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SECOND ORDER SYMMETRIC DUALITY IN FRACTIONAL VARIATIONAL PROBLEMS OVER CONE CONSTRAINTS

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Abstract: In the present paper, we introduce a pair of second order fractional symmetric variational programs over cone constraints and derive weak, strong, and converse duality theorems under second order $\mathcal{F}$-convexity assumptions. Moreover, self duality theorem is also discussed. Our results give natural unification and extension of some previously known results in the literature.

Keywords: Variational problem, Second order $\mathcal{F}$-convexity, Second order duality.

MSC: 90C26, 90C29, 90C30, 90C46.

1. INTRODUCTION

Duality results in calculus of variations arise in various fields of engineering science such as mechanics, physics, filtering and optimal control theory. It allows us to associate a dual problem with variational problem and to study the relationship between the two problems. In mechanics, duality allows us to describe precisely the relationship between different energy principles which govern certain nonlinear problems. The primal and the dual problems are two well known forms of the conservation principles characterizing the displacements and the constraints, respectively.

The notion of symmetric duality received several impulse after poineering work of Dorn [7]. Mond and Hanson [15] applied the concept of symmetric duality to variational problems. Kim and Lee [14] formulated a pair of multiobjective nonlinear generalized symmetric dual variational problems involving vector-valued functions which unify the Wolfe and Mond-Weir models and established weak, strong, and converse duality theorems by using the concept of efficiency. Ahmad [1] extended the concept of symmetric duality for multiobjective fractional problems to the class of multiobjective variational problems. When approximations are used, second order duality provides tighter bounds as compared to the first. Motivated with this idea, Husain et al. [12] introduced the concept of second order invexity and generalized invexity to formulate a Mond-Weir type second order dual to a variational problem. Later on, Gulati and Mehndiratta [10] modified the problems given in Husain et al. [12] and obtained necessary optimality conditions and duality relations.

Fractional programming method has received particular attention in the last three decades due to its frequent appearance in diverse fields such as blending problems, minimum risk problems in stochastic programming, macro-economic planning, information theory, mathematical taxonomy etc. Economic applications include maximization of productivity, maximization of return on investment, maximization of return/risk, minimization of cost/time, which often leads to optimization problems whose objective function is a ratio. Gupta and Kailey [9] formulated a pair of second order multiobjective symmetric duality results under K-η-bonvexity assumption. For detailed study on fractional programming, readers are advised to see [18, 19, 20].

Chen [6] considered symmetric dual problem for a class of multiobjective fractional variational problems and duality results are proved through a parametric approach under partial invexity. Ahmad et al. [2] formulated a pair of symmetric fractional variational programming problems over cones and established weak, strong, converse and self duality theorems under pseudoinvexity assumptions. Kailey and Gupta [13] studied symmetric nondifferentiable multiobjective fractional variational problems and derived duality results under generalized (F, α, ρ, d)-convexity assumptions.

In the present paper, we consider fractional symmetric variational programs over cone constraints and establish weak, strong, and converse duality theorems under second order $\mathcal{F}$-convexity assumptions. The construction of the paper is as follows. In Section 2, we recall some basic definitions and results needed in the sequel of the paper. In Section 3, we formulate a pair of second order fractional symmetric variational programs over cone constraints and derive appropriate duality theorems in Section 4. Moreover, we discuss self duality and static symmetric duality in Section 5 and Section 6, respectively. Finally, we conclude our paper in Section 7.

2. NOTATIONS and PRELIMINARIES

The following convention for vector inequalities will be used. If $a,\ b \in \mathbb{R}^n$, then

$$a \geqq b \Rightarrow a_i \geqq b_i,\ i = 1, \ldots, n;$$

$$a \geq b \Rightarrow a \geqq b \text{ and } a \neq b;$$

$$a > b \Rightarrow a_i > b_i,\ i = 1, \ldots, n.$$

Let $I = [a, b]$ be a real interval and X denote the space of n-dimensional piecewise smooth functions $x : I \mapsto \mathbb{R}^n$ with norm

$$\| x \| = \| x \|_\infty + \| Dx \|_\infty,$$

where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_0^t u(s)\, ds,$$

where α is a given boundary value. Therefore, $\frac{d}{dt} = D$ except at discontinuities. Also, suppose that $\Psi(t, x(t), \dot{x}(t)) : I \times X \times X \mapsto \mathbb{R}$ is continuously differentiable function where $\dot{x}(t)$ represents the derivative of $x(t)$. From now onwards, we write x and $\dot{x}$ in place of $x(t)$ and $\dot{x}(t)$, respectively. Let Ψ_x and $\Psi_{\dot{x}}$ denote the gradient vectors of Ψ with respect to x and $\dot{x}$, respectively,
i.e.,

$$\Psi_x = \begin{pmatrix} \frac{\partial \Psi}{\partial x^1} \\ \vdots \\ \frac{\partial \Psi}{\partial x^n} \end{pmatrix}, \quad \Psi_{\dot{x}} = \begin{pmatrix} \frac{\partial \Psi}{\partial \dot{x}^1} \\ \vdots \\ \frac{\partial \Psi}{\partial \dot{x}^n} \end{pmatrix}.$$

In the same way, Ψ_{xx} denotes the Hessian matrix of Ψ with respect to x, which is a symmetric $n \times n$ matrix.

We shall use the following definitions in proving the fundamental results.

Definition 1. *A subset C of $\mathbb{R}^n$ is called cone, if for each $x \in C$ and $\lambda \in \mathbb{R}$, $\lambda \geqq 0$, we have $\lambda x \in C$. Moreover, if C is convex, then it is a convex cone.*

Definition 2. *The polar cone C^* of a cone C is defined by*

$$C^* = \{z : x^T z \leqq 0 \text{ for all } x \in C\}.$$

Definition 3. *A functional $\mathcal{F} : I \times X \times X \times X \times X \times \mathbb{R}^n \to \mathbb{R}$ is sublinear, with respect to its sixth argument, if for all $x, \dot{x}, u, \dot{u} \in X$,* *(i)* $\mathcal{F}(t,x,\dot{x},u,\dot{u};\theta_1+\theta_2) \leqq \mathcal{F}(t,x,\dot{x},u,\dot{u};\theta_1)+\mathcal{F}(t,x,\dot{x},u,\dot{u};\theta_2)$, *for any* $\theta_1,\ \theta_2 \in \mathbb{R}^n$,

(ii) $\mathcal{F}(t,x,\dot{x},u,\dot{u};a\theta) = a\mathcal{F}(t,x,\dot{x},u,\dot{u};\theta)$, *for any* $a \geqq 0$ *and* $\theta \in \mathbb{R}^n$.

From (ii), *it is clear that* $\mathcal{F}(t,x,\dot{x},u,\dot{u};0) = 0$. *For notational convenience, we write* $\mathcal{F}(t,x,\dot{x},u,\dot{u};\theta) = \mathcal{F}(t,x,u;\theta)$.

Now, we introduce the definition of second order $\mathcal{F}$-convex function.

Definition 4. *The functional* $\int_a^b \Psi(t,x,\dot{x})\,dt$ *is said to be second order* $\mathcal{F}$*-convex at* u *if*

$$\int_a^b \Psi(t,x,\dot{x})\,dt - \int_a^b \Psi(t,u,\dot{u})\,dt + \frac{1}{2}\int_a^b q(t)^T M q(t)\,dt$$
$$\geqq \int_a^b \mathcal{F}(t,x,u;\Psi_x(t,u,\dot{u}) - D\Psi_{\dot{x}}(t,u,\dot{u}) + Mq(t))\,dt$$

for all $x \in X,\ q(t) \in R^n,\ t \in I$ *and for some arbitrary sublinear functional* $\mathcal{F}$. *Here it is to be noted that* $M = \Psi_{xx}(t,u,\dot{u}) - 2D\Psi_{x\dot{x}}(t,u,\dot{u}) + D^2\Psi_{\dot{x}\dot{x}}(t,u,\dot{u}) - D^3\Psi_{\dot{x}\ddot{x}}(t,u,\dot{u})$ *and* T *denotes the transpose of a matrix.*

Remark 5. *(i) If* $\mathcal{F}(t,x,u;a) = \eta(t,x,u)^T a$, *then the above definition reduce to second order invex with respect to* η *given in Husain et al. [12].*

(ii) In addition to (i) above, if $M(t,x,\dot{x}) = 0$, *then we obtain the definition of invexity discussed in Mond et al. [16].*

Now, we give an example to show the existence of second-order $\mathcal{F}$-convex which is neither second-order invex nor invex with respect to the same η.

Example 6. *Let* $I = [0,1]$ *and* X *be the space of piecewise smooth function* $x : I \mapsto [0,1]$. *Consider the functional* $\Psi : I \times X \times X \mapsto \mathbb{R}$, *defined by*

$$\Psi(t,x,\dot{x}) = x(t) + x^2(t) - x^3(t) - \cos(x(t)) + \arctan(x(t)).$$

Suppose $\mathcal{F} : I \times X \times X \times X \times X \times \mathbb{R} \mapsto \mathbb{R}$ *is given by*

$$\mathcal{F}(t,x,\dot{x},u,\dot{u};a) = a(u(t) - x(t)).$$

Then the functional $\int_0^1 \Psi(t,x,\dot{x})dt$ *is second-order* $\mathcal{F}$*-convex at* $u(t) = 0$, *for all* $x(t) \in X,\ q(t) \in \mathbb{R}$.

Explanation:

$$\int_0^1 \Psi(t,x,\dot{x})dt - \int_0^1 \Psi(t,u,\dot{u})\,dt + \frac{1}{2}\int_0^1 q(t)^T M q(t)\,dt$$
$$= \int_0^1 \{x(t) + x^2(t) - x^3(t) - \cos(x(t)) + \arctan(x(t)) + 1 + \frac{3}{2}q^2(t)\}\,dt$$

and

$$\int_0^1 \mathcal{F}(t,x,u;\Psi_x(t,u,\dot{u}) - D\Psi_{\dot{x}}(t,u,\dot{u}) + Mq(t))\,dt$$
$$= \int_0^1 F(t,x,u;2+3q(t))\,dt$$
$$= -\int_0^1 (2+3q(t))x(t)\,dt$$

Therefore, we have

$$\int_0^1 \Psi(t,x,\dot{x})dt - \int_0^1 \Psi(t,u,\dot{u})\,dt + \frac{1}{2}\int_0^1 q(t)^T M q(t)\,dt$$
$$- \int_0^1 \mathcal{F}(t,x,u;\Psi_x(t,u,\dot{u}) - D\Psi_{\dot{x}}(t,u,\dot{u}) + Mq(t))\,dt$$
$$= \int_0^1 [x(t) + x^2(t) - x^3(t) - \cos(x(t)) + \arctan x(t) + 1$$
$$+ \frac{3}{2}q^2(t) + (2+3q(t))x(t)]dt \geqq 0,\ \forall x(t) \in X,\ q(t) \in \mathbb{R}.$$

Consequently, we conclude that $\int_0^1 \Psi(t,x,\dot{x})dt$ *is second-order* $\mathcal{F}$*-convex at* $u(t) = 0$.

But, if we define $\eta(t,x,u) = 2(x(t) - u(t))$*, then*

$$\int_0^1 \eta(t,x,u)(\Psi_x(t,u,\dot{u}) - D\Psi_{\dot{x}}(t,u,\dot{u}) + Mq(t))\,dt = \int_0^1 2x(t)(2+3q(t))\,dt.$$

Moreover,

$$\int_0^1 \Psi(t,x,\dot{x})\,dt - \int_0^1 \Psi(t,u,\dot{u})\,dt$$
$$= \int_0^1 \{x(t) + x^2(t) - x^3(t) - \cos(x(t)) + \arctan(x(t)) + 1\}\,dt$$

and

$$\int_0^1 \eta(t,x,u)(\Psi_x(t,u,\dot{u}) - D\Psi_{\dot{x}}(t,u,\dot{u}))\,dt = \int_0^1 4x(t)\,dt.$$

From the above results, it follows that $\int_0^1 \Psi(t,x,\dot{x})\,dt$ *is neither second-order invex nor invex with respect to the same* η*, for all* $x(t) \in X,\ q(t) \in \mathbb{R}$.

Let C_1 and C_2 be closed convex cones with nonempty interiors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

3. SECOND ORDER VARIATIONAL FRACTIONAL SYMMETRIC DUALITY

In this section, we introduce the following second order variational fractional symmetric dual programs over cone constraints.

Primal Problem:

$$\text{(PP)} \qquad \text{Minimize} \quad \frac{\int_a^b (f(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T A p(t))\, dt}{\int_a^b (g(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T B p(t))\, dt}$$

subject to

$$\begin{aligned}
&x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),\\
&y(a) = 0 = y(b), \quad \dot{y}(a) = 0 = \dot{y}(b),
\end{aligned}$$

$$\begin{aligned}
&\left(\int_a^b (g(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T B p(t))\, dt\right)(f_y - Df_{\dot{y}} + Ap(t))\\
&\quad - \left(\int_a^b (f(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T A p(t))\, dt\right)(g_y - Dg_{\dot{y}} + Bp(t)) \in C_2^*, \ t \in I,\\
&y^T\left[\left(\int_a^b (g(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T B p(t))\, dt\right)(f_y - Df_{\dot{y}} + Ap(t))\right.\\
&\quad \left.- \left(\int_a^b (f(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T A p(t))\, dt\right)(g_y - Dg_{\dot{y}} + Bp(t))\right] \geqq 0, \ t \in I,
\end{aligned}$$

$$x \in C_1.$$

Dual Problem:

$$\text{(DP)} \qquad \text{Maximize} \quad \frac{\int_a^b (f(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Y q(t))\, dt}{\int_a^b (g(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Z q(t))\, dt}$$

subject to

$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b),$$
$$v(a) = 0 = v(b), \quad \dot{v}(a) = 0 = \dot{v}(b),$$

$$-\left[\left(\int_a^b (g(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Zq(t))\,dt\right)(f_x - Df_{\dot{x}} + Yq(t))\right.$$
$$\left. - \left(\int_a^b (f(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Yq(t))\,dt\right)(g_x - Dg_{\dot{x}} + Zq(t))\right] \in C_1^*,\ t \in I,$$

$$u^T\left[\left(\int_a^b (g(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Zq(t))\,dt\right)(f_x - Df_{\dot{x}} + Yq(t))\right.$$
$$\left. - \left(\int_a^b (f(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}p(t)^T Yp(t))\,dt\right)(g_x - Dg_{\dot{x}} + Zq(t))\right] \leqq 0,\ t \in I,$$

$$v \in C_2,$$

where
(i) $f : I \times C_1 \times C_1 \times C_2 \times C_2 \to \mathbb{R}_+$, and $g : I \times C_1 \times C_1 \times C_2 \times C_2 \to \mathbb{R}_+ \setminus \{0\}$,
(ii) $A(t,x,\dot{x},y,\dot{y}) = f_{yy}(t,x,\dot{x},y,\dot{y}) - 2Df_{y\dot{y}}(t,x,\dot{x},y,\dot{y})$
$\qquad + D^2 f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y}) - D^3 f_{\dot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y}),\ t \in I,$
(iii) $B(t,x,\dot{x},y,\dot{y}) = g_{yy}(t,x,\dot{x},y,\dot{y}) - 2Dg_{y\dot{y}}(t,x,\dot{x},y,\dot{y})$
$\qquad + D^2 g_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y}) - D^3 g_{\dot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y}),\ t \in I,$
(iv) $Y(t,u,\dot{u},v,\dot{v}) = f_{xx}(t,u,\dot{u},v,\dot{v}) - 2Df_{x\dot{x}}(t,u,\dot{u},v,\dot{v})$
$\qquad + D^2 f_{\dot{x}\dot{x}}(t,u,\dot{u},v,\dot{v}) - D^3 f_{\dot{x}\ddot{x}}(t,u,\dot{u},v,\dot{v}),\ t \in I,$
(v) $Z(t,u,\dot{u},v,\dot{v}) = g_{xx}(t,u,\dot{u},v,\dot{v}) - 2Dg_{x\dot{x}}(t,u,\dot{u},v,\dot{v})$
$\qquad + D^2 g_{\dot{x}\dot{x}}(t,u,\dot{u},v,\dot{v}) - D^3 g_{\dot{x}\ddot{x}}(t,u,\dot{u},v,\dot{v}),\ t \in I,$
(vi) $p : I \to \mathbb{R}^m,\ q : I \to \mathbb{R}^n,$
It is convenient to parametrize the problems (PP) and (DP) in the sense of Dinkelbach [8] by choosing

$$l = \frac{\int_a^b (f(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Ap(t))\,dt}{\int_a^b (g(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Bp(t))\,dt},$$

$$m = \frac{\int_a^b (f(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Yq(t))\,dt}{\int_a^b (g(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Zq(t))\,dt}.$$

Therefore, the primal problem (PP) and the dual problem (DP) can be written as follows:
Primal Problem:

(PP') Minimize l

subject to

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),$$
$$y(a) = 0 = y(b), \quad \dot{y}(a) = 0 = \dot{y}(b),$$

$$\int_a^b (f(t, x, \dot{x}, y, \dot{y}) - \frac{1}{2} p(t)^T A p(t))\, dt$$
$$- l \int_a^b (g(t, x, \dot{x}, y, \dot{y}) - \frac{1}{2} p(t)^T B p(t))\, dt = 0, \tag{1}$$
$$(f_y - Df_{\dot{y}} + Ap(t)) - l(g_y - Dg_{\dot{y}} + Bp(t)) \in C_2^*, \ t \in I, \tag{2}$$
$$y^T \{(f_y - Df_{\dot{y}} + Ap(t)) - l(g_y - Dg_{\dot{y}} + Bp(t))\} \geqq 0, \ t \in I, \tag{3}$$
$$x \in C_1.$$

Dual Problem:

$(\mathrm{DP}^{'})$ Maximize m

subject to

$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b),$$
$$v(a) = 0 = v(b), \quad \dot{v}(a) = 0 = \dot{v}(b),$$

$$\int_a^b (f(t, u, \dot{u}, v, \dot{v}) - \frac{1}{2} q(t)^T Y q(t))\, dt$$
$$- m \int_a^b (g(t, u, \dot{u}, v, \dot{v}) - \frac{1}{2} q(t)^T Z q(t))\, dt = 0, \tag{4}$$
$$- [(f_x - Df_{\dot{x}} + Yq(t)) - m(g_x - Dg_{\dot{x}} + Zq(t))] \in C_1^*, \ t \in I, \tag{5}$$
$$u^T \{(f_x - Df_{\dot{x}} + Yq(t)) - m(g_x - Dg_{\dot{x}} + Zq(t))\} \leqq 0, \ t \in I, \tag{6}$$
$$v \in C_2.$$

Remark 7. *If $A = B = Y = Z = 0$, then the problems* (PP) *and* (DP) *reduce to the problems* (FP) *and* (FD) *discussed in Ahmad et al. [2].*

In the subsequent analysis, we consider $\mathcal{F} : I \times C_1 \times C_1 \times C_1 \times C_1 \times \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathcal{G} : I \times C_2 \times C_2 \times C_2 \times C_2 \times \mathbb{R}^n \mapsto \mathbb{R}$ are sublinear functionals.

4. DUALITY THEOREMS

In this section, we establish weak, strong, and converse duality theorems for the problems, (PP′) and (DP′), which are equally applicable to (PP) and (DP), respectively.

Theorem 8. *(Weak duality). Let (x, y, l, p) and (u, v, m, q) be feasible solutions to primal problem* $(\mathrm{PP}^{'})$ *and dual problem* $(\mathrm{DP}^{'})$*, respectively. Further, assume that* *(a)* $\int_a^b (f(t,.,.,v,\dot v) - mg(t,.,.,v,\dot v))\,dt$ *is second order* $\mathcal{F}$*-convex at* u *for fixed* v,

(b) $-\int_a^b (f(t,x,\dot x,.,.) - lg(t,x,\dot x,.,.))\,dt$ *is second order* $\mathcal{G}$*-convex at* y *for fixed* x,

(c) $\mathcal{F}(t,x,u;\xi) + u^T\xi \geqq 0,\ \forall x,u \in C_1,\ \ -\xi \in C_1^*,\ t \in I,$ *and*

(d) $\mathcal{G}(t,v,y;\zeta) + y^T\zeta \geqq 0,\ \forall v,y \in C_2,\ \ -\zeta \in C_2^*,\ t \in I.$

Then $l \geqq m$.

Proof. From the assumption (c) and constraint (5), we have

$$\mathcal{F}(t,x,u;(f_x - Df_{\dot x} + Yq(t)) - m(g_x - Dg_{\dot x} + Zq(t))) \\ + u^T\{(f_x - Df_{\dot x} + Yq(t)) - m(g_x - Dg_{\dot x} + Zq(t))\} \geqq 0,$$

which by the virtue of (6) becomes

$$\mathcal{F}(t,x,u;(f_x - Df_{\dot x} + Yq(t)) - m(g_x - Dg_{\dot x} + Zq(t))) \geqq 0.$$

Since $\int_a^b f(t,.,.,v,\dot v) - mg(t,.,.,v,\dot v)\,dt$ is second order $\mathcal{F}$-convex at u for fixed v, we have

$$\int_a^b (f(t,x,\dot x,v,\dot v) - f(t,u,\dot u,v,\dot v) + \frac{1}{2}q(t)^TYq(t) \\ - m(g(t,x,\dot x,v,\dot v) - g(t,u,\dot u,v,\dot v) + \frac{1}{2}q(t)^TZq(t)))\,dt \geqq 0.$$

The above inequality by virtue of relation (4) yields

$$\int_a^b (f(t,x,\dot x,v,\dot v) - mg(t,x,\dot x,v,\dot v))\,dt \geqq 0. \tag{7}$$

From the assumption (d) and constraint (2), we have

$$\mathcal{G}(t,v,y;-\{(f_y - Df_{\dot y} + Ap(t)) - l(g_y - Dg_{\dot y} + Bp(t))\}) \\ - y^T\{(f_y - Df_{\dot y} + Ap(t)) - l(g_y - Dg_{\dot y} + Bp(t))\} \geqq 0,$$

which by the virtue of (3) becomes

$$\mathcal{G}(t,v,y;-\{(f_y - Df_{\dot y} + Ap(t)) - l(g_y - Dg_{\dot y} + Bp(t))\}) \geqq 0,$$

Since $-\int_a^b f(t,x,\dot x,.,.) - lg(t,x,\dot x,.,.)\,dt$ is second order $\mathcal{G}$-convex at y for fixed x, we have

$$\int_a^b (f(t,x,\dot x,y,\dot y) - f(t,x,\dot x,v,\dot v) + \frac{1}{2}p(t)^TAp(t) \\ - l(g(t,x,\dot x,y,\dot y) - g(t,x,\dot x,v,\dot v) + \frac{1}{2}p(t)^TBp(t)))\,dt \geqq 0.$$

The above inequality by virtue of relation (1) yields

$$\int_a^b (lg(t,x,\dot{x},v,\dot{v}) - f(t,x,\dot{x},v,\dot{v}))\,dt \geqq 0. \tag{8}$$

On adding (7) and (8), we get

$$\int_a^b (l-m)g(t,x,\dot{x},v,\dot{v})\,dt \geqq 0,$$

which implies

$$l \geqq m.$$

This completes the proof. □

Theorem 9. *(Strong Duality). Let us assume that*

(i) $(\bar{x},\bar{y},\bar{l},\bar{p}(t))$ *be an optimal solution of (PP′),*

(ii) *the matrices* $A - \bar{l}B$ *be nonsingular,*

(iii) $f_y - \bar{l}g_y - D(f_{\dot{y}} - \bar{l}g_{\dot{y}}) + (A - \bar{l}B)\bar{p}(t) \neq 0$, *and*

(iv) *the matrix*

$$\Big((A\bar{p}(t))_y - \bar{l}(B\bar{p}(t))_y - D(A\bar{p}(t))_{\dot{y}} + \bar{l}D(B\bar{p}(t))_{\dot{y}} + D^2(A\bar{p}(t))_{\ddot{y}} - \bar{l}D^2(B\bar{p}(t))_{\ddot{y}}$$
$$-D^3(A\bar{p}(t))_{\dddot{y}} + \bar{l}D^3(B\bar{p}(t))_{\dddot{y}} + D^4(A\bar{p}(t))_{\ddddot{y}} - \bar{l}D^4(B\bar{p}(t))_{\ddddot{y}}\Big)$$

be positive or negative definite.

Then $(\bar{x},\bar{y},\bar{l},\bar{p}(t)=0)$ *is a solution of* (DP′). *If, in addition, the conditions of Theorem 8 are satisfied, then* $(\bar{x},\bar{y},\bar{l},\bar{p}(t)=0)$ *is an optimal solution of* (DP′).

Proof. Since $(\bar{x},\bar{y},\bar{l},\bar{p}(t))$ is an optimal solution of (PP′), there exist $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \in C_2$, and $\xi \in \mathbb{R}$ such that the following Fritz John conditions are satisfied at $(\bar{x},\bar{y},\bar{l},\bar{p}(t))$:

$$\Big[\beta\Big(f_x - \bar{l}g_x - D(f_{\dot{x}} - \bar{l}g_{\dot{x}}) - \frac{1}{2}(\bar{p}(t)^T A\bar{p}(t))_x + \frac{\bar{l}}{2}(\bar{p}(t)^T B\bar{p}(t))_x + \frac{1}{2}D(\bar{p}(t)^T A\bar{p}(t))_{\dot{x}}$$
$$-\frac{\bar{l}}{2}D(\bar{p}(t)^T B\bar{p}(t))_{\dot{x}} - \frac{1}{2}D^2(\bar{p}(t)^T A\bar{p}(t))_{\ddot{x}} + \frac{\bar{l}}{2}D^2(\bar{p}(t)^T B\bar{p}(t))_{\ddot{x}}$$
$$+\frac{1}{2}D^3(\bar{p}(t)^T A\bar{p}(t))_{\dddot{x}} - \frac{\bar{l}}{2}D^3(\bar{p}(t)^T B\bar{p}(t))_{\dddot{x}} - \frac{1}{2}D^4(\bar{p}(t)^T A\bar{p}(t))_{\ddddot{x}}$$
$$+\frac{\bar{l}}{2}D^4(\bar{p}(t)^T B\bar{p}(t))_{\ddddot{x}}\Big) + (\gamma - \xi\bar{y})\Big(f_{yx} - \bar{l}g_{yx} - D(f_{y\dot{x}} - \bar{l}g_{y\dot{x}})$$
$$-D(f_{\dot{y}x} - \bar{l}g_{\dot{y}x}) + D^2(f_{\dot{y}\dot{x}} - \bar{l}g_{\dot{y}\dot{x}}) - D^3(f_{\dot{y}\ddot{x}} - \bar{l}^3 g_{\dot{y}\ddot{x}}) + (A\bar{p}(t))_x$$

$$-\bar{l}(B\bar{p}(t))_x - D((A\bar{p}(t))_{\dot{x}} - \bar{l}(B\bar{p}(t))_{\dot{x}}) + D^2((A\bar{p}(t))_{\ddot{x}} - \bar{l}(B\bar{p}(t))_{\ddot{x}})$$
$$-D^3((A\bar{p}(t))_{\dddot{x}} - \bar{l}(B\bar{p}(t))_{\dddot{x}}) + D^4((A\bar{p}(t))_{\ddddot{x}} - \bar{l}(B\bar{p}^i(t))_{\ddddot{x}}\Big)\Big]$$
$$(x-\bar{x}) \geqq 0, \ \forall\, x \in C_1, \ t \in I, \tag{9}$$

$$\beta\Big(f_y - \bar{l}g_y - D(f_{\dot{y}} - \bar{l}g_{\dot{y}}) - \frac{1}{2}(\bar{p}(t)^T A)_y + \frac{\bar{l}}{2}(\bar{p}(t)^T B)_y + \frac{1}{2}D(\bar{p}(t)^T A\bar{p}(t))_{\dot{y}}$$
$$-\frac{\bar{l}}{2}D(\bar{p}^T(t)B\bar{p}(t))_{\dot{y}} - \frac{1}{2}D^2(\bar{p}(t)^T A\bar{p}(t))_{\ddot{y}} + \frac{\bar{l}}{2}D^2(\bar{p}(t)^T B\bar{p}(t))_{\ddot{y}}$$
$$+\frac{1}{2}D^3(\bar{p}(t)^T A\bar{p}(t))_{\dddot{y}} - \frac{\bar{l}}{2}D^3(\bar{p}(t)^T B\bar{p}(t))_{\dddot{y}} - \frac{1}{2}D^4(\bar{p}(t)^T A\bar{p}(t))_{\ddddot{y}}$$
$$+\frac{\bar{l}}{2}D^4(\bar{p}(t)^T B\bar{p}(t))_{\ddddot{y}}\Big) + (\gamma - \xi\bar{y})\Big(A - \bar{l}B + (A\bar{p}(t))_y - \bar{l}(B\bar{p}(t))_y$$
$$-D(A\bar{p}(t))_{\dot{y}} + \bar{l}D(B\bar{p}(t))_{\dot{y}} + D^2(A\bar{p}(t))_{\ddot{y}} + \bar{l}D(B\bar{p}(t))_{\ddot{y}} - D^3(A\bar{p}(t))_{\dddot{y}}$$
$$+\bar{l}D(B\bar{p}(t))_{\dddot{y}} + D^4(A\bar{p}(t))_{\ddddot{y}} - \bar{l}D(B\bar{p}(t))_{\ddddot{y}}\Big) - \xi(f_y - \bar{l}g_y$$
$$-D(f_{\dot{y}} - \bar{l}g_{\dot{y}}) - A\bar{p}(t) + \bar{l}B\bar{p}(t)) = 0, \ t \in I, \tag{10}$$

$$\alpha - \beta(g - \frac{\bar{l}}{2}\bar{p}^T(t)Bp(t)) + (\gamma - \xi\bar{y})(-g_y + Dg_{\dot{y}} - B\bar{p}(t)) = 0, \ t \in I, \tag{11}$$

$$-\beta(A\bar{p}(t) - \bar{l}B\bar{p}(t)) + (\gamma - \xi\bar{y})(A - \bar{l}B) = 0, \ t \in I, \tag{12}$$

$$\gamma(f_y - \bar{l}g_y - D(f_{\dot{y}} - \bar{l}g_{\dot{y}}) + Ap(t) - \bar{l}Bp(t)) = 0, \ t \in I, \tag{13}$$

$$\xi\bar{y}(f_y - \bar{l}g_y - D(f_{\dot{y}} - \bar{l}g_{\dot{y}}) + Ap(t) - \bar{l}Bp(t)) = 0, \ t \in I, \tag{14}$$

$$(\alpha, \beta, \gamma, \xi) \neq 0, \ t \in I, \tag{15}$$

$$(\alpha, \beta, \gamma, \xi) \geqq 0, \ t \in I. \tag{16}$$

Since $A - \bar{l}B$ is nonsingular, from (12) we get

$$\gamma - \xi\bar{y} = \beta\bar{p}(t). \tag{17}$$

On rearranging (10), we obtain

$$(\beta - \xi)(f_y - \bar{l}g_y - D(f_{\dot{y}} - \bar{l}g_{\dot{y}})) + (A - \bar{l}B)(\gamma - \xi\bar{y} - \xi\bar{p}(t))$$
$$+\Big((A\bar{p}(t))_y - \bar{l}(B\bar{p}(t))_y - D(A\bar{p}(t))_{\dot{y}} + \bar{l}D(B\bar{p}(t))_{\dot{y}}$$
$$+D^2(A\bar{p}(t))_{\ddot{y}} - \bar{l}D^2(B\bar{p}(t))_{\ddot{y}} - D^3(A\bar{p}(t))_{\dddot{y}}$$
$$+\bar{l}D^3(B\bar{p}(t))_{\dddot{y}} + D^4(A\bar{p}(t))_{\ddddot{y}} - \bar{l}D^4(B\bar{p}(t))_{\ddddot{y}}\Big)$$
$$(\gamma - \xi\bar{y} - \frac{1}{2}\beta\bar{p}(t)) = 0.$$

The above equation in view of (17) can be written as

$$(\beta-\xi)(f_y-\bar{l}g_y-D(f_{\dot{y}}-\bar{l}g_{\dot{y}})+(A-\bar{l}B)\bar{p}(t))+\frac{1}{2}(\gamma-\xi\bar{y}(t))((A\bar{p}(t))_y-\bar{l}(B\bar{p}(t))_y$$
$$-D(A\bar{p}(t))_{\dot{y}}+\bar{l}D(B\bar{p}(t))_{\dot{y}}+D^2(A\bar{p}(t))_{\ddot{y}}-\bar{l}D^2(B\bar{p}(t))_{\ddot{y}}-D^3(A\bar{p}(t))_{\dddot{y}}$$
$$+\bar{l}D^3(B\bar{p}(t))_{\dddot{y}}+D^4(A\bar{p}(t))_{\ddddot{y}}-\bar{l}D^4(B\bar{p}(t))_{\ddddot{y}})=0. \tag{18}$$

Multiplying (18) by $\gamma-\xi\bar{y}$ and using (13) and (14), the above relation yields

$$\frac{1}{2}(\gamma-\xi\bar{y}(t))\Big((A\bar{p}(t)_y-\bar{l}(B\bar{p}(t))_y-D(A\bar{p}(t))_{\dot{y}}+\bar{l}D(B\bar{p}(t))_{\dot{y}}$$
$$+D^2(A\bar{p}(t))_{\ddot{y}}-\bar{l}D^2(B\bar{p}(t))_{\ddot{y}}-D^3(A\bar{p}(t))_{\dddot{y}}+\bar{l}D^3(B\bar{p}(t))_{\dddot{y}}$$
$$+D^4(A\bar{p}(t))_{\ddddot{y}}-\bar{l}D^4(B\bar{p}(t))_{\ddddot{y}}\Big)=0,$$

which by hypothesis (iv) gives

$$\gamma=\xi\bar{y}. \tag{19}$$

On using (19) in (18), we get

$$(\beta-\xi)(f_y-\bar{l}g_y-D(f_{\dot{y}}-\bar{l}g_{\dot{y}})+(A-\bar{l}B)\bar{p}(t))=0, \tag{20}$$

which by hypothesis (iii) gives

$$\beta=\xi. \tag{21}$$

Now, if we take $\xi=0$, then from (21), $\beta=0$ and from (19), we conclude that $\gamma=0$. Also, by (11), we get $\alpha=0$. Hence, $(\alpha,\beta,\gamma,\xi)\neq 0$ contradicting (15). Thus, $\xi>0$ and consequently $\beta>0$. Since $\xi>0,\ t\in I$, from (19), we have

$$\bar{y}=\frac{\gamma}{\xi}\in C_2.$$

Using (19) and (21) in (9), we obtain

$$\beta(f_x-\bar{l}g_x-D(f_{\dot{x}}-\bar{l}g_{\dot{x}}))(x-\bar{x})\geqq 0,\ t\in I. \tag{22}$$

Let $x=C_1$. Then $x+\bar{x}\in C_1$. On substituting $x+\bar{x}$ in place of x in (22), we have

$$x^T(f_x-\bar{l}g_x-D(f_{\dot{x}}-\bar{l}g_{\dot{x}}))(x-\bar{x})\geqq 0,$$

which by definition of polar cone yields

$$-(f_x-\bar{l}g_x-D(f_{\dot{x}}-\bar{l}g_{\dot{x}}))(x-\bar{x})\in C_1^*.$$

Again, letting $x=0$ and $x=2\bar{x}$ simultaneously in relation (22), we have

$$\bar{x}(f_x-\bar{l}g_x-D(f_{\dot{x}}-\bar{l}g_{\dot{x}}))(x-\bar{x})=0,\ t\in I.$$

From what has been done, it follows that $(\bar{x},\bar{y},\bar{l},\bar{p}(t)=0)$ is a feasible solution to (DP′). The optimality for (DP′) follows from weak duality theorem (Theorem 8). $\square$

A converse duality theorem may be stated as its proof would be analogous to Theorem 9.

Theorem 10. *(Converse Duality). Let us assume that*

(i) $(\bar{u}, \bar{v}, \bar{m}, \bar{q}(t))$ *be an optimal solution of (DP′),*

(ii) the matrices $Y - \bar{m}Z$ *be nonsingular,*

(iii) $f_x - \bar{m}g_x - D(f_{\dot{x}} - \bar{m}g_{\dot{x}}) + (Y - \bar{m}Z)\bar{q}(t) \neq 0$, *and*

(iv) the matrix

$$\Big((Y\bar{q}(t)_x - \bar{m}(Z\bar{q}(t))_x - D(B\bar{q}(t))_{\dot{x}} + \bar{m}D(Z\bar{q}(t))_{\dot{x}} + D^2(Y\bar{q}(t))_{\ddot{x}} - \bar{m}D^2(Z\bar{q}(t))_{\ddot{x}}$$

$$-D^3(Y\bar{q}(t))_{\dddot{x}} + \bar{m}D^3(Z\bar{q}(t))_{\dddot{x}} + D^4(Y\bar{q}(t))_{\ddddot{x}} - \bar{m}D^4(Z\bar{q}(t))_{\ddddot{x}}\Big)$$

be positive or negative definite.

Then $(\bar{u}, \bar{v}, \bar{m}, \bar{q}(t) = 0)$ *is a solution of* (PP′). *If, in addition, the conditions of Theorem 8 are satisfied, then* $(\bar{u}, \bar{v}, \bar{m}, \bar{q}(t) = 0)$ *is an optimal solution of* (PP′).

5. SELF DUALITY

A mathematical programming problem is said to be self dual if the dual can be recast in the form of primal. The programs (PP′) and (DP′) turns to be a self-dual, if we take $C = C_1 = C_2$ and impose additional restrictions of skew symmetry on f and symmetry on g, that is,

$$f(t, x, \dot{x}, y, \dot{y}) = -f(t, y, \dot{y}, x, \dot{x}), \; g(t, x, \dot{x}, y, \dot{y}) = g(t, y, \dot{y}, x, \dot{x}).$$

As a consequence of skew symmetry and symmetry imposed on f and g, respectively, we have

$$\begin{aligned}
f_x(t, x, \dot{x}, y, \dot{y}) &= -f_y(t, y, \dot{y}, x, \dot{x}), \; g_x(t, x, \dot{x}, y, \dot{y}) = g_y(t, y, \dot{y}, x, \dot{x}),\\
f_y(t, x, \dot{x}, y, \dot{y}) &= -f_x(t, y, \dot{y}, x, \dot{x}), \; g_y(t, x, \dot{x}, y, \dot{y}) = g_x(t, y, \dot{y}, x, \dot{x}),\\
f_{\dot{x}}(t, x, \dot{x}, y, \dot{y}) &= -f_{\dot{y}}(t, y, \dot{y}, x, \dot{x}), \; g_{\dot{x}}(t, x, \dot{x}, y, \dot{y}) = g_{\dot{y}}(t, y, \dot{y}, x, \dot{x}),\\
f_{\dot{y}}(t, x, \dot{x}, y, \dot{y}) &= -f_{\dot{x}}(t, y, \dot{y}, x, \dot{x}), \; g_{\dot{y}}(t, x, \dot{x}, y, \dot{y}) = g_{\dot{x}}(t, y, \dot{y}, x, \dot{x}),
\end{aligned}$$

and so on.
Now, we will show that (PP′) and (DP′) are self duals. By recasting the dual problem (DP′) as a minimization problem, we have
Dual Problem:

(DP′) Minimize $-m$

subject to

$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b),$$
$$v(a) = 0 = v(b), \quad \dot{v}(a) = 0 = \dot{v}(b),$$

$$\int_a^b (f(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Y q(t))\,dt - m\int_a^b (g(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Z q(t))\,dt = 0,$$

$$-\,[(f_x(t,u,\dot{u},v,\dot{v}) - Df_{\dot{x}}(t,u,\dot{u},v,\dot{v}) + Yq(t)) - m(g_x(t,u,\dot{u},v,\dot{v}) - Dg_{\dot{x}}(t,u,\dot{u},v,\dot{v}) + Zq(t))] \in C^*,\ t \in I,$$

$$u^T\{(f_x(t,u,\dot{u},v,\dot{v}) - Df_{\dot{x}}(t,u,\dot{u},v,\dot{v}) + Y(t,u,\dot{u},v,\dot{v})q(t)) - m(g_x(t,u,\dot{u},v,\dot{v}) - Dg_{\dot{x}}(t,u,\dot{u},v,\dot{v}) + Zq(t))\} \leqq 0,\ t \in I,$$

$$v \in C.$$

On using the skew symmetry and symmetry of f and g, respectively, the above problem is transformed to
Dual Problem:

(DP′) Minimize z

subject to

$$u(a) = 0 = u(b),\ \dot{u}(a) = 0 = \dot{u}(b),$$
$$v(a) = 0 = v(b),\ \dot{v}(a) = 0 = \dot{v}(b),$$

$$\int_a^b (f(t,v,\dot{v},u,\dot{u}) - \frac{1}{2}q(t)^T A' q(t))\,dt - z\int_a^b (g(t,v,\dot{v},u,\dot{u}) - \frac{1}{2}q(t)^T B' q(t))\,dt = 0,$$

$$\Big[(f_y(t,v,\dot{v},u,\dot{u}) - Df_{\dot{y}}(t,v,\dot{v},u,\dot{u}) + A'q(t)) - z(g_y(t,v,\dot{v},u,\dot{u}) - Dg_{\dot{y}}(t,v,\dot{v},u,\dot{u}) + B'q(t))\Big] \in C^*,\ t \in I,$$

$$u^T\Big\{(f_y(t,v,\dot{v},u,\dot{u}) - Df_{\dot{y}}(t,v,\dot{v},u,\dot{u}) + A'q(t)) - z(g_y(t,v,\dot{v},u,\dot{u}) - Dg_{\dot{y}}(t,v,\dot{v},u,\dot{u}) + B'q(t))\Big\} \geqq 0,\ t \in I,$$

$$v \in C,$$

where

$$(i)\ A^{'} = f_{yy}(t,v,\dot{v},u,\dot{u}) - 2Df_{y\dot{y}}(t,v,\dot{v},u,\dot{u}) + D^2 f_{\dot{y}\dot{y}}(t,v,\dot{v},u,\dot{u}) - D^3 f_{\dot{y}\ddot{y}}(t,v,\dot{v},u,\dot{u}),\ t \in I,$$

$$(ii)\ B^{'} = g_{yy}(t,v,\dot{v},u,\dot{u}) - 2Dg_{y\dot{y}}(t,v,\dot{v},u,\dot{u}) + D^2 g_{\dot{y}\dot{y}}(t,v,\dot{v},u,\dot{u}) - D^3 g_{\dot{y}\ddot{y}}(t,v,\dot{v},u,\dot{u}),\ t \in I,$$

$$(iii)\ z = \frac{\int_a^b (f(t,v,\dot{v},u,\dot{u}) - \frac{1}{2}q(t)^T A^{'} q(t))\, dt}{\int_a^b (g(t,v,\dot{v},u,\dot{u}) - \frac{1}{2}q(t)^T B^{\prime} q(t))\, dt}.$$

This shows that the dual problem (DP′) is identical to (PP′). Hence, feasibility of (u, v, m, q) to (DP′) implies the feasibility of (v, u, m, q) to (PP′).

Remark 11. *It is easy to see that Y and Z transform to $A^{'}$ and $B^{'}$, if we assume the functions f and g to be skew symmetric and symmetric, respectively.*

We now state the following self-duality theorem.

Theorem 12. *(Self duality). Let $f(t,x,\dot{x},y,\dot{y})$ be skew symmetric, $g(t,x,\dot{x},y,\dot{y})$ symmetric and $C = C_1 = C_2$. Then* (DP′) *is self dual. If* (PP′) *and* (DP′) *are dual problems and $(\bar{x},\bar{y},m,\bar{p}(t))$ is a joint optimal solution, then so is $(\bar{y},\bar{x},m,\bar{p}(t))$ and the common optimal value of the objective function is 0.*

6. STATIC SYMMETRIC DUAL PROGRAM

If we drop the time dependency from (PP) and (DP), we get the following second order fractional dual symmetric programs over cones:
Primal Problem:

$$\textbf{(SPP)} \qquad \text{Minimize} \qquad \frac{f(x,y) - \frac{1}{2}p^T \nabla_{yy} f(x,y)p}{g(x,y) - \frac{1}{2}p^T \nabla_{yy} g(x,y)p}$$

subject to

$$(g(x,y) - \frac{1}{2}p^T \nabla_{yy} g(x,y)p)(\nabla_y f(x,y) + \nabla_{yy} f(x,y)p) - (f(x,y) - \frac{1}{2}p^T \nabla_{yy} f(x,y)p)(\nabla_y g(x,y) + \nabla_{yy} g(x,y)p) \in C_2^*,$$

$$y^T \Big[(g(x,y) - \frac{1}{2}p^T \nabla_{yy} g(x,y)p)(\nabla_y f(x,y) + \nabla_{yy} f(x,y)p) - (f(x,y) - \frac{1}{2}p^T \nabla_{yy} f(x,y)p)(\nabla_y g(x,y) + \nabla_{yy} g(x,y)p)\Big] \geqq 0,$$

$$x \in C_1.$$

Dual Problem:

$$\textbf{(SDP)} \quad \text{Maximize} \quad \frac{f(u,v)-\frac{1}{2}q^T\nabla_{xx}f(u,v)q}{g(u,v)-\frac{1}{2}q^T\nabla_{xx}g(u,v)q}$$

subject to

$$-\left[(g(u,v)-\frac{1}{2}q^T\nabla_{xx}g(u,v)q)(\nabla_x f(u,v)+\nabla_{xx}f(u,v)q)\right.$$
$$\left.-(f(u,v)-\frac{1}{2}q^T\nabla_{xx}f(u,v)q)(\nabla_x g(u,v)+\nabla_{xx}g(u,v)q)\right]\in C_1^*,$$
$$u^T\Big[(g(u,v)-\frac{1}{2}q^T\nabla_{xx}g(u,v)q)(\nabla_x f(u,v)+\nabla_{xx}f(u,v)q)$$
$$-(f(u,v)-\frac{1}{2}q^T\nabla_{xx}f(u,v)q)(\nabla_x g(u,v)+\nabla_{xx}g(u,v)q)\Big]\leqq 0,$$
$$v\in C_2.$$

The equivalent form of the above problem can be written as
Primal Problem:

$$\textbf{(SPP}^{'}\textbf{)} \qquad \text{Minimize} \qquad r$$

subject to

$$f(x,y)-\frac{1}{2}p^T\nabla_{yy}f(x,y)p-r(g(x,y)-\frac{1}{2}p^T\nabla_{yy}g(x,y)p)=0,$$
$$(\nabla_y f(x,y)+\nabla_{yy}f(x,y)p)-r(\nabla_y g(x,y)+\nabla_{yy}g(x,y)p)\in C_2^*,$$
$$y^T\Big[(\nabla_y f(x,y)+\nabla_{yy}f(x,y)p)-r(\nabla_y g(x,y)+\nabla_{yy}g(x,y)p)\Big]\geqq 0,$$
$$x\in C_1.$$

Dual Problem:

$$\textbf{(SDP}^{'}\textbf{)} \qquad \text{Maximize} \qquad s$$

subject to

$$f(u,v)-\frac{1}{2}q^T\nabla_{xx}f(u,v)q-s(g(u,v)-\frac{1}{2}q^T\nabla_{xx}g(u,v)q)=0,$$
$$-[(\nabla_x f(u,v)+\nabla_{xx}f(u,v)q)-s(\nabla_x g(u,v)+\nabla_{xx}g(u,v)q)]\in C_1^*,$$
$$u^T\Big[(\nabla_x f(u,v)+\nabla_{xx}f(u,v)q)-s(\nabla_x g(u,v)+\nabla_{xx}g(u,v)q)\Big]\leqq 0,$$
$$v\in C_2,$$

where

$$r=\frac{f(x,y)-\frac{1}{2}p^T\nabla_{yy}f(x,y)p}{g(x,y)-\frac{1}{2}p^T\nabla_{yy}g(x,y)p},$$

$$s = \frac{f(u,v) - \frac{1}{2}q^T \nabla_{xx} f(u,v) q}{g(u,v) - \frac{1}{2}q^T \nabla_{xx} g(u,v) q}.$$

The following theorems may be proved along the lines of Theorem 3.1, Theorem 3.2, Theorem 3.3 given in Gulati et al. [11].

Theorem 13. *(Weak duality). Let (x, y, l, p) and (u, v, m, q) be feasible solutions to primal* $(\mathrm{SPP}^{'})$ *and dual* $(\mathrm{SDP}^{'})$*, respectively. Further, assume that*

(a) $f(.,v) - sg(.,v)$ is second order $\mathcal{F}$-pseudoconvex in first variable at u for fixed v,

(b) $-f(x,.) + rg(x,.)$ is second order $\mathcal{G}$-pseudoconvex in the second variable at y for fixed x,

(c) $\mathcal{F}_{x,u}(\xi) + \xi^T u \geqq 0,\ \forall\ -\xi \in C_1^*$, and

(d) $\mathcal{G}_{v,y}(\zeta) + \zeta^T y \geqq 0,\ \forall\ -\zeta \in C_2^*$.

Then $l \geqq m$.

Theorem 14. *(Strong duality). Let f and g be thrice continuously differentiable functions and let $(\bar{x}, \bar{y}, \bar{r}, \bar{p})$ be an optimal solution of* $(\mathrm{SPP}^{'})$*. Assume that*

(i) $\nabla_{yy} f(\bar{x},\bar{y}) - \bar{r}\nabla_{yy} g(\bar{x},\bar{y})$ is positive definite and $\bar{p}^T(\nabla_y f(\bar{x},\bar{y}) - \bar{r}\nabla_y g(\bar{x},\bar{y})) \geqq 0$, or
$\nabla_{yy} f(\bar{x},\bar{y}) - \bar{r}\nabla_{yy} g(\bar{x},\bar{y})$ is negative definite and $\bar{p}^T(\nabla_y f(\bar{x},\bar{y}) - \bar{r}\nabla_y g(\bar{x},\bar{y})) \leqq 0$, and

(ii) $\nabla_y f(\bar{x},\bar{y}) + \nabla_{yy} f(\bar{x},\bar{y})\bar{p} - \bar{r}(\nabla_y g(\bar{x},\bar{y}) + \nabla_{yy} g(\bar{x},\bar{y})\bar{p}) \neq 0$.

Then $(\bar{x}, \bar{y}, \bar{r}, \bar{q} = 0)$ is a feasible solution for $(\mathrm{SDP}^{'})$ *and the objective values of* $(\mathrm{SPP}^{'})$ *and* $(\mathrm{SDP}^{'})$ *are equal. Furthermore, if the hypotheses of Theorem 13 are satisfied for all feasible solutions of* $(\mathrm{SPP}^{'})$ *and* $(\mathrm{SDP}^{'})$*, then $(\bar{x}, \bar{y}, \bar{r}, \bar{q} = 0)$ is an optimal solution of* $(\mathrm{SDP}^{'})$*.*

Theorem 15. *(Converse duality). Let f and g be thrice continuously differentiable functions and let $(\bar{u}, \bar{v}, \bar{s}, \bar{q})$ be an optimal solution of* $(\mathrm{SDP}^{'})$*. Assume that*

(i) $\nabla_{xx} f(\bar{u},\bar{v}) - \bar{s}\nabla_{xx} g(\bar{u},\bar{v})$ is positive definite and $\bar{q}^T(\nabla_x f(\bar{u},\bar{v}) - \bar{s}\nabla_x g(\bar{u},\bar{v})) \geqq 0$, or
$\nabla_{xx} f(\bar{u},\bar{v}) - \bar{s}\nabla_{xx} g(\bar{u},\bar{v})$ is negative definite and $\bar{q}^T(\nabla_x f(\bar{u},\bar{v}) - \bar{s}\nabla_x g(\bar{u},\bar{v})) \leqq 0$, and

(ii) $\nabla_x f(\bar{u},\bar{v}) + \nabla_{xx} f(\bar{u},\bar{v})\bar{q} - \bar{s}(\nabla_x g(\bar{u},\bar{v}) + \nabla_{xx} g(\bar{u},\bar{v})\bar{q}) \neq 0$.

Then $(\bar{u}, \bar{v}, \bar{s}, \bar{p} = 0)$ is a feasible solution for $(\mathrm{SPP}^{'})$ *and the objective values of* $(\mathrm{SPP}^{'})$ *and* $(\mathrm{SDP}^{'})$ *are equal. Furthermore, if the hypotheses of Theorem 13 are satisfied for all feasible solutions of* $(\mathrm{SPP}^{'})$ *and* $(\mathrm{SDP}^{'})$*, then $(\bar{u}, \bar{v}, \bar{s}, \bar{p} = 0)$ is an optimal solution of* $(\mathrm{SPP}^{'})$*.*

Special cases of the static problem

(i) If we take $C_1 = \mathbb{R}^n_+$, $C_2 = \mathbb{R}^m_+$, we obtain the symmetric dual programs (FP) and (FD) given in Gulati et al. [11]

(ii) If we set $p = 0$ and $q = 0$, we get the programs considered in Chandra et al. [4, 5].

(iii) If $g = 1$ for all x, y, then the problems (FP) and (FD) reduce to the problems studied in Bector and Chandra [3]. Also, if $p = 0$ and $q = 0$, then we obtain the programs considered in Mond and Weir [17].

7. CONCLUSIONS

In this paper, we have introduced the concept of second order $\mathcal{F}$-convexity and discussed a pair of second order variational fractional symmetric dual programs over cone constraints. Weak, strong and converse duality theorems are established under second order $\mathcal{F}$-convexity assumptions. The present work can be further extended to second order nondifferentiable variational fractional symmetric dual problems over cone constraints. It will be more interesting if we consider the higher order analogues of these problems. These may be taken as the future tasks of the authors.

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