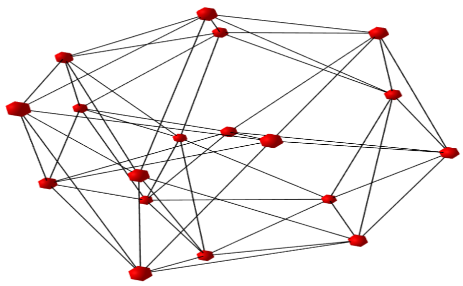

Fig. 1 Line graph of Tensor product $K_3 \times K_3$

4.1.2. Line graph of Tensor product $K_n \times K_{p,p}$: Consider $G_2 = L(K_n \times K_{p,p})$ = Line graph of Tensor product $K_n \times K_{p,p}$, for $n > 2$. All relevant parameters of G_2 are summarized in Table 4.

Table. 4 Line graph of Tensor product $K_n \times K_{p,p}$, for $n > 2$.

N	D	m	Δ	λ_1
$n^2p^2 - np^2$	≤ 3	≤ 5	$2np - 2p - 2$	$2np - 2p - 2$

Table 5 presents some properties of the Tensor product $K_n \times K_{p,p}$, for $n > 2$:

Table. 5 Tensor product $K_n \times K_{p,p}$, for $n > 2$.

N^{OG}	Δ	Is Regular?
$2 * n * p$	$np - p$	Yes

The chromatic number of the Line graph of Tensor product $K_n \times K_{p,p}$ is calculated as follows: from Theorem 2.23, it can be observed that the edge chromatic number (chromatic index) of $K_n \times K_{p,p}$ is $np - p$. Also, from Definition 2.2, the edge chromatic number $\chi^{'}(K_n \times K_{p,p}) = \chi(L(K_n \times K_{p,p}))$. If $n = p$, then the tightness values are given as follows:

$$t_1(G_2) \leq 5(2n^2 - 2n - 2) = O(\sqrt{N});$$
$$stt(G_2) \leq 4(2n^2 - 2n - 2) = O(\sqrt{N});$$
$$spt(G_2) \leq 5(2n^2 - 2n - 2) = O(\sqrt{N});$$
$$t_2(G_2) \leq 4(2n^2 - 2n - 2) = O(\sqrt{N}).$$

If $n = p$, the new tightness values $t_3(G_2)$ and $t_4(G_2)$ are also given as follows:

$$t_3(G_2) = \mathsf{m}(\chi - 1) \leq 5(n^2 - n - 1) \leq 5n^2 - 5n - 5 = O(\sqrt{N});$$
$$t_4(G_2) = (D + 1)(\chi - 1) \leq 4(n^2 - n - 1) \leq 4n^2 - 4n - 4 = O(\sqrt{N}).$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt{N})$, and hence G_2 can be used as a model for MINs. The Line graph of Tensor product $K_3 \times K_{2,2}$ is given in Fig. 2.

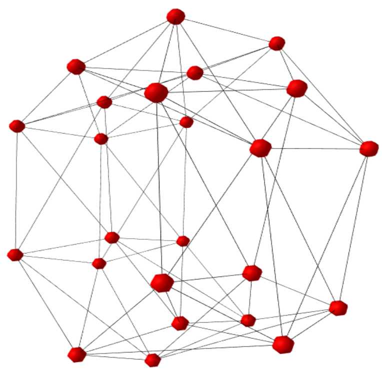

Fig. 2 Line graph of Tensor product $K_3 \times K_{2,2}$

4.1.3. Line graph of Cartesian product $K_{1,n-1}\square K_{1,p-1}$: Consider $G_3 = L(K_{1,n-1}\square K_{1,p-1})$ = Line graph of Cartesian product $K_{1,n-1}\square K_{1,p-1}$. Table 6 summarizes all the relevant parameters of G_3 .

Table. 6 Line graph of Cartesian product $K_{1,n-1}\square K_{1,p-1}$.

N	D	m	Δ	λ_1
$2np - n - p$	≤ 3	≤ 8	$2p + n - 4$	$n + p - 2$

. The properties of Cartesian product $K_{1,n-1}\square K_{1,p-1}$ are given in Table 7.

Table. 7 Cartesian product $K_{1,n-1}\square K_{1,p-1}$.

N^{OG}	Δ	Is Regular?
$n * p$	$n + p - 2$	No

The chromatic number of the Line graph of Cartesian product $K_{1,n-1}\square K_{1,p-1}$ is calculated as follows: from Theorem 2.22 the edge chromatic number (chromatic index) of $K_{1,n-1}\square K_{1,p-1}$ is $n+p-2$. Also, from Definition 2.2, the edge chromatic number $\chi^{'}(K_{1,n-1}\square K_{1,p-1}) = \chi(L(K_{1,n-1}\square K_{1,p-1}))$. If $n = p$, then the tightness values are given as follows:

$$t_1(G_3) \leq 8(2n+n-4) \leq 8(3n-4) = O(\sqrt{N});$$
$$stt(G_3) \leq 4(2n+n-4) \leq 8(3n-4) = O(\sqrt{N});$$
$$spt(G_3) \leq 8(n+n-2) \leq 8(2n-2) = O(\sqrt{N});$$
$$t_2(G_3) \leq 4(n+n-2) \leq 4(2n-2) = O(\sqrt{N}).$$

If $n = p$, the new tightness values $t_3(G_3)$ and $t_4(G_3)$ are also given as follows:

$$t_3(G_3) = \mathsf{m}(\chi-1) \leq 8(2n-3) = O(\sqrt{N});$$
$$t_4(G_3) = (D+1)(\chi-1) \leq 4(2n-3) = O(\sqrt{N}).$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt{N})$, and hence G_3 can be used as a model for MINs. Fig. 3 gives the Line graph of Cartesian product $K_{1,2}\square K_{1,3}$.

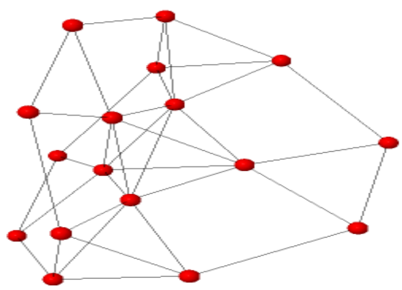

Fig. 3 Line graph of Cartesian product $K_{1,2}\square K_{1,3}$

4.1.4. Line graph of Complete graph K_n: Consider $G_4 = L(K_n)$ = Line graph of Complete graph of K_n, for $n > 2$. Table 8 summarizes all the relevant properties of G_4 .

Table. 8 Line graph of Complete graph K_n.

N	D	m	Δ	λ_1
$\frac{n^2-n}{2}$	≤ 2	≤ 3	$2(n-2)$	$2(n-2)$

The properties of the Complete graph K_n are given in Table 9.

Table. 9 Complete graph K_n

N^{OG}	Δ	Is Regular?
n	$n-1$	Yes

The chromatic number of the Line graph of Complete graph K_n is calculated as follows: from Theorem 2.20 and Corollary 2.21, if N^{OG} is odd, the edge chromatic number (chromatic index) of K_n is n, and $n-1$ if N^{OG} is even. Also, from Definition 2.2, the edge chromatic number $\chi^{'}(K_n) = \chi(L(K_n))$. The tightness values are given as follows: :

$$t_1(G_4) \leq 3 \times 2(n-2) \leq 6(n-2) = O(\sqrt{N});$$
$$stt(G_4) \leq 3 \times 2(n-2) \leq 6(n-2) = O(\sqrt{N});$$
$$spt(G_4) \leq 3 \times 2(n-2) \leq 6(n-2) = O(\sqrt{N});$$
$$t_2(G_4) \leq 3 \times 2(n-2) \leq 6(n-2) = O(\sqrt{N}).$$

The new tightness values $t_3(G_4)$ and $t_4(G_4)$ are also given as follows:

$$t_3(G_4) = \mathsf{m}(\chi-1) \leq 3(n-2) \leq 3n-6 = O(\sqrt{N}); \text{ if n is even}$$
$$t_3(G_4) = \mathsf{m}(\chi-1) \leq 3(n-1) \leq 3n-3 = O(\sqrt{N}); \text{ if n is odd}$$
$$t_4(G_4) = (D+1)(\chi-1) \leq 3(n-2) \leq 3n-6 = O(\sqrt{N}); \text{ if n is even}$$
$$t_4(G_4) = (D+1)(\chi-1) \leq 3(n-1) \leq 3n-3 = O(\sqrt{N}); \text{ if n is odd}$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt{N})$, and hence G_4 can be used as a model for MINs. The Line graph of Complete graph K_5 is given in Fig. 4.

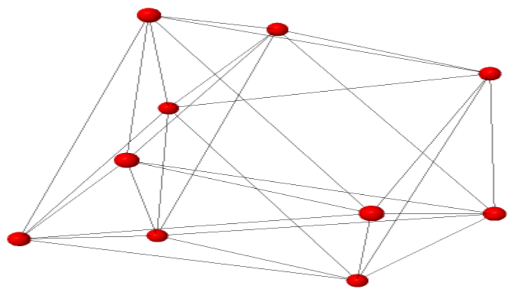

Fig. 4 Line graph of Complete graph K_5

4.1.5. Line graph of Complete Bipartite graph $K_{n,n}$: Consider $G_5 = L(K_{n,n})$ = Line graph of Complete Bipartite graph $K_{n,n}$. Table 10 summarizes all the relevant properties of G_5 .

Table. 10 Line graph of Complete Bipartite graph $K_{n,n}$.

N	D	m	Δ	λ_1
n^2	2	3	$2(n-1)$	$2(n-1)$

The properties of the Complete Bipartite graph $K_{n,n}$ are given in Table 11.

Table. 11 Complete Bipartite graph $K_{n,n}$

N^{OG}	Δ	Is Regular?
$2*n$	n	Yes

The chromatic number of the Line graph of Complete Bipartite graph $K_{n,n}$ is calculated as follows: from Theorem 2.19, the edge chromatic number (chromatic index) of $K_{n,n}$ is n. Also, from Definition 2.2, the edge chromatic number $\chi^{'}(K_{n,n})$ = $\chi(L(K_{n,n}))$. The tightness values are given as follows:

$$t_1(G_5) = 3 \times 2(n-1) = 6(n-1) = O(\sqrt{N});$$
$$stt(G_5) = 3 \times 2(n-1) = 6(n-1) = O(\sqrt{N});$$
$$spt(G_5) = 3 \times 2(n-1) = 6(n-1) = O(\sqrt{N});$$
$$t_2(G_5) = 3 \times 2(n-1) = 6(n-1) = O(\sqrt{N}).$$

The new tightness values $t_3(G_5)$ and $t_4(G_5)$ are also given as follows:

$$t_3(G_5) = \mathsf{m}(\chi - 1) = 3(n-1) = 3n-3 = O(\sqrt{N});$$
$$t_4(G_5) = (D+1)(\chi - 1) = 3(n-1) = 3n-3 = O(\sqrt{N}).$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt{N})$, and hence G_5 can be used as a model for MINs. The Line graph of Complete Bipartite graph $K_{3,3}$ is shown in Fig. 5.

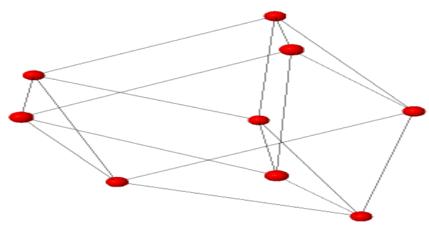

Fig. 5 Line graph of Complete Bipartite graph $K_{3,3}$

4.1.6. Line graph of Crown graph $K_{n,n} - I$: Consider $G_6 = K_{n,n} - I =$ Line graph of Crown graph $K_{n,n} - I$. Table 12 summarizes all the relevant properties of G_6 .

Table. 12 Line graph of Crown graph $K_{n,n} - I$.

N	D	m	Δ	λ_1
$n^2 - n$	3	4	$2(n-2)$	$2(n-2)$

The properties of the Crown graph $K_{n,n} - I$ are given in Table 13.

Table. 13 Crown graph $K_{n,n} - I$

N^{OG}	Δ	Is Regular?
$2 * n$	$n-1$	Yes

The chromatic number of the Line graph of Crown graph $K_{n,n} - I$ is calculated as shown below. The Crown graph $K_{n,n} - I$ is a $(n-1)$-regular bipartite graph with the number of vertices is given as $2n$. It can be observed that the edge chromatic number (chromatic index) of $K_{n,n} - I$ is $n-1$ (from Theorem 2.19). Also, from Definition 2.2 the edge chromatic number $\chi^{'}(K_{n,n} - I) = \chi(L(K_{n,n} - I))$. The tightness values are given as follows:

$$t_1(G_6) = 4 \times 2(n-2) = 8n - 8 = O(\sqrt{N});$$
$$stt(G_6) = 4 \times 2(n-2) = 8n - 8 = O(\sqrt{N});$$
$$spt(G_6) = 4 \times 2(n-2) = 8n - 8 = O(\sqrt{N});$$
$$t_2(G_6) = 4 \times 2(n-2) = 8n - 8 = O(\sqrt{N}).$$

The new tightness values $t_3(G_6)$ and $t_4(G_6)$ are also given as follows:

$$t_3(G_6) = \mathsf{m}(\chi - 1) = 4(n-2) = 4n - 8 = O(\sqrt{N});$$
$$t_4(G_6) = (D+1)(\chi - 1) = 4(n-2) = 4n - 8 = O(\sqrt{N}).$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt{N})$, and hence G_6 can be used as a model for MINs. In Fig. 6 the Line graph of Crown graph $K_{4,4} - I$ is shown.

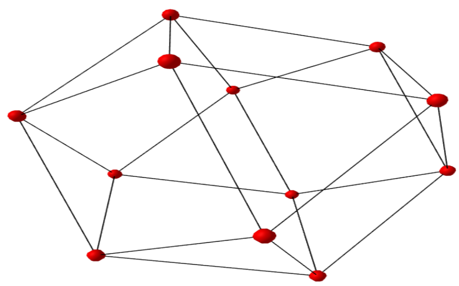

Fig. 6 Line graph of Crown graph $K_{4,4} - I$

4.1.7. Line graph of Complete Tripartite graph $K_{n,n,n}$: Consider $G_7 = L(K_{n,n,n})$ = Line graph of Complete Tripartite graph $K_{n,n,n}$. Table 14 summarizes all the relevant properties of G_7.

Table. 14 Line graph of Complete Tripartite graph $K_{n,n,n}$.

N	D	m	Δ	λ_1
$3n^2$	2	4	$4n-2$	$4n-2$

The properties of the Complete Tripartite graph $K_{n,n,n}$ are given in Table 15.

Table. 15 Complete Tripartite graph $K_{n,n,n}$

N^{OG}	Δ	Is Regular?
$3*n$	$2*n$	Yes

The chromatic number of the Line graph of Complete Tripartite graph $K_{n,n,n}$ is calculated as shown below. It can be observed that the edge chromatic number (chromatic index) of Complete Tripartite graph $K_{n,n,n}$ is $2n+1$ if the number of vertices is odd (from Corollary 2.21), and $2n$ if number of vertices is even (from Theorem 2.20). Also, from Definition 2.2 the edge chromatic number $\chi^{'}(K_{n,n,n}) = \chi(L(K_{n,n,n}))$. The tightness values are given as follows:

$$\begin{aligned} t_1(G_7) &= 4 \times (4n-2) = 16n-8 = O(\sqrt{N}); \\ stt(G_7) &= 3 \times (4n-2) = 12n-6 = O(\sqrt{N}); \\ spt(G_7) &= 4 \times (4n-2) = 16n-8 = O(\sqrt{N}); \\ t_2(G_7) &= 3 \times (4n-2) = 12n-6 = O(\sqrt{N}). \end{aligned}$$

The new tightness values $t_3(G_7)$ and $t_4(G_7)$ are also given as follows:

$$\begin{aligned} t_3(G_7) &= \mathsf{m}(\chi-1) = 4(2n-1) = 8n-4 = O(\sqrt{N}); \text{ if n is even} \\ t_3(G_7) &= \mathsf{m}(\chi-1) = 4(2n) = 8n = O(\sqrt{N}); \text{ if n is odd} \\ t_4(G_7) &= (D+1)(\chi-1) = 3(2n-1) = 6n-3 = O(\sqrt{N}); \text{ if n is even} \\ t_4(G_7) &= (D+1)(\chi-1) = 3(2n) = 6n = O(\sqrt{N}); \text{ if n is odd} \end{aligned}$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt{N})$, and hence G_7 can be used as a model for MINs. The Line graph of Complete Tripartite graph $K_{2,2,2}$ is given in Fig. 7.

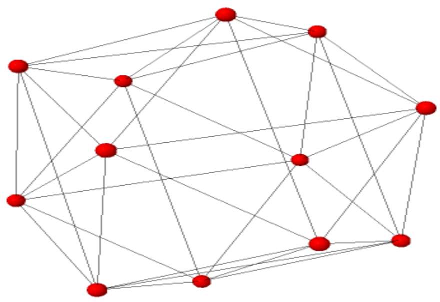

Fig. 7 Line graph of Complete Tripartite graph $K_{2,2,2}$

Example 4.2. The set $S^{O(\sqrt[3]{N})}$ contains the following graphs:
4.2.1. Line graph of Johnson graph $J(n,2)$
4.2.2. Line graph of Cartesian product $K_{1,n-1} \Box K_p$
4.2.3. Line graph of Rook graph $K_n \Box K_n$
4.2.4. Line graph of Total graph of complete bipartite graph $K_{n,n}$
4.2.5. Line graph of Total graph of complete graph K_n

4.2.1. Line graph of Johnson graph $J(n,2)$: Consider $G_1 = L(J(n,2)) =$ Line graph of Johnson graph $J(n,2)$, for $n > 3$. The graph parameters of G_1 are given in Table 16.

Table. 16 Line graph of Johnson graph $J(n,2)$.

N	D	m	Δ	λ_1
$\frac{n^3-3n^2+2n}{2}$	≤ 3	≤ 4	$4n-10$	$4n-10$

The properties of Johnson graph $J(n,2)$ are given in Table 17.

Table. 17 Johnson graph $J(n, 2)$

N^{OG}	Δ	Is Regular?
$\binom{n}{2}$	$2(n-2)$	Yes

It can be observed that the edge chromatic number (chromatic index) of $J(n, 2)$ is $2(n-2)+1$ if the number of vertices is odd (from Corollary 2.21), and $2(n-2)$ if number of vertices is even (from Theorem 2.20). Also, from Definition 2.2 the edge chromatic number $\chi^{'}(J(n, 2)) = \chi(L(J(n, 2)))$. The tightness values are given as follows:

$$t_1(G_1) \leq 4(4n-10) \leq 16n-40 = O(\sqrt[3]{N});$$
$$stt(G_1) \leq 4(4n-10) \leq 16n-40 = O(\sqrt[3]{N});$$
$$spt(G_1) \leq 4(4n-10) \leq 16n-40 = O(\sqrt[3]{N});$$
$$t_2(G_1) \leq 4(4n-10) \leq 16n-40 = O(\sqrt[3]{N}).$$

The new tightness values $t_3(G_1)$ and $t_4(G_1)$ are also given as follows:

$t_3(G_1) = \mathsf{m}(\chi-1) \leq 4(2n-5) \leq 8n-20 = O(\sqrt[3]{N})$; if no. of vertices is even
$t_3(G_1) = \mathsf{m}(\chi-1) \leq 4(2n-4) \leq 8n-16 = O(\sqrt[3]{N})$; if no. of vertices is odd
$t_4(G_1) = (D+1)(\chi-1) \leq 4(2n-5) \leq 8n-20 = O(\sqrt[3]{N})$; if no. of vertices is even
$t_4(G_1) = (D+1)(\chi-1) \leq 4(2n-4) \leq 8n-16 = O(\sqrt[3]{N})$; if no. of vertices is odd

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt[3]{N})$, and hence G_1 can be used as a model for MINs. The Line graph of Johnson graph $J(4, 2)$ is provided in Fig. 8.

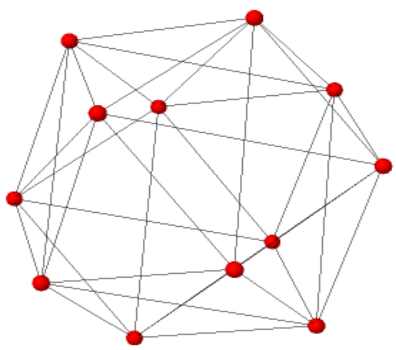

Fig. 8 Line graph of Johnson graph $J(4, 2)$

4.2.2. Line graph of Cartesian product $K_{1,n-1}\Box K_p$: Consider $G_2 = L(K_{1,n-1}\Box K_p)$ = Line graph of Cartesian product $K_{1,n-1}\Box K_p$. The graph parameters of G_2 are given in Table 18.

Table. 18 Line graph of Cartesian product $K_{1,n-1}\square K_p$.

N	D	m	Δ	λ_1
$\frac{np^2+np-2p}{2}$	≤ 4	≤ 7	$2n+2p-6$	$n+2p-4$

The properties of Cartesian product $K_{1,n-1}\square K_p$ are given in Table 19.

Table. 19 Cartesian product $K_{1,n-1}\square K_p$

N^{OG}	Δ	Is Regular?
$n*p$	$n+p-2$	No

The chromatic number of the Line graph of Cartesian product $K_{1,n-1}\square K_p$ is calculated as follows: from Theorem 2.22, the edge chromatic number (chromatic index) of $K_{1,n-1}\square K_p$ is $n+p-2$. Also, from Definition 2.2, the edge chromatic number $\chi^{'}(K_{1,n-1}\square K_p) = \chi(L(K_{1,n-1}\square K_p))$. If $n = p$, then the tightness values are given as follows:

$$t_1(G_2) \leq 7(2n+2n-6) \leq 7(4n-6) = O(\sqrt[3]{N});$$
$$stt(G_2) \leq 5(2n+2n-6) \leq 5(4n-6) = O(\sqrt[3]{N});$$
$$spt(G_2) \leq 7(n+2n-4) \leq 7(3n-4) = O(\sqrt[3]{N});$$
$$t_2(G_2) \leq 5(n+2n-4) \leq 5(3n-4) = O(\sqrt[3]{N}).$$

If $n = p$, the new tightness values $t_3(G_2)$ and $t_4(G_2)$ are also given as follows:

$$t_3(G_2) = \mathsf{m}(\chi-1) \leq 7(n+n-3) \leq 7(2n-3) = O(\sqrt[3]{N});$$
$$t_4(G_2) = (D+1)(\chi-1) \leq 5(n+n-3) \leq 5(2n-3) = O(\sqrt[3]{N})$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt[3]{N})$, and hence G_2 can be used as a model for MINs. In Fig. 9, the Line graph of Cartesian product $K_{1,2}\square K_3$ is shown.

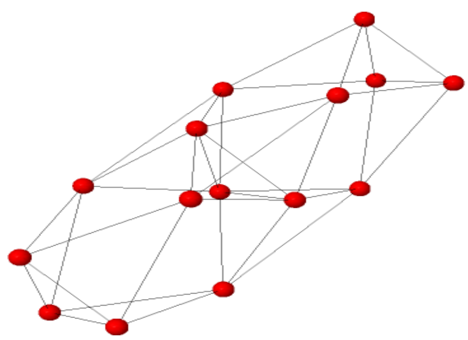

Fig. 9 Line graph of Cartesian product $K_{1,2}\square K_3$

4.2.3. Line graph of Rook graph $K_n\square K_n$: Consider $G_3 = L(K_n\square K_n) =$ Line graph of Rook graph $K_n\square K_n$, for $(n > 2)$. The graph parameters of G_3 are given in Table 20.

Table. 20 Line graph of Rook graph $K_n \Box K_n$.

N	D	m	Δ	λ_1
$n^3 - n^2$	3	4	$4n - 6$	$4n - 6$

The properties of Rook graph $K_n \Box K_n$ are given in Table 21.

Table. 21 Rook graph $K_n \Box K_n$

N^{OG}	Δ	Is Regular?
n^2	$2 * n - 2$	Yes

The chromatic number of the Line graph of Rook graph $K_n \Box K_n$ is calculated as follows: it can be observed that the edge chromatic number (chromatic index) of $K_n \Box K_n$ is $2n - 1$ if the number of vertices is odd (from Corollary 2.21), and $2n - 2$ if number of vertices is even (from Theorem 2.20). Also, from Definition 2.2 the edge chromatic number $\chi^{'}(K_n \Box K_n) = \chi(L(K_n \Box K_n))$. The tightness values are given as follows:

$$t_1(G_3) = 4(4n - 6) = 16n - 24 = O(\sqrt[3]{N});$$
$$stt(G_3) = 4(4n - 6) = 16n - 24 = O(\sqrt[3]{N});$$
$$spt(G_3) = 4(4n - 6) = 16n - 24 = O(\sqrt[3]{N});$$
$$t_2(G_3) = 4(4n - 6) = 16n - 24 = O(\sqrt[3]{N}).$$

The new tightness values $t_3(G_3)$ and $t_4(G_3)$ are also given as follows:

$$t_3(G_3) = \mathsf{m}(\chi - 1) = 4(2n - 3) = 8n - 12 = O(\sqrt[3]{N}); \text{ if n is even}$$
$$t_3(G_3) = \mathsf{m}(\chi - 1) = 4(2n - 2) = 8n - 8 = O(\sqrt[3]{N}); \text{ if n is odd}$$
$$t_4(G_3) = (D + 1)(\chi - 1) = 4(2n - 3) = 8n - 12 = O(\sqrt[3]{N}); \text{ if n is even}$$
$$t_4(G_3) = (D + 1)(\chi - 1) = 4(2n - 2) = 8n - 8 = O(\sqrt[3]{N}); \text{ if n is odd}$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt[3]{N})$, and hence G_3 can be used as a model for MINs. The Line graph of Rook graph $K_3 \Box K_3$ is shown in Fig. 10.

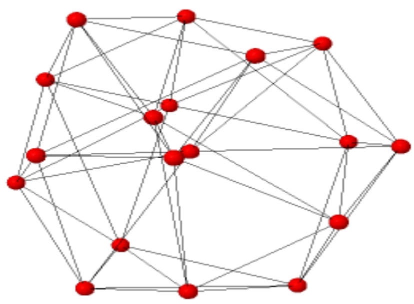

Fig. 10 Line graph of Rook graph $K_3 \Box K_3$

4.2.4. Line graph of Total graph of complete bipartite graph $K_{n,n}$: Consider $G_4 = L(T(K_{n,n}))$ = Line graph of Total graph of complete bipartite graph $K_{n,n}$. The graph parameters of G_4 are given in Table 22.

Table. 22 Line graph of Total graph of complete bipartite graph $K_{n,n}$.

N	D	m	Δ	λ_1
$n^3 + 2n^2$	≤ 3	≤ 7	$4n - 2$	$4n - 2$

The properties of Total graph of complete bipartite graph $K_{n,n}$ are given in Table 23.

Table. 23 Total graph of complete bipartite graph $K_{n,n}$

N^{OG}	Δ	Is Regular?
$n^2 + 2n$	$2 * n$	Yes

The chromatic number of the Line graph of Total graph of complete bipartite graph $K_{n,n}$ is calculated as follows: it can be observed that the edge chromatic number (chromatic index) of Total graph of complete bipartite graph $K_{n,n}$ is $2n+1$ if the number of vertices is odd (from Corollary 2.21), and $2n$ if number of vertices is even (from Theorem 2.20). Also, from Definition 2.2 the edge chromatic number $\chi^{'}(T(K_{n,n})) = \chi(L(T(K_{n,n})))$. The tightness values are given as follows:

$$t_1(G_4) \leq 7(4n-2) \leq 28n - 14 = O(\sqrt[3]{N});$$
$$stt(G_4) \leq 4(4n-2) \leq 16n - 8 = O(\sqrt[3]{N});$$
$$spt(G_4) \leq 7(4n-2) \leq 28n - 14 = O(\sqrt[3]{N});$$
$$t_2(G_4) \leq 4(4n-2) \leq 16n - 8 = O(\sqrt[3]{N}).$$

The new tightness values $t_3(G_4)$ and $t_4(G_4)$ are also given as follows:

$$t_3(G_4) = \mathsf{m}(\chi - 1) \leq 7(2n-1) \leq 14n - 7 = O(\sqrt[3]{N}); \text{ if n is even}$$
$$t_3(G_4) = \mathsf{m}(\chi - 1) \leq 7(2n) \leq 14n = O(\sqrt[3]{N}); \text{ if n is odd}$$
$$t_4(G_4) = (D+1)(\chi - 1) \leq 4(2n-1) \leq 8n - 4 = O(\sqrt[3]{N}); \text{ if n is even}$$
$$t_4(G_4) = (D+1)(\chi - 1) \leq 4(2n) \leq 8n = O(\sqrt[3]{N}); \text{ if n is odd}$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt[3]{N})$, and hence G_4 can be used as a model for MINs. The Line graph of Total graph of complete bipartite graph $K_{2,2}$ is given in Fig. 11.

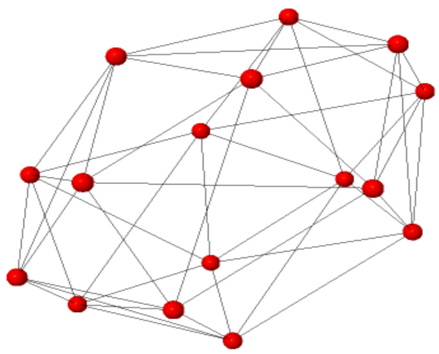

Fig. 11 Line graph of Total graph of complete bipartite graph $K_{2,2}$

4.2.5. Line graph of Total graph of complete graph K_n: Consider $G_5 = L(T(K_n))$ = Line graph of Total graph of complete graph K_n. The graph parameters of G_5 are given in Table 24.

Table. 24 Line graph of Total graph of complete graph K_n.

N	D	m	Δ	λ_1
$\frac{n^3-n}{2}$	≤ 2	≤ 4	$4n-6$	$4n-6$

The properties of Total graph of complete graph K_n are given in Table 25.

Table. 25 Total graph of complete graph K_n

N^{OG}	Δ	Is Regular?
$\frac{n^2+n}{2}$	$2n-2$	Yes

The chromatic number of the Line graph of Total graph of complete graph K_n is calculated as follows: it can be observed that the edge chromatic number (chromatic index) of Total graph of complete graph K_n is $2n-1$ if the number of vertices is odd (from Corollary 2.21), and $2n-2$ if number of vertices is even (from Theorem 2.20). Also, from Definition 2.2 the edge chromatic number $\chi^{'}(T(K_n)) = \chi(L(T(K_n)))$. The tightness values are given as follows:

$$t_1(G_5) \leq 4(4n-6) \leq 16n-24 = O(\sqrt[3]{N});$$
$$stt(G_5) \leq 3(4n-6) \leq 12n-18 = O(\sqrt[3]{N});$$
$$spt(G_5) \leq 4(4n-6) \leq 16n-24 = O(\sqrt[3]{N});$$
$$t_2(G_5) \leq 3(4n-6) \leq 12n-18 = O(\sqrt[3]{N}).$$

The new tightness values $t_3(G_5)$ and $t_4(G_5)$ are also given as follows:

$$t_3(G_5) = \mathsf{m}(\chi-1) \leq 4(2n-3) \leq 8n-12 = O(\sqrt[3]{N}); \text{ if n is even}$$
$$t_3(G_5) = \mathsf{m}(\chi-1) \leq 4(2n-2) \leq 8n-8 = O(\sqrt[3]{N}); \text{ if n is odd}$$

$$t_4(G_5) = (D+1)(\chi-1) \leq 3(2n-3) \leq 6n-9 = O(\sqrt[3]{N}); \text{ if n is even}$$
$$t_4(G_5) = (D+1)(\chi-1) \leq 3(2n-2) \leq 6n-6 = O(\sqrt[3]{N}); \text{ if n is odd}$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt[3]{N})$, and hence G_5 can be used as a model for MINs. Fig. 12 shows the Line graph of Total graph of complete graph K_3.

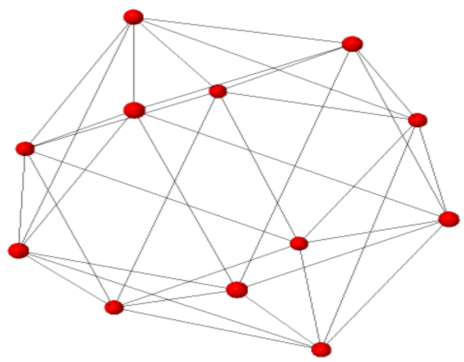

Fig. 12 Line graph of Total graph of complete graph K_3

Example 4.3. The Line graph of Johnson graph $J(n,3)$ belongs to the set $S^{O(\sqrt[4]{N})}$

Consider $G_1 = L(J(n,3)) =$ Line graph of Johnson graph $J(n,3)$, for $n > 5$. The graph parameters of G_1 are given in Table 26.

Table. 26 The Line graph of Johnson graph $J(n,3)$.

N	D	m	Δ	λ_1
$\frac{n^4-6n^3+11n^2-6n}{4}$	≤ 4	≤ 5	$6n-20$	$6n-20$

The properties of Johnson graph $J(n,3)$ are given in Table 27.

Table. 27 Johnson graph $J(n,3)$

N^{OG}	Δ	Is Regular?
$\binom{n}{3}$	$3(n-3)$	Yes

The chromatic number of the Line graph of Johnson graph $J(n,3)$ is calculated as follows: it can be observed that the edge chromatic number (chromatic index) of $J(n,3)$ is $3(n-3)+1$ if the number of vertices is odd (from Corollary 2.21), and $3(n-3)$ if number of vertices is even(from Theorem 2.20). Also, from Definition 2.2 the edge chromatic number $\chi'(J(n,3)) = \chi(L(J(n,3)))$. The tightness values are given as follows:

$$t_1(G_1) \leq 5(6n-20) \leq 30n-100 = O(\sqrt[4]{N});$$
$$stt(G_1) \leq 5(6n-20) \leq 30n-100 = O(\sqrt[4]{N});$$

$$spt(G_1) \leq 5(6n-20) \leq 30n-100 = O(\sqrt[4]{N});$$
$$t_2(G_1) \leq 5(6n-20) \leq 30n-100 = O(\sqrt[4]{N}).$$

The new tightness values $t_3(G_1)$ and $t_4(G_1)$ are also given as follows:

$$t_3(G_1) = \mathsf{m}(\chi-1) \leq 5(3n-10) = O(\sqrt[4]{N}); \text{ if no. of vertices is even}$$
$$t_3(G_1) = \mathsf{m}(\chi-1) \leq 5(3n-9) = O(\sqrt[4]{N}); \text{ if no. of vertices is odd}$$
$$t_4(G_1) = (D+1)(\chi-1) \leq 5(3n-10) = O(\sqrt[4]{N}); \text{ if no. of vertices is even}$$
$$t_4(G_1) = (D+1)(\chi-1) \leq 5(3n-9) = O(\sqrt[4]{N}); \text{ if no. of vertices is odd}$$

The tightness values t_1, stt, spt, t_2, t_3, and t_4 are bounded by $O(\sqrt[4]{N})$, and hence G_1 can be used as a model for MINs. The Line graph of Johnson graph $J(6,3)$ is shown in Fig. 13.

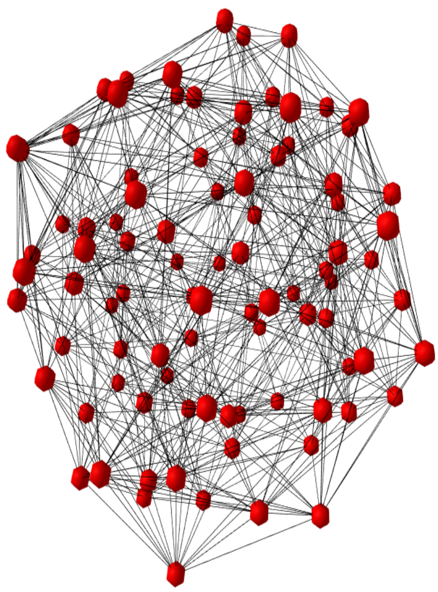

Fig. 13 Line graph of Johnson graph $J(6,3)$

5. CONCLUSION

We have studied some well-known classes of graphs, such as the line graphs of Johnson graphs, Rook graphs, Crown graphs, complete graphs, and complete bipartite graphs. Also, graphs resulting from various graph operations, such as line graphs of the Cartesian product of graphs and line graphs of the Tensor product of graphs, are included in our study. All these graphs turned out to be well-suited interconnection network models because their tightness values range from $O(\sqrt[4]{N})$ to $O(\sqrt{N})$, where N is the number of vertices in the graph under consideration. Based on some theorems from the literature, we noticed that for these graphs it is easier to compute chromatic number than the spectrum, and we proposed the new tightness values $t_3(G)$ and $t_4(G)$ for these and similar graphs. In addition, the resulting values of $t_3(G)$ and $t_4(G)$ are significantly smaller than the corresponding values for $t_2(G)$. As a result, we propose using the tightness values $t_3(G)$ and $t_4(G)$ whenever it seems more appropriate. In the future, we can extend examples by comparing the tightness values of various graphs obtained from different graph operations.

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