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OPTIMALITY CONDITIONS AND DUALITY FOR MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING WITH DATA UNCERTAINTY VIA MORDUKHOVICH SUBDIFFERENTIAL

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Abstract: Based on the notation of Mordukhovich subdifferential in [27], we propose some of new concepts of convexity to establish optimality conditions for quasi $\varepsilon-$solutions for nonlinear semi-infinite optimization problems with data uncertainty in constraints. Moreover, some examples are given to illustrate the obtained results.

Keywords: Robust Optimality Condition, Semi-Infinite Programming, Constraint Qualification, Generalized Convexity, Mordukhovich subdifferential.

MSC: 49K99, 65K10, 90C34, 90C26, 90C46.

1. INTRODUCTION

In this paper, we consider the following semi-infinite optimization problem in the absence of data uncertainty

$$\text{(SIP)} \quad \begin{array}{l} \min f(x), \\ \text{s.t. } g_t(x) \leq 0, \forall t \in T, \end{array}$$

where T is a nonempty infinite index set and $f, g_t : \mathbb{R}^n \to \mathbb{R}, t \in T$ are locally Lipschitz functions.

The semi-infinite optimization problem (SIP) in the face of data uncertainty in the constraints can be captured by the problem

$$\text{(USIP)} \quad \begin{array}{l} \min f(x), \\ \text{s.t. } g_t(x, v_t) \leq 0, \forall t \in T, \end{array}$$

where $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, t \in T$ are locally Lipschitz functions, and for each $t \in T, v_t \in \mathbb{R}^q$ is an uncertain parameter, which belongs to some convex compact set $\mathcal{V}_t \subset \mathbb{R}^q$.

The uncertainty set-valued mapping $\mathcal{V} : T \rightrightarrows \mathbb{R}^q$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$. The notation $v \in \mathcal{V}$ means that v is a selection of $\mathcal{V}$, i.e., $v : T \to \mathbb{R}^q$ and $v_t \in \mathcal{V}_t$ for all $t \in T$. So, the uncertainty set is the graph of $\mathcal{V}$, that is, $\text{gph}\mathcal{V} := \{(t, v_t) \mid v_t \in \mathcal{V}_t, t \in T\}$.

The robust counterpart of (USIP) is as follows:

$$\text{(RSIP)} \quad \begin{array}{l} \min f(x), \\ \text{s.t. } g_t(x, v_t) \leq 0, \forall v_t \in \mathcal{V}_t, \forall t \in T. \end{array}$$

The robust feasible set of (RSIP) is defined by

$$F := \{x \in \Omega \mid g_t(x, v_t) \leq 0, \forall v_t \in \mathcal{V}_t, \forall t \in T\},$$

where $\Omega \subset \mathbb{R}^n$ is locally closed and nonempty (not necessarily convex).

Let $\mathbb{R}^{(T)}$ be the linear space given below

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

Let $\mathbb{R}_+^{(T)}$ be the positive cone in $\mathbb{R}^{(T)}$ defined by

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0 \text{ for all } t \in T\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T.

For $g_t, t \in T$,

$$\sum_{t \in T} \lambda_t g_t = \begin{cases} \sum_{t \in T(\lambda)} \lambda_t g_t, & \text{if } T(\lambda) \neq \emptyset, \\ 0, & \text{if } T(\lambda) = \emptyset. \end{cases}$$

Now, we introduce some concepts of robust approximate minima for (RSIP).

Definition 1. *Let $\varepsilon \geq 0$. A point $\bar{x} \in F$ is said to be*

(i) [14] a quasi $\varepsilon-$solution to problem (RSIP) if

$$f(\bar{x}) \leq f(x) + \sqrt{\varepsilon}||x - \bar{x}||, \forall x \in F.$$

(ii) a quasi weakly $\varepsilon-$solution to problem (RSIP) if

$$f(\bar{x}) < f(x) + \sqrt{\varepsilon}||x - \bar{x}||, \forall x \in F.$$

In recent years, the study of one among more a semi-infinite programming problem (SIP), which is an optimization problem on a feasible set described by an infinite number of inequality constraints, has occupied attention of researches. Many successful treatments of deterministic semi-infinite programming have been investigated from several different perspectives. We refer the readers to the book [10], the survey papers [12, 24], and the references therein.

Semi-infinite programming problems could be applied in various fields such as in engineering design, mathematical physics, robotics, optimal control, transportation problems, fuzzy sets, cooperative games, see [10, 24].

On the other hand, robust optimization has emerged as a remarkable deterministic framework for studying optimization problems with uncertain data [1, 2]. Many researchers have been attracted to work on the real-world application of robust optimization in engineering, business and management. Many interesting results could be found in [3, 7, 8, 11, 21] and the references therein.

Furthermore, since sometimes the exact solutions do not exist while the approximate ones do, even in the convex case [25, 26] and other references therein, the results on optimality conditions and duality theorems for approximate solutions to multiobjective optimization problems and problem SIP have been investigated in [6, 29]. Recently, many researchers have worked on the theory of approximate optimal solutions for various types of uncertain optimization problems, such as uncertain convex optimization problems in [30], uncertain convex multiobjective optimization problems in [32], uncertain nonconvex multiobjective optimization problems in [9] and uncertain convex semidefinite optimization problems in [13]. However, the results on optimality conditions as well as duality for approximate solutions to semi-infinite programming problems under uncertainty have been studied in few papers. More precisely, robust approximate optimality theorems and duality results for an uncertain convex semi-infinite programming problem have been obtained in [22]. By using robust optimization technique, results on necessary and sufficient optimality conditions for robust quasi approximate optimal solution of problem SIP have been established in [31]. Recently, some new characterizations of robust quasi epsilon-solutions for a nonsmooth semi-infinite optimization problems with data uncertainty have been given in [20]. More recently, robust approximate quasi optimal solutions for a class of nonlinear semi-infinite programming with data uncertainty in both the objective and constraints have been studied in [33]. By using techniques of variational analysis, Mordukhovich and Nghia [28] established necessary optimality conditions for SIP via Mordukhovich/limiting subdifferential. Kanzi [18] considered nonsmooth semi-infinite programming with mixed constraints in terms of the Michel–Penot subdifferential. For nonsmooth semi-infinite multiobjective programming (SIMP) with inequality constraints, by using the Clarke subdifferential, Kanzi and Nobakhtian [19] established necessary conditions for weakly efficient solutions . Chuong and Kim [4], and Chuong and Yao [5] proposed a basic constraint qualification in terms of the Mordukhovich/limiting subdifferential and applied it to optimality conditions for nonsmooth SIMP. Jiao *et al.* [14] investigated necessary condition, sufficient condition and type dual model for a quasi $\varepsilon-$solution to a semi-infinite programming problem (SIP) in terms of the Mordukhovich/limiting subdifferential. In [15, 16], Joshi implied sufficient conditions and weak and strong duality theorems for semi-infinite mathematical programming problems with equilibrium constraints by using convexificators. Recently, some concepts of approximate efficient solutions for vector optimization problem have been introduced in [17]. Joshi formulated approximate vector variational inequalities in terms of the Clarke subdifferentials and applied them to characterize an approximate efficient solution of the vector optimization problem. By using the Clarke subdifferential, Khantree and Wangkeeree [20], and Sun *et al.* [33] established approximate optimality conditions and approximate duality theorems for semi-infinite optimization problems with data uncertainty. However, to the best of our knowledge, there is no paper dealing with the Mordukhovich/limiting subdifferential for approximate optimality conditions and approximate duality theorems of semi-infinite optimization problems with data uncertainty.

Inspired by the above observations, we provide some new results for quasi $\varepsilon-$solutions of nonsmooth semi-infinite programming problems with data uncertainty in constraints (RSIP) via Mordukhovich/limiting subdifferential.

The rest of the paper is organized as follows. Section 1, and Section 2 present introduction, notations and preliminaries. In Section 3, we establishe necessary and sufficient conditions for $\varepsilon-$quasi-solution to problem (RSIP). In Section 4, we obtain $\varepsilon-$Mond-Weir type duality of semi-infinite optimization problem under uncertainty in constraints. Finally, conclusions are given in Section 5.

2. PRELIMINARIES

Throughout the paper we use the standard notation of variational analysis in [27]. Let $\mathbb{R}^n$ denote the Euclidean space equipped with the usual Euclidean norm $||.||$, and $\mathbb{R}^n$ for its topological dual, because $(\mathbb{R}^n)^* = \mathbb{R}^n$. The nonnegative (resp., nonpostive) orthant cone of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \mid x_i \geq 0, i = 1, ..., n\}$ (resp., $\mathbb{R}^n_-$). The inner product is defined by $\langle ., . \rangle$. The closed unit ball of $\mathbb{R}^n$ is denoted by $\mathbb{B}$. For a given set $\Omega \subset \mathbb{R}^n$, we use conv$\Omega$ to indicate the convex hull of Ω, and the notation $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$.

A given set-valued mapping $F : \Omega \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be closed at $\bar{x} \in \Omega$ if for any sequence $\{x_k\} \subset \Omega, x_k \to \bar{x}$ and any sequence $\{y_k\} \subset \mathbb{R}^m, y_k \to \bar{y}$, one has $\bar{y} \in F(\bar{x})$.

Given a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with values $F(x) \subset \mathbb{R}^m$ in the collection of all the subsets, we denote by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \{y \in \mathbb{R}^m \mid \exists x_k \to \bar{x}, y_k \to y \text{ with } y_k \in F(x_k), \forall k = 1, 2, ...\}$$

the sequential Painlevé Kuratowski upper limit of F at $\bar{x}$.

Let $\Omega \subset \mathbb{R}^n$ be a nonempty set. Given any $\bar{x} \in \Omega$, we define the Fréchet/regular normal cone to Ω at $\bar{x}$ by

$$N^F(\bar{x}; \Omega) := \left\{ v \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\| x - \bar{x} \|} \leq 0 \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $N^F(\bar{x};\Omega) := \emptyset$.

The Mordukhovich/limiting [27] normal cone $N^M(\bar{x};\Omega)$ to Ω at $\bar{x} \in \Omega \subset \mathbb{R}$ is obtained from Fréchet/regular normal cones by taking the sequential Painlevé Kuratowski upper limit as

$$N^M(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} N^F(\bar{x};\Omega).$$

If $\bar{x} \notin \Omega$, we put $N^M(\bar{x};\Omega) := \emptyset$.

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}} := [-\infty, +\infty]$ be an extended real-valued function. The domain and epigraph of f are given by

$$\text{dom} f := \{x \in \mathbb{R}^n \mid f(x) < \infty\}$$

and

$$\text{epi} f := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \geq f(x)\}.$$

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}} := [-\infty, +\infty]$ be finite at $\bar{x} \in \text{dom} f$, then the Mordukhovich/limiting subdifferential [27] of f at $\bar{x}$ is defined by

$$\partial^M f(\bar{x}) := \left\{v \in \mathbb{R}^n \mid (v,-1) \in N^M\left((\bar{x}, f(\bar{x})); \text{epi} f\right)\right\}.$$

If $|f(\bar{x})| = \infty$, then one puts $\partial^M f(\bar{x}) := \emptyset$.

Given $\Omega \subset \mathbb{R}^n$ and consider the indicator function $\delta(.,\Omega)$ defined by

$$\delta(x;\Omega) := \begin{cases} 0, & \text{if } x \in \Omega, \\ +\infty, & \text{otherwise.} \end{cases}$$

Furthermore, we have a relation between the Mordukhovich/limiting normal cone and the Mordukhovich/limiting subdifferential of the indicator function as follows

$$N^M(\bar{x};\Omega) = \partial^M \delta(\bar{x};\Omega), \forall x \in \Omega.$$

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a given real valued function. We say that f is locally Lipschitz, if for any $x \in \mathbb{R}^n$, there exist a positive constant L and an open neighbourhood $\mathcal{N}(x)$ of x, such that for any $x_1, x_2 \in \mathcal{N}(x)$,

$$||f(x_1) - f(x_2)|| \leq L||x_1 - x_2||.$$

Lemma 2. *[27] Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x}$. If $\bar{x}$ is a local minimizer of f then*

$$0 \in \partial^M(\bar{x}).$$

Lemma 3. *[27] Suppose that $f_i : \mathbb{R}^n \to \bar{\mathbb{R}}, i = 1,2,...,n, n \geq 2$ is lower semicontinuous around $\bar{x} \in \mathbb{R}^n$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then, for any $x \in \mathbb{R}^n$,*

$$\partial^M(f_1 + f_2 + ... + f_n)(\bar{x}) \subset \partial^M f_1(\bar{x}) + \partial^M f_2(\bar{x}) + ... + \partial^M f_n(\bar{x}).$$

3. ROBUST APPROXIMATE OPTIMALITY CONDITIONS

In this section, we establish the necessary and sufficient optimality conditions for approximate solutions to problem(RSIP).

The following constraint qualification is an extension of Definition 3.1 in [14].

Definition 4. *Let $\bar{x} \in F$. We say that the following robust-type Mordukhovich constraint qualification (RMCQ) is satisfied at $\bar{x}$ if*

$$N^M(\bar{x}; F) \subseteq \bigcup_{\substack{\lambda \in A(\bar{x}) \\ v_t \in \mathcal{V}_t}} \left[\sum_{t \in T} \lambda_t \partial_x^M g_t(\bar{x}, v_t) \right] + N^M(\bar{x}; \Omega),$$

where $A(\bar{x}) := \{\lambda \in \mathbb{R}_+^{(T)} \mid \lambda_t g_t(\bar{x}, v_t) = 0, \forall v_t \in \mathcal{V}_t, t \in T\}$ is set of active constraint multipliers at $\bar{x}$.

If $\mathcal{V}_t$ is a singleton, (RMCQ) becomes the constraint qualification (CQ) for (SIP). The qualification conditions of (CQ) type have been introduced and used in [14] and the references therein.

Now, we propose necessary optimality condition for robust approximate solution of (RSIP) under the (RMCQ).

Theorem 5. *Let $\varepsilon \geq 0$ and let $\bar{x} \in F$ be a quasi $\varepsilon-$solution of (RSIP). Suppose that (RMCQ) at $\bar{x}$ holds. Then, there exist $(\bar{\lambda}_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ and $\bar{v}_t \in \mathcal{V}_t, t \in T$, such that*

$$0 \in \partial^M f(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \partial_x^M g_t(\bar{x}, \bar{v}_t) + N^M(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0. \tag{1}$$

Proof. Suppose that $\bar{x} \in F$ is a quasi $\varepsilon-$solution of (RSIP). Then, for any $x \in F$,

$$f(\bar{x}) \leq f(x) + \sqrt{\varepsilon}||x - \bar{x}||. \tag{2}$$

From (2), $\bar{x}$ is a minimizer of the following problem

$$\min_{x \in F} \{f(x) + \sqrt{\varepsilon}||x - \bar{x}||\}.$$

Equivalently, $\bar{x}$ is an optimal solution of the following unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \{f(x) + \sqrt{\varepsilon}||x - \bar{x}|| + \delta_F(x)\}.$$

By Lemma 2, we have

$$0 \in \partial^M \left(f(.) + \sqrt{\varepsilon}||. - \bar{x}|| + \delta_F(.) \right)(\bar{x}). \tag{3}$$

From the fact that $\partial^M||. - \bar{x}|| = \mathbb{B}$ at $\bar{x}$, (3) and from the Lemma 3, we have

$$0 \in \partial^M f(\bar{x}) + N^M(\bar{x}; F) + \sqrt{\varepsilon}\mathbb{B}. \tag{4}$$

By (RMCQ), (4) is equivalent to

$$0 \in \partial^M f(\bar{x}) + \bigcup_{\substack{\lambda \in A(\bar{x}) \\ v_t \in \mathcal{V}_t}} \left[\sum_{t \in T} \lambda_t \partial_x^M g_t(\bar{x}, v_t) \right] + N^M(\bar{x}; \Omega), +\sqrt{\varepsilon}\mathbb{B},$$

where

$$A(\bar{x}) := \{\lambda \in \mathbb{R}_+^{(T)} \mid \lambda_t g_t(\bar{x}, v_t) = 0, \forall v_t \in \mathcal{V}_t, t \in T\}.$$

Then, there exist $\bar{\lambda}_t \in \mathbb{R}_+^{(T)}$ and $\bar{v}_t \in \mathcal{V}_t, t \in T$ such that

$$0 \in \partial^M f(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \partial_x^M g_t(\bar{x}, \bar{v}_t) + N^M(\bar{x}; \Omega), +\sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0.$$

This completes the proof. □

The following simple example shows that the condition (RMCQ) is essential in Theorem 5.

Example 6. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V}_t \to \mathbb{R}$ be defined by*

$$f(x) = 2x \text{ and } g_t(x, v_t) = t v_t x^2,$$

where $\Omega = \mathbb{R}, x \in \mathbb{R}, t \in T = [0, 1]$ and $v_t \in \mathcal{V}_t = [2 - t, 2 + t]$ for any $t \in T$. By simple computation, $F = \{0\}$. Now, take $\bar{x} = 0, \varepsilon = \frac{1}{4}$. Then, it is easy to show that $\bar{x} = 0$ is a quasi $\varepsilon-$solution of (RSIP). Indeed, we have

$$f(x) + \sqrt{\varepsilon}|x - \bar{x}| = 2x + \frac{1}{2}|x| \geq 0 = f(\bar{x}), \forall x \in F.$$

On the other hand, we have $N^M(\bar{x}; \Omega) = N^M(\bar{x}; \mathbb{R}) = \{0\}$ and $\partial_x^M g_t(\bar{x}, v_t) = \{0\}, v_t \in \mathcal{V}_t$, for any $t \in T$, one has

$$\bigcup_{\substack{\lambda \in A(\bar{x}) \\ v_t \in \mathcal{V}_t}} \left[\sum_{t \in T} \lambda_t \partial_x^M g_t(\bar{x}, v_t) \right] + N^M(\bar{x}; \mathbb{R}) = \{0\}.$$

Moreover, $N^M(\bar{x}; F) = \mathbb{R}$. Clearly, the (RMCQ) is not satisfied at $\bar{x}$. On the other hand, take $\varepsilon = \frac{1}{4}$ and $\mathbb{B} = [-1, 1]$. It is easy to see that

$$0 \notin \left[\frac{3}{2}, \frac{5}{2}\right] = \{2\} + \left[-\frac{1}{2}, \frac{1}{2}\right] = \partial^M f(\bar{x}) + \sum_{t \in T} \lambda_t \partial_x^M g_t(\bar{x}, v_t) + N^M(\bar{x}; \mathbb{R}) + \sqrt{\varepsilon}\mathbb{B},$$

for any $\lambda_t \in \mathbb{R}_+^{(T)}$ and $v_t \in \mathcal{V}_t, t \in T$. Then, (1) is not valid. Thus, Theorem 5 is not valid either.

In the special case when $\mathcal{V}_t$ is a singleton, we have the following result in [14].

Corollary 7. *Let $\varepsilon \geq 0$ and let $\bar{x}$ be a quasi $\varepsilon-$solution of (SIP). Suppose that CQ at $\bar{x}$ holds. Then, there exists $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, such that,*

$$0 \in \partial^M f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \partial^M g_t(\bar{x}) + N^M(\bar{x};\Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}) = 0.$$

Now, we introduce a new concept of Karush-Kahn-Tucker (KKT) type conditions for (RSIP).

Definition 8. *Let $\varepsilon \geq 0$ and let F be the robust feasible set of (RSIP). A point $\bar{x} \in F$ is said to satisfy the robust approximate KKT condition on F if there exist $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $\bar{v}_t \in \mathcal{V}_t, t \in T$, which $(\bar{\lambda}_t)_{t\in T}$ are not all zero such that*

$$0 \in \partial^M f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \partial_x^M g_t(\bar{x}, \bar{v}_t) + N^M(\bar{x};\Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0.$$

Remark 9. *In the special case that $\mathcal{V}_t$ is a singleton, the robust approximate KKT conditions with respect to ε for (RSIP) reduces to following the approximate KKT condition with respect to ε for (SIP), i.e.*

$$0 \in \partial^M f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \partial^M g_t(\bar{x}) + N^M(\bar{x};\Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}) = 0.$$

Before we discuss the sufficient conditions for quasi $\varepsilon-$solutions to problem (RSIP), we introduce the concepts of convexity, which is inspired by [23].

Definition 10. *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty subset. A local Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be*

(i) Mordukhovich pseudo-convex at $x \in \Omega$ if for all $y \in \Omega$,

$$\langle \xi, y - x\rangle \geq 0, \exists \xi \in \partial^M f(x) \Rightarrow f(y) \geq f(x).$$

(ii) Mordukhovich quasi-convex at $x \in \Omega$ if for all $y \in \Omega$,

$$f(y) \leq f(x) \Rightarrow \langle \xi, y - x\rangle \leq 0, \forall \xi \in \partial^M f(x).$$

(iii) Mordukhovich $\varepsilon-$quasi-convex at $x \in \Omega$ if for all $y \in \Omega$,

$$f(y) \leq f(x) \Rightarrow \langle \xi, y - x\rangle + \sqrt{\varepsilon}||y - x|| \leq 0, \forall \xi \in \partial^M f(x),$$

for all positive numbers ε.

Definition 11. *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty subset and $\varepsilon \geq 0$. A local Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be*

(i) Mordukhovich $\varepsilon-$pseudo-convex of type I at $x \in \Omega$ if for all $y \in \Omega$,

$$\langle \xi, y - x\rangle + \sqrt{\varepsilon}||y - x|| \geq 0, \exists \xi \in \partial^M f(x) \Rightarrow f(y) + \sqrt{\varepsilon}||y - x|| \geq f(x).$$

(ii) Mordukhovich $\varepsilon-$pseudo-convex of type II at $x \in \Omega$ if for all $y \in \Omega$,

$$\langle \xi, y-x \rangle \geq 0, \exists \xi \in \partial^M f(x) \Rightarrow f(y) + \sqrt{\varepsilon}||y-x|| \geq f(x).$$

Remark 12. *If f is Mordukhovich $\varepsilon-$pseudo-convex of type I at x, then it is also Mordukhovich $\varepsilon-$pseudo-convex of type II at x. But the converse is not true in general.*

Example 13. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} \frac{x}{4}, & \text{if } x \geq 0, \\ x, & \text{if } x < 0. \end{cases}$$

By simple computation, we have

$$\partial^M f(0) = \{\frac{1}{4}, 1\}.$$

It is easy to see that f is $\varepsilon-$pseudo-convex of type II but not $\varepsilon-$pseudo-convex of type I at $x = 0$. We fisrt prove that f is $\varepsilon-$pseudo-convex of type II at $x = 0$. Indeed, take $y = -1, \xi = \frac{1}{4} \in \partial^M f(0) = \{\frac{1}{4}, 1\}$ and $\varepsilon = \frac{1}{4}$. Clearly,

$$f(y) + \sqrt{\varepsilon}|y-x| = -1 + \frac{1}{2} = -\frac{1}{2} \leq 0 = f(x),$$

which implies

$$\langle \xi, y-x \rangle = -\frac{1}{4} \leq 0.$$

We now prove that f is not $\varepsilon-$pseudo-convex of type I at $x = 0$. Indeed, take $y = -1, \xi = \frac{1}{4} \in \partial^M f(0) = \{\frac{1}{4}, 1\}$ and $\varepsilon = \frac{1}{4}$. Clearly,

$$f(y) + \sqrt{\varepsilon}|y-x| = -1 + \frac{1}{2} = -\frac{1}{2} \leq 0 = f(x).$$

However,

$$\langle \xi, y-x \rangle + \sqrt{\varepsilon}|y-x| = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} \geq 0.$$

Next, we can derive the following sufficient condition for a quasi $\varepsilon-$solution of (RSIP).

Theorem 14. *Let $\varepsilon \geq 0$ and Ω be convex set. Assume that $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT condition with respect to ε. If $f(.)$ is Mordukhovich $\varepsilon-$pseudo-convex of type I at $\bar{x}$ and $g_t(., \bar{v}_t), t \in T$ is Mordukhovich quasi-convex at $\bar{x}$, then $\bar{x} \in F$ is a quasi $\varepsilon-$solution of (RSIP).*

Proof. Let $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ be satisfied regarding the robust approximate KKT condition with respect to ε. Therefore, there exist $\xi_0 \in \partial^M f(\bar{x}), \xi_t \in \partial_x^M g(\bar{x}, \bar{v}_t), \forall t \in T$ with $w \in N^M(\bar{x}; \Omega)$ and $b \in \mathbb{B}$, such that

$$\xi_0 + \sum_{t \in T} \bar{\lambda}_t \xi_t + w + \sqrt{\varepsilon} b = 0. \tag{5}$$

Since $b \in \mathbb{B}, w \in N^M(\bar{x}; \Omega)$ and Ω is convex set, it follows that, for any $x \in F$,

$$\langle w, x - \bar{x} \rangle \leq 0, \langle b, x - \bar{x} \rangle \leq ||x - \bar{x}||.$$

From (5), we have

$$\left\langle \xi_0 + \sum_{t \in T} \bar{\lambda}_t \xi_t, x - \bar{x} \right\rangle + \sqrt{\varepsilon} ||x - \bar{x}|| \geq 0,$$

which means that

$$\langle \xi_0, x - \bar{x} \rangle + \sqrt{\varepsilon} ||x - \bar{x}|| \geq - \left\langle \sum_{t \in T} \bar{\lambda}_t \xi_t, x - \bar{x} \right\rangle. \tag{6}$$

Moreover, if $t \in T(\lambda)$, then $g_t(\bar{x}, \bar{v}_t) = 0$. Note that for any $x \in F$, then $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. It follows that $g_t(x, \bar{v}_t) \leq g_t(\bar{x}, \bar{v}_t)$ for any $x \in F$ and $t \in T(\lambda)$. By the Mordukhovich quasi-convexity of $g_t(., \bar{v}_t)$ at $\bar{x}$ and $\xi_t \in \partial_x^M g_t(\bar{x}, \bar{v}_t)$, we obtain

$$\langle \xi_t, x - \bar{x} \rangle \leq 0. \tag{7}$$

Combining (6) and (7), we obtain

$$\langle \xi_0, x - \bar{x} \rangle + \sqrt{\varepsilon} ||x - \bar{x}|| \geq 0.$$

Since $f(., \bar{u})$ is Mordukhovich $\varepsilon-$pseudo-convex of type I at $\bar{x}$, it follows from Definition 11 that

$$f(x) + \sqrt{\varepsilon} ||x - \bar{x}|| \geq f(\bar{x}).$$

Therefore, $\bar{x}$ is a quasi $\varepsilon-$solution of (RSIP). This completes the proof. □

Now, we present an example to show the importance of the Mordukhovich $\varepsilon-$pseudo-convexity of type I in Theorem 14 (function $f(.)$ is given in [27] page 87).

Example 15. *Let $x \in \mathbb{R}, t \in T = [0, 1], \Omega = [0, +\infty)$ and $v_t \in \mathcal{V}_t = [2 - t, 2 + t]$ for any $t \in T$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V}_t \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and

$$g_t(x, v_t) = tx^2 - 2v_t x.$$

Then, $F = [0,2]$ *and* $N^M(\bar{x};\Omega) = N^M(\bar{x};[0,+\infty)) = (-\infty, 0]$. *Let us consider* $\bar{x} = 0, \bar{\lambda}_t = 0$ *and* $\bar{v}_t = 2-t$. *Note that* $f(.)$ *is locally Lipschitz at* $\bar{x}$ *and* $g_t(., \bar{v}_t)$ *is convex at* $\bar{x}$. *We have,*

$$\partial^M f(\bar{x}) = [-1,1] (\text{ see [27] page 87}) \text{ and } \partial_x^M g_t(\bar{x}, \bar{v}_t) = \{2(t-2)\}.$$

We prove that $f(.)$ *is not Mordukhovich* $\varepsilon-$*pseudo-convex of type I at* $\bar{x}$. *Indeed, take* $\bar{y} = \frac{2}{3\pi}, \xi = 0 \in \partial^M f(\bar{x}) = [-1,1]$ *and* $0 \le \sqrt{\varepsilon} \le \frac{2}{3\pi}$. *Clearly,*

$$\langle \xi, \bar{y} - \bar{x}\rangle + \sqrt{\varepsilon}|\bar{y} - \bar{x}| = \sqrt{\varepsilon}|\bar{y} - \bar{x}| \ge 0.$$

However,

$$f(\bar{y}) + \sqrt{\varepsilon}|\bar{y} - \bar{x}| = -\frac{4}{9\pi^2} + \sqrt{\varepsilon}.\frac{2}{3\pi} \le 0 = f(\bar{x}).$$

Now, take an arbitrarily $0 \le \sqrt{\varepsilon} \le \frac{2}{3\pi}$. *Then,* $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ *satisfies the robust approximate KKT conditions with respect to* ε. *Indeed, let us select* $\sqrt{\varepsilon} = \frac{1}{9}, \bar{x} = 0, \bar{\lambda}_t = 0, \bar{v}_t = 2-t$ *and* $\mathbb{B} = [-1,1]$. *Then,*

$$0 \in \left(-\infty, \frac{4}{3}\right] = \partial^M f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \partial_x^M g_t(\bar{x}, \bar{v}_t) + N^M(\bar{x};\mathbb{R}) + \sqrt{\varepsilon}\mathbb{B},$$

and $\bar{\lambda}_t g(\bar{x}, \bar{v}_t) = 0$.

However, $\bar{x} = 0$ *is not a quasi* $\varepsilon-$*solution of (RSIP). In order to see this, let us take* $x = \frac{2}{3\pi} \in F$ *and* $\sqrt{\varepsilon} = \frac{1}{9}$. *Then,*

$$f(x) + \sqrt{\varepsilon}|x - \bar{x}| = -\frac{4}{9\pi^2} + \frac{2}{27\pi} < 0 = f(\bar{x}).$$

In the special case when $\mathcal{V}_t$ is a singleton, we can obtain the following result.

Corollary 16. *Consider problem (SIP). Let* $\varepsilon \ge 0$ *and* Ω *be convex set. Assume that* $(\bar{x}, \bar{\lambda}_t) \in F \times \mathbb{R}_+^{(T)}$ *satisfies approximate KKT condition with respect to* ε. *If* f *is Mordukhovich* $\varepsilon-$*pseudo-convex of type I at* $\bar{x}$ *and* $g_t, t \in T$ *is Mordukhovich quasi-convex at* $\bar{x}$, *then* $\bar{x} \in F$ *is a quasi* $\varepsilon-$*solution of (SIP).*

In the following theorem, we give another sufficient optimality condition for robust $\varepsilon-$quasi-minimum of (RSIP).

Theorem 17. *Let* $\varepsilon \ge 0$ *and* Ω *be convex set. Assume that* $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ *satisfies the robust approximate KKT condition with respect to* ε. *If* $f(.)$ *is Mordukhovich* $\varepsilon-$*pseudo-convex of type II at* $\bar{x}$ *and* $g_t(., \bar{v}_t), t \in T$ *is Mordukhovich* $\varepsilon-$*quasi-convex at* $\bar{x}$, *then* $\bar{x} \in F$ *is a quasi* $\varepsilon-$*solution of (RSIP).*

Proof. Similarly to the proof of Theorem 14, there exist $\xi_0 \in \partial^M f(\bar{x})$, $\xi_t \in \partial_x^M g(\bar{x}, \bar{v}_t), \forall t \in T$ with $w \in N^M(\bar{x}; \Omega)$ and $b \in \mathbb{B}$, such that

$$\langle \xi_0, x - \bar{x} \rangle \geq -\sqrt{\varepsilon}||x - \bar{x}|| - \left\langle \sum_{t \in T} \bar{\lambda}_t \xi_t, x - \bar{x} \right\rangle. \tag{8}$$

On the other hand, if $t \in T(\lambda)$, then $g_t(\bar{x}, \bar{v}_t) = 0$. Note that for any $x \in F$, $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. It follows that $g_t(x, \bar{v}_t) \leq g_t(\bar{x}, \bar{v}_t)$ for any $x \in F$ and $t \in T(\lambda)$. By the Mordukhovich $\varepsilon-$quasi-convexity of $g_t(., \bar{v}_t)$ at $\bar{x}$ and $\xi_t \in \partial_x^M g_t(\bar{x}, \bar{v}_t)$, we obtain

$$\langle \xi_t, x - \bar{x} \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \leq 0. \tag{9}$$

Combining (8) and (9), we obtain

$$\langle \xi_0, x - \bar{x} \rangle \geq 0.$$

Since $f(., \bar{u})$ is Mordukhovich $\varepsilon-$pseudo-convex of type II at $\bar{x}$, it follow from Definition 11 that

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| \geq f(\bar{x}).$$

Therefore, $\bar{x}$ is a quasi $\varepsilon-$solution of (RSIP). This completes the proof. □

Now, we present an example to show the importance of the Mordukhovich $\varepsilon-$pseudo-convexity of type II in Theorem 17.

Example 18. *Let $f, g_t, t \in T, \Omega$ and $\mathcal{V}_t$ be defined as in Example 15. Then, $F = [0, 2]$ and $N^M(\bar{x}; \Omega) = N^M(\bar{x}; [0, +\infty)) = (-\infty, 0]$. Let us consider $\bar{x} = 0, \bar{\lambda}_t = 0$, and $\bar{v}_t = 2 - t$. Note that $f(.)$ is locally Lipschitz at $\bar{x}$ and $g_t(., \bar{v}_t)$ is convex at $\bar{x}$. We have,*

$$\partial^M f(\bar{x}) = [-1, 1] \text{ and } \partial_x^M g_t(\bar{x}, \bar{v}_t) = \{2(t - 2)\}.$$

We prove that $f(., \bar{u})$ is not Mordukhovich $\varepsilon-$pseudo-convex of type II at $\bar{x}$. Indeed, take $\bar{y} = \frac{2}{3\pi}, \xi = 0 \in \partial^M f(\bar{x}) = [-1, 1]$ and $0 \leq \sqrt{\varepsilon} \leq \frac{2}{3\pi}$. Clearly,

$$\langle \xi, \bar{y} - \bar{x} \rangle = 0 \geq 0.$$

However,

$$f(\bar{y}) + \sqrt{\varepsilon}|\bar{y} - \bar{x}| = -\frac{4}{9\pi^2} + \sqrt{\varepsilon}.\frac{2}{3\pi} \leq 0 = f(\bar{x}).$$

Now, take an arbitrarily $0 \leq \sqrt{\varepsilon} \leq \frac{2}{3\pi}$. From Example 15, $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT conditions with respect to ε. By virtue of Example 15, $\bar{x} = 0$ is not a quasi $\varepsilon-$solution of (RSIP).

In the special case when $\mathcal{V}_t$ is a singleton, we can obtain the following result.

Corollary 19. *Consider problem (SIP). Let $\varepsilon \geq 0$ and Ω be convex set. Assume that $(\bar{x}, \bar{\lambda}_t) \in F \times \mathbb{R}_+^{(T)}$ satisfies approximate KKT condition with respect to ε. If f is Mordukhovich $\varepsilon-$pseudo-convex of type II at $\bar{x}$ and $g_t, t \in T$ is Mordukhovich $\varepsilon-$quasi-convex at $\bar{x}$, then $\bar{x} \in F$ is an $\varepsilon-$quasi-minimum of (SIP).*

Motivated by the definition of generalized convexity due to [8, 9] and [20], we introduce a new concept of generalized convexity as follows:

Definition 20. *Let $g_T := (g_t)_{t \in T}, \varepsilon \geq 0$.*

(i) We say that (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized convex on F at $\bar{x}$, if for any $x \in F, \xi_0 \in \partial^M f(\bar{x})$ and $\xi_t \in \partial_x^M g_t(\bar{x}, v_t), v_t \in \mathcal{V}_t, t \in T$, there exists $w \in \mathbb{R}^n$ such that

$$\langle \xi_0, w \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \geq 0 \Rightarrow f(x) + \sqrt{\varepsilon}||x - \bar{x}|| \geq f(\bar{x}),$$

$$g_t(x, v_t) \leq g_t(\bar{x}, v_t) \Rightarrow \langle \xi_t, w \rangle \leq 0, \forall t \in T,$$

and

$$\langle b, w \rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}.$$

(ii) We say that (f, g_T) is Mordukhovich strictly $\varepsilon-$quasi generalized convex on F at $\bar{x}$, if for any $x \in F, \xi_0 \in \partial^M f(\bar{x})$ and $\xi_t \in \partial_x^M g_t(\bar{x}, v_t), v_t \in \mathcal{V}_t, t \in T$, there exists $w \in \mathbb{R}^n$ such that

$$\langle \xi_0, w \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \geq 0 \Rightarrow f(x) + \sqrt{\varepsilon}||x - \bar{x}|| > f(\bar{x}),$$

$$g_t(x, v_t) \leq g_t(\bar{x}, v_t) \Rightarrow \langle \xi_t, w \rangle \leq 0, \forall t \in T,$$

and

$$\langle b, w \rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}.$$

Now, let us provide an example illustrating our Definition 20 (i).

Example 21. *Let $x \in \mathbb{R}, t \in T = [0, 1]$ and $v_t \in \mathcal{V}_t = [-t - 1, -t]$ for any $t \in T, \mathbb{B} = [-1, 1]$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V}_t \to \mathbb{R}$ be defined by*

$$f(x) = |x| + x^3 \text{ and } g_t(x, v_t) = v_t x^2.$$

Then $F = \mathbb{R}$. Let us consider $\bar{x} = 0$, we have $\partial^M f(\bar{x}) = [-1, 1]$ and $\partial_x^M g(\bar{x}, v_t) = \{0\}$. Let us consider $x = -1 \in F = \mathbb{R}, \xi_0 = 0 \in \partial^M f(\bar{x}), \xi_t \in \partial_x^M g(\bar{x}, v_t), 0 \leq \varepsilon \leq 1$, by taking $w = x = -1$, it follows that $w \in \mathbb{R}$,

$$\langle \xi_0, w \rangle + \sqrt{\varepsilon}|x - \bar{x}| = \sqrt{\varepsilon} \geq 0 \Rightarrow f(x) + \sqrt{\varepsilon}|x - \bar{x}| = \sqrt{\varepsilon} \geq 0 = f(\bar{x}),$$

$$g_t(x, v_t) = v_t \leq g_t(\bar{x}, v_t) = 0 \Rightarrow \langle \xi_t, w \rangle = 0 \leq 0, t \in T,$$

and

$$\langle b, w \rangle = -b \leq ||x - \bar{x}|| = 1, \forall b \in [-1, 1].$$

This shows that (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized convex on F at $\bar{x} \in F$.

Next, we give sufficient conditions for a feasible point of problem (RSIP) to be a quasi $\varepsilon-$solution and a quasi weakly $\varepsilon-$solution.

Theorem 22. *Let $\varepsilon \geq 0$. Assume that $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT conditions with respect to ε.*

- *(i) If (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized convex on F at $\bar{x}$, then $\bar{x}$ is a quasi weakly $\varepsilon-$solution of (RSIP).*
- *(ii) If (f, g_T) is Mordukhovich strictly $\varepsilon-$quasi generalized convex on F at $\bar{x}$, then $\bar{x}$ is a quasi $\varepsilon-$solution of (RSIP).*

Proof. Since $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT condition with respect to ε, there exists $\xi_0 \in \partial^M f(\bar{x}), \xi_t \in \partial_x^M g(\bar{x}, \bar{v}_t), \forall t \in T$ with $w \in N^M(\bar{x}; \Omega)$ and $b \in \mathbb{B}$, such that

$$\xi_0 + \sum_{t \in T} \bar{\lambda}_t \xi_t + w + \sqrt{\varepsilon} b = 0, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0.$$

or, equivalent

$$\xi_0 + \sum_{t \in T} \bar{\lambda}_t \xi_t + \sqrt{\varepsilon} b = -w. \tag{10}$$

We first prove (i). Suppose on contrary that $\bar{x}$ is not a quasi weakly $\varepsilon-$solution of (RSIP). It then follows that there exists $x \in F$ satisfying

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| \leq f(\bar{x}). \tag{11}$$

On the other hand, if $t \in T(\lambda)$, then $g_t(\bar{x}, \bar{v}_t) = 0$. Note that for any $x \in F$, then $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. It follows that

$$g_t(x, \bar{v}_t) \leq g_t(\bar{x}, \bar{v}_t), \text{ for any } x \in F \text{ and } t \in T(\lambda). \tag{12}$$

By the Mordukhovich $\varepsilon-$quasi generalized convexity of (f, g_T) on $\mathcal{F}$ at $\bar{x}$ and (11), (12), there exists $d \in \mathbb{R}^n$ such that $(x \neq \bar{x})$

$$\langle \xi_0, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0,$$

$$\langle \xi_t, d \rangle \leq 0, t \in T,$$

and

$$\langle b, d \rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}. \tag{13}$$

Therefore, we have

$$\langle \xi_0, d \rangle + \sum_{t \in T} \bar{\lambda}_t \langle \xi_t, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0.$$

On the other hand, by (13), one has

$$\left\langle \xi_0 + \sum_{t\in T} \bar{\lambda}_t \xi_t + \sqrt{\varepsilon} b, d \right\rangle < 0,$$

which contradicts (10).

We now prove (ii). Suppose on contrary that $\bar{x}$ is not a quasi $\varepsilon-$solution of (RSIP). It then follows that there exists $x \in F$ satisfying

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| < f(\bar{x}). \tag{14}$$

On the other hand, if $t \in T(\lambda)$, then $g_t(\bar{x}, \bar{v}_t) = 0$. Note that for any $x \in F$, then $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. It follows that

$$g_t(x, \bar{v}_t) \leq g_t(\bar{x}, \bar{v}_t), \text{ for any } x \in F \text{ and } t \in T(\lambda). \tag{15}$$

By the Mordukhovich strictly $\varepsilon-$quasi generalized convexity of (f, g_T) on F at $\bar{x}$ and (14), (15), there exists $d \in \mathbb{R}^n$ such that

$$\langle \xi_0, d\rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0,$$

$$\langle \xi_t, d\rangle \leq 0, t \in T,$$

and

$$\langle b, d\rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}. \tag{16}$$

Therefore, we have

$$\langle \xi_0, d\rangle + \sum_{t\in T} \bar{\lambda}_t \langle \xi_t, d\rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0.$$

On the other hand, by (16), one has

$$\left\langle \xi_0 + \sum_{t\in T} \bar{\lambda}_t \xi_t + \sqrt{\varepsilon} b, d \right\rangle < 0,$$

which contradicts (10). This completes the proof. □

4. MOND-WEIR TYPE DUALITY IN ROBUST APPROXIMATE OPTIMIZATION PROBLEM

In this section, we investigate some results for $\varepsilon-$Mond-Weir type robust duality for robust optimization problems under Mordukhovich $\varepsilon-$quasi generalized convexity assumptions.

Now, we consider the Mond–Weir type dual problem (RUD) of (RSIP) as given by

$$
\text{(RUD)} \quad \begin{cases} \max & f(y) \\ \text{s.t.} & 0 \in \partial^M f(y) + \sum_{t\in T} \lambda_t \partial_x^M g_t(y, v_t) + N^M(y;\Omega) + \sqrt{\varepsilon}\mathbb{B}, \\ & \lambda_t g_t(y, v_t) \geq 0, \\ & y \in \Omega, \lambda_t \in \mathbb{R}_+^{(T)}, \varepsilon \geq 0, v_t \in \mathcal{V}_t, t \in T. \end{cases}
$$

The feasible set of (RUD) is defined by

$$
F_{\text{RUD}} = \{(y, \lambda_t, v_t) \in \Omega \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t \mid 0 \in \partial^M f(y) + \sum_{t\in T} \lambda_t \partial_x^M g_t(y, v_t) + N^M(y;\Omega)
$$
$$
+\sqrt{\varepsilon}\mathbb{B}, \lambda_t g_t(y, v_t) \geq 0.\}.
$$

Now, we give the following definition of a robust approximate quasi-solution for (RUD).

Definition 23. *Let $\varepsilon \geq 0$.*

(i) We say that $(\bar{y}, \bar{\lambda}_t, \bar{v}_t) \in F_{\text{RUD}}$ is a quasi $\varepsilon-$solution of (RUD) if for any $(y, \lambda_t, v_t) \in F_{\text{RUD}}$,

$$
f(\bar{y}) + \sqrt{\varepsilon}||y - \bar{y}|| \geq f(y).
$$

(ii) We say that $(\bar{y}, \bar{\lambda}_t, \bar{v}_t) \in F_{\text{RUD}}$ is a quasi weakly $\varepsilon-$solution of (RUD) if for any $(y, \lambda_t, v_t) \in F_{\text{RUD}}$,

$$
f(\bar{y}) + \sqrt{\varepsilon}||y - \bar{y}|| > f(y).
$$

Now, we establish the following approximate weak duality theorem, which holds between (RSIP) and (RUD).

Theorem 24. *Let $\varepsilon \geq 0$ and $x \in F$. Suppose that $(\bar{x}, \bar{\lambda}_t, \bar{v}_T) \in F_{\text{RUD}}$.*

(i) If (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized convex on F at $\bar{x}$, then

$$
f(x) > f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||.
$$

(ii) If (f, g_T) is Mordukhovich strictly $\varepsilon-$quasi generalized convex on F at $\bar{x}$, then

$$
f(x) \geq f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||.
$$

Proof. Since $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F_{RUD}$, we have $\bar{x} \in \Omega, \bar{v}_t \in \mathcal{V}_t, \bar{\lambda}_t \geq 0, t \in T$ and

$$
0 \in \partial^M f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \partial_x^M g_t(\bar{x}, \bar{v}_t) + N^M(\bar{x};\Omega) + \sqrt{\varepsilon}B, \tag{17}
$$

From (17), there exist $\xi_0 \in \partial^M f(x), \xi_t \in \partial_x^M g(x, v_t), \forall t \in T$ with $w \in N^M(x; \Omega)$ and $b \in \mathbb{B}$, such that

$$\xi_0 + \sum_{t \in T} \lambda_t \xi + \sqrt{\varepsilon} b = -w. \tag{18}$$

We first prove (i). Let $x \in F$. Suppose on contrary that

$$f(x) \leq f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||. \tag{19}$$

Note that for any $x \in F$, $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$ and $\bar{\lambda}_t \geq 0, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \geq 0, \bar{v}_t \in \mathcal{V}_t, t \in T$. It follows that

$$g_t(x, \bar{v}_t) \leq 0 \leq g_t(\bar{x}, \bar{v}_t). \tag{20}$$

By the Mordukhovich $\varepsilon-$quasi generalized convexity of (f, g_T) on F at $\bar{x}$ and (19), (20), there exists $d \in \mathbb{R}^n$ such that $(x \neq \bar{x})$

$$\langle \xi_0, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0,$$

$$\langle \xi_t, d \rangle \leq 0, t \in T,$$

$$\langle b, d \rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}.$$

Therefore, we have

$$\langle \xi_0, d \rangle + \sum_{t \in T} \bar{\lambda}_t \langle \xi_t, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0. \tag{21}$$

On the other hand, by (18), one has

$$\langle \xi_0, d \rangle + \sum_{t \in T} \bar{\lambda}_t \langle \xi_t, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| = -\langle w, d \rangle \geq 0,$$

which contradicts (21). Thus,

$$f(x) > f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||.$$

We now prove (ii). Let $x \in F$. Suppose on contrary that

$$f(x) < f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||. \tag{22}$$

Note that for any $x \in F$, $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$ and $\bar{\lambda}_t \geq 0, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \geq 0, \bar{v}_t \in \mathcal{V}_t, t \in T$. It follows that

$$g_t(x, \bar{v}_t) \leq 0 \leq g_t(\bar{x}, \bar{v}_t). \tag{23}$$

By the Mordukhovich strictly $\varepsilon-$quasi generalized convexity of (f, g_T) on F at $\bar{x}$ and (22), (23), there exists $d \in \mathbb{R}^n$ such that

$$\langle \xi_0, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0,$$

$$\langle \xi_t, d\rangle \le 0, t \in T,$$
$$\langle b, d\rangle \le ||x - \bar{x}||, \forall b \in \mathbb{B}.$$

Therefore, we have

$$\langle \xi_0, d\rangle + \sum_{t\in T} \bar{\lambda}_t \langle \xi_t, d\rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0. \tag{24}$$

On the other hand, by (18), one has

$$\langle \xi_0, d\rangle + \sum_{t\in T} \bar{\lambda}_t \langle \xi_t, d\rangle + \sqrt{\varepsilon}||x - \bar{x}|| = -\langle w, d\rangle \ge 0,$$

which contradicts (24). Thus,

$$f(x) \ge f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||.$$

This completes the proof. □

Theorem 25. *Let $\varepsilon \ge 0$. Assume that $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT condition with respect to ε. Let $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F_{\text{RUD}}$ and suppose that the objective values of (RSIP) and (RUD) at this point are equal. In addition,*

- *(i) if (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized convex on F at $\bar{x}$, then $(\bar{x}, \bar{\lambda}_t, \bar{v}_t)$ is a quasi weakly $\varepsilon-$solution of (RUD) and $\bar{x}$ is a quasi weakly $\varepsilon-$solution of (RSIP).*
- *(ii) if (f, g_T) is Mordukhovich strictly $\varepsilon-$quasi generalized convex on F at $\bar{x}$, then $(\bar{x}, \bar{\lambda}_t, \bar{v}_t)$ is a quasi $\varepsilon-$solution of (RUD) and $\bar{x}$ is a quasi $\varepsilon-$solution of (RSIP).*

Proof. We first prove (i). Since $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT condition with respest to ε, there exist $\xi_0 \in \partial^M f(\bar{x}), \xi_t \in \partial_x^M g(\bar{x}, \bar{v}_t), \forall t \in T$ with $w \in N^M(\bar{x}; \Omega)$ and $b \in \mathbb{B}$, such that

$$\xi_0 + \sum_{t\in T} \bar{\lambda}_t \xi_t + w + \sqrt{\varepsilon} b = 0, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0.$$

Note that for any $x \in F$, $g_t(x, \bar{v}_t) \le 0$ for any $t \in T$ and $\bar{\lambda}_t \ge 0, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0, \bar{v}_t \in \mathcal{V}_t, t \in T$. It follows that

$$g_t(x, \bar{v}_t) \le 0 = g_t(\bar{x}, \bar{v}_t).$$

From $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F_{\text{RUD}}$, clearly the objective values of (RSIP) and (RUD) at $(\bar{x}, \bar{u})$ are equal to $f(\bar{x})$. If (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized at $\bar{x}$, by Theorem 24 for all $(x, \lambda_t, v_t) \in F_{\text{RUD}}$ and $\bar{x} \in F$, we obtain

$$f(\bar{x}) > f(x) - \sqrt{\varepsilon}||\bar{x} - x||.$$

Thus, x is a quasi weakly $\varepsilon-$solution of (RSIP). From Definition 23, we have $(\bar{x}, \bar{\lambda}_t, \bar{v}_t)$ is a quasi weakly $\varepsilon-$solution of (RUD).

The proof of (ii) is similar to the one of (i). This completes the proof. □

5. CONCLUSIONS

In this paper, we obtained a necessary optimality condition for a quasi $\varepsilon-$solution to a semi-infinite programming problem with data uncertainty (RSIP). In order to formulate sufficient conditions for quasi $\varepsilon-$solution to a semi-infinite programming problem with data uncertainty (RSIP), we give concepts of Mordukhovich (strictly) $\varepsilon-$quasi generalized convex functions defined in terms of the Mordukhovich subdifferential of locally Lipschitz functions. Moreover, sufficient optimality conditions for such $\varepsilon-$quasi-solution to a semi-infinite programming problem with data uncertainty (RSIP) was proposed in term of Mordukhovich $\varepsilon-$pseudo-convex of type I (and type II) functions of locally Lipschitz functions. In addition, we establish robust $\varepsilon-$ type duality of uncertain optimization problem under new generalized convexity assumptions. Finally, we give some examples to illustrate the obtained results.

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