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A CLASS OF NEW TYPE UNIFIED NON-DIFFERENTIABLE HIGHER ORDER SYMMETRIC DUALITY THEOREMS OVER ARBITRARY CONES UNDER GENERALIZED ASSUMPTIONS

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Abstract: In the present paper, a newly combined higher-order non-differentiable symmetric duality in scalar-objective programming over arbitrary cones is formulated. In literature we have discussed primal-dual results with arbitrary cones, while in this article, we have derived combined result with one model over arbitrary cones. The theorems of duality are derived for these problems under η-pseudoinvexity/η-invexity/C-pseudoconvexity/C-convexity speculations over arbitrary cones.

Keywords: Symmetric Duality, Non-Differentiable Programming, Mixed Duality, Arbitrary Cones.

MSC: 90C26, 90C30, 90C32, 90C46.

1. INTRODUCTION

Duality mathematical programming is used in Economics, Control Theory, Business and other diverse fields. In mathematical programming, a pair of primal and dual problems is said to be symmetric when the dual problem is expressed in the form of the primal problem, then it does happen that its dual is the primal problem. Symmetric duality in nonlinear programming was introduced by Dorn [10]. The notion of symmetric duality was developed significantly by Dantzig et al.[11]. Mangasarian [16] introduced the concept of second and higher-order duality for nonlinear problems. The study of higher-order duality is significant due to the computational advantage over the first order duality as it provides tighter bounds for the value of the objective function when approximations are used.

In recent past, several definitions such as nonsmooth univex, nonsmooth quasiunivex and nonsmooth pseudoinvex functions have been introduced by Xianjun [20]. Mond and Zhang [17] obtained duality results for various higher-order dual problems under higher-order invexity assumptions. Chandra et al. [3] and Yang et al. [21] discussed a mixed symmetric dual formulation for a nonlinear programming problem and for a class of non-differentiable nonlinear programming problems, respectively. Later on, Chen [1] studied duality relations for Mond-Weir type multi-objective higher-order symmetric dual programs under F-convexity assumptions.

Khurana [15] defined the cone-pseudobonvex / strongly cone-pseudobonvex functions and formulated a pair of Mond-Weir type symmetric dual multiobjective programs over arbitrary cones, also established the duality theorems by using these defined functions. Recently, Kaseem [14] introduced second order (K, F)-pseudoconvex and strongly second order (K, F)-pseudoconvex functions and formulated a pair of second order multiobjective symmetric dual nonlinear programs over arbitrary cones in order to prove weak, strong and converse duality theorems. For more information, readers are advised to see [4, 5, 6, 7, 8, 9].

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We have formulated a new mixed type higher-order non-differentiable symmetric duality in scalar-objective programming problem. In literature, we have discussed the results either Wolfe or Mond-Weir type dual or separately, while in this we have combined result over one model over arbitrary cones. The duality theorems are proved for these programs over arbitrary cones under η-pseudoinvexity/η-invexity/C-pseudoconvexity/C-convexity assumptions.

2. PRELIMINARIES AND DEFINITIONS

We examine the subsequent scalar objective programming problem:

(P) Minimize $F(x)$ and $x \in X$
where $X \subseteq R^{n+m}$. Let $F : X \to R$.

The subsequent pattern for vector inequalities will be used: If a, $b \in R^n$, then

$$a \geqq b \Leftrightarrow a_i \geqq b_i, i = 1, 2, ..., n;$$

$$a \geqslant b \Leftrightarrow a \geqq b \text{ and } a \neq b;$$

$$a > b \Leftrightarrow a_i > b_i, i = 1, 2, ..., n.$$

2.1. Definition

Let C be a nonempty compact convex set in R^n. The support function $s(x|C)$ of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

The sub-differential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For any convex set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0 \text{ for all } z \in S\}.$$

It is readily verified that for a compact convex set E, y is in $N_E(x)$ if and only if

$$s(y|E) = x^T y.$$

2.2. Definition

The positive polar cone P^* of a cone P is defined by

$$P^* = \{y \in R^p : x^T y \geqq 0, \forall x \in P\}$$

2.3. Definition

Let $C : X \times X \times R^n \to R$ $(X \subseteq R^n)$ be a function which satisfies $C_{x,u}(0) = 0$, $\forall (x, u) \in X \times X$. Then, the function C is said to be convex on R^n with respect to third argument iff for any fixed $(x, u) \in X \times X$,

$$C_{x,u}(\lambda x_1 + (1-\lambda)x_2) \leqq \lambda C_{x,u}(x_1) + (1-\lambda)C_{x,u}(x_2),\ \forall \lambda \in (0,1),\ \forall x_1, x_2 \in R^n.$$

Many generalizations [2, 12, 13, 18] of the definition of a convex function have been introduced in optimization theory in order to weak the assumption of convexity for establishing duality results for new classes of nonconvex optimization problems, including vector optimization problems. One of such a generalization of convexity in the vectorial case, we introduce the following concept of higher-order C-convex/C-pseudoconvex functions:

2.4. Definition

$\Phi : X \mapsto R$ is higher-order invex at $u \in X$ with respect to $\eta : X \times X \mapsto R^n$ and $H : X \times R^n \mapsto R$, $(X \subseteq R^n)$ if for all $(x, p) \in X \times R^n$,

$$\Phi(x) - \Phi(u) - H(u,p) + p^T \nabla_p H(u,p) \geqq \eta^T(x,u)\{\nabla_x \Phi(u) + \nabla_p H(u,p)\}.$$

2.5. Definition

$\Phi : X \mapsto R$ is higher-order C-convex at $u \in X$ with respect to $H : X \times R^n \mapsto R$, $(X \subseteq R^n)$ if for all $(x, p) \in X \times R^n$,

$$\Phi(x) - \Phi(u) - H(u,p) + p^T \nabla_p H(u,p) \geqq C_{x,u}\Big\{\nabla_x \Phi(u) + \nabla_p H(u,p)\Big\}.$$

2.6. Example

Let $X = [0, 2] \subseteq R$, $n = m = 1$ and $k = 1$. Consider the function $\Phi : X \to R$ is given by

$$\Phi(x) = \left(\frac{e^x - e^{-x}}{2}\right)^5.$$

Next, $H : X \times X \to R$ is given as

$$H(u,p) = \frac{u^2 p}{3}$$

and $\eta(x, u) = x^2 u^2 + u$.

We have to claim that Φ is higher-order invex at $u \in X$ with respect to η and H. For this, it is sufficient to prove that the following expression is nonnegative i.e.

$$\Upsilon = \Phi(x) - \Phi(u) - H(u,p) + p^T \nabla_p H(u,p) - \eta^T(x,u)\{\nabla_x \Phi(u) + \nabla_p H(u,p)\} \geqq 0.$$

Substituting the values of Φ, H and η in above expression, we have

$$\Upsilon = \left(\frac{e^x - e^{-x}}{2}\right)^5 - \frac{u^2 p}{3} + \frac{u^2 p}{3} - \left(\frac{e^u - e^{-u}}{2}\right)^5 - (x^2u^2 + u)\left(5\left(\frac{e^u - e^{-u}}{2}\right)^4 \left(\frac{e^u + e^{-u}}{2}\right) + \frac{u^2}{3}\right).$$

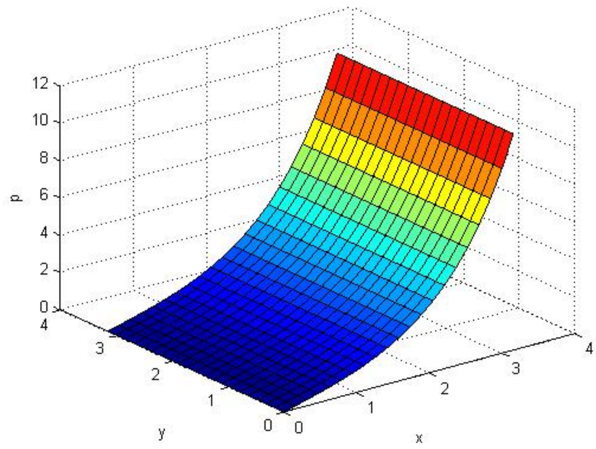

Figure 1: The function $\Upsilon = \left(\frac{e^x - e^{-x}}{2}\right)^5, \ \forall\ p, \ \forall\ x \in \big[0, 2\big]$ is non-negative

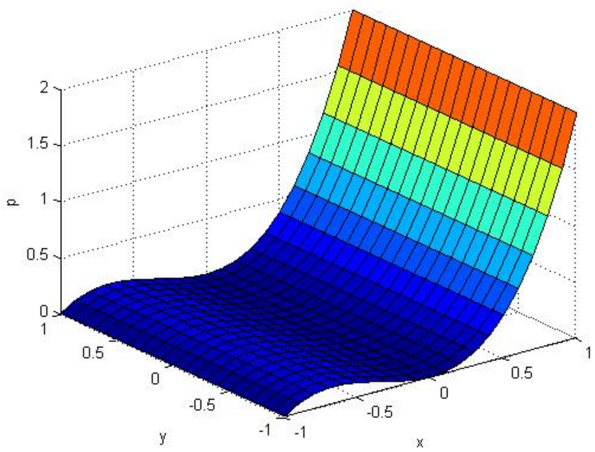

Figure 2: The function $\Upsilon_1 = (x^3 + x^2), \ \forall\ p, \ \forall\ x \in [-1, 1]$ is non-negative

Simplifying the above equation at the point $u = 0 \in X$, we obtain

$$\Upsilon = \left(\frac{e^x - e^{-x}}{2}\right)^5, \ \forall\ x \in X.$$

From Figure 1, it is clear that $\Upsilon \geqq 0, \ \forall\ x \in X$. Therefore, Φ is higher-order invex at $u = 0 \in X$ with respect to η and H.

2.7. Example

Let $X = [-1, 1]$. Consider the function $\Phi : X \to R$ is given by

$$\Phi(x) = x^3 + x^2.$$

Next, $H : X \times X \to R$ is given as

$$H(u, p) = -\frac{u^4 p}{5}.$$

Consider $\eta(x, u) = x^2 u^2$.
We have to claim that Φ is higher-order invex at $u = 0 \in X$ with respect to η and H. For this, it is enough to show that

$$\Upsilon_1 = \Phi(x) - \Phi(u) - H(u,p) + p^T \nabla_p H(u,p) - \eta^T(x,u)\{\nabla_x \Phi(u) + \nabla_p H(u,p)\} \geqq 0.$$

Obviously, from Figure 2, we have $\Upsilon_1 \geqq 0, \ \forall\ x \in X$. Hence, Φ is higher-order invex at $u = 0 \in X$ with respect to η and H.

2.8. Definition

$\Phi : X \mapsto R$ is higher-order pseudo-invex at $u \in X$ with respect to $\eta : X \times X \mapsto R^n$ and $H : X \times R^n \mapsto R, \ (X \subseteq R^n)$ if for all $(x, p) \in X \times R^n$,

$$\eta^T(x,u)\Big\{\nabla_x \Phi(u) + \nabla_p H(u,p)\Big\} \geqq 0 \Rightarrow \Big(\Phi(x) - \Phi(u) - H(u,p) + p^T \nabla_p H(u,p)\Big) \geqq 0.$$

2.9. Definition

$\Phi : X \mapsto R$ is higher-order C-pseudo-convex at $u \in X$ with respect to $H : X \times R^n \mapsto R, \ (X \subseteq R^n)$ if for all $(x, p) \in R^n \times R^n$,

$$C_{x,u}\Big\{\nabla_x \Phi(u) + \nabla_p H(u,p)\Big\} \geqq 0 \Rightarrow \Big(\Phi(x) - \Phi(u) - H(u,p) + p^T \nabla_p H(u,p)\Big) \geqq 0.$$

2.10. Example

Consider a function $\Phi(x) = e^{-x} - x^2, \ \ C_{x,u}(a) = a^2(x^2 - u^2)$,
$H(u, p) = p(1 + u)^{-1}$, where $X = [1, \infty)$.

The above example 2.10 shows that the function Φ is higher-order C-pseudoconvex at $u = 1 \in [1, \infty)$, but it is not higher-order F-pseudoconvex at $u = 1 \in [1, \infty)$ because it is not sublinear in its third position.

3. NON-DIFFERENTIABLE HIGHER ORDER MIXED TYPE SYMMETRIC DUALITY MODEL OVER ARBITRARY CONES

For N= {1, 2, 3, ..., n} and M= {1, 2, 3, ..., m}, let us assume $J_1 \subset N$, $K_1 \subset M$ and $J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$, where $|J_1|$ denotes the number of elements in the set J_1. The other numbers $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Notice that if $J_1 = \varnothing$, then $J_2 = N$, that is $|J_1| = 0$ and $|J_2| = n$ then $R^{|J_1|}$ is zero dimensional Euclidean space and $R^{|J_2|}$ is n-dimensional Euclidean space. It is clear that any $x \in R^n$ can be written as $x = \{x_1,\ x_2\},\ x_1 \in R^{|J_1|},\ x_2 \in R^{|J_2|}$. Similarly, any $y \in R^m$ can be written as $y = \{y_1,\ y_2\},\ y_1 \in R^{|K_1|},\ y_2 \in R^{|K_2|}$. Let
(i) $f_1 : R^{|J_1|} \times R^{|K_1|} \to R$,
(ii) $f_2 : R^{|J_2|} \times R^{|K_2|} \to R$,
(iii) $g_1 : R^{|J_1|} \times R^{|K_1|} \times R^{|J_1|} \to R$,
(iv) $g_2 : R^{|J_2|} \times R^{|K_2|} \times R^{|J_2|} \to R$,
(v) $h_1 : R^{|J_1|} \times R^{|K_1|} \times R^{|K_1|} \to R$,
(vi) $h_2 : R^{|J_2|} \times R^{|K_2|} \times R^{|K_2|} \to R$, be twice differentiable functions, respectively.

In this section, we introduce the following pair of non-differentiable higher order symmetric duality model over arbitrary cones and derive duality theorems.

Primal Problem (MNHP):

Minimize $L(x, y, z, p) =$
$$f_1(x_1, y_1) + s(x_1|E_1) + f_2(x_2, y_2) + s(x_2|E_2) - y_1^T z_1 + h_1(x_1, y_1, p_1) + h_2(x_2, y_2, p_2)$$

$$-p_1^T \nabla_{p_1} h_1(x_1, y_1, p_1) - p_2^T \nabla_{p_2} h_2(x_2, y_2, p_2) - (y_2)^T [\nabla_{y_2} f_2(x_2, y_2) + \nabla_{p_2} h_2(x_2, y_2, p_2)]$$

subject to

$$-\Big(\nabla_{y_1} f_1(x_1, y_1) - z_1 + \nabla_{p_1} h_1(x_1, y_1, p_1)\Big) \in C_1^*, \tag{1}$$

$$-\Big(\nabla_{y_2} f_2(x_2, y_2) - z_2 + \nabla_{p_2} h_2(x_2, y_2, p_2)\Big) \in C_2^*, \tag{2}$$

$$y_1^T [\nabla_{y_1} f_1(x_1, y_1) - z_1 + \nabla_{p_1} h_1(x_1, y_1, p_1)] \geqq 0, \tag{3}$$

$$p_1^T [\nabla_{y_1} f_1(x_1, y_1) - z_1 + \nabla_{p_1} h_1(x_1, y_1, p_1)] \geqq 0, \tag{4}$$

$$p_2^T [\nabla_{y_2} f_2(x_2, y_2) - z_2 + \nabla_{p_2} h_2(x_2, y_2, p_2)] \geqq 0, \tag{5}$$

$$x_1 \in C_3, x_2 \in C_4, y_2 \geqq 0, \tag{6}$$

$$z_1 \in D_1, z_2 \in D_2. \tag{7}$$

Dual Problem (MNHD):

Minimize $M(u,v,w,r) =$
$f_1(u_1,v_1) - s(v_1|D_1) + f_2(u_2,v_2) - s(v_2|D_2) + u_1^T w_1 + g_1(u_1,v_1,r_1) + g_2(u_2,v_2,r_2)$

$-r_1^T \nabla_{r_1} g_1(u_1,v_1,r_1) - (r_2)^T \nabla_{r_2} g_2(u_2,v_2,r_2) - (u_2)^T [\nabla_{u_2} f_2(u_2,v_2) + \nabla_{r_2} g_2(u_2,v_2,r_2)]$
subject to

$$\Big(\nabla_{u_1} f_1(u_1,v_1) + w_1 + \nabla_{r_1} g_1(u_1,v_1,r_1)\Big) \in C_3^*, \tag{8}$$

$$\Big(\nabla_{u_2} f_2(u_2,v_2) + w_2 + \nabla_{r_2} g_2(u_2,v_2,r_2)\Big) \in C_4^*, \tag{9}$$

$$u_1^T [\nabla_{u_1} f_1(u_1,v_1) + w_1 + \nabla_{r_1} g_1(u_1,v_1,r_1)] \leqq 0, \tag{10}$$

$$r_1^T [\nabla_{u_1} f_1(u_1,v_1) + w_1 + \nabla_{r_1} g_1(u_1,v_1,r_1)] \leqq 0, \tag{11}$$

$$(r_2)^T [\nabla_{u_2} f_2(u_2,v_2) + w_2 + \nabla_{r_2} g_2(u_2,v_2,r_2)] \leqq 0, \tag{12}$$

$$v_1 \in C_1, v_2 \in C_2, u_2 \geqq 0, \tag{13}$$

$$w_1 \in E_1, w_2 \in E_2, \tag{14}$$

where $p_1 \in R^{|K_1|}$, $p_2 \in R^{|K_2|}$, $r_1 \in R^{|J_1|}$ and $r_2 \in R^{|J_2|}$ and E_1, E_2, D_1 and D_2 are compact convex sets in $R^{|J_1|}, R^{|J_2|}, R^{|K_1|}$ and $R^{|K_2|}$, respectively.

Let P^0 and Q^0 be feasible set of (MNHP) and (MNHD), respectively.

Theorem 3.1 (Weak Duality). Let $(x_1,x_2,y_1,y_2,z_1,z_2,p_1,p_2) \in P^0$ and $(u_1,u_2,v_1,v_2,w_1,w_2,r_1,r_2) \in Q^0$. Let

(i) $f_1(.,v_1) + (.)^T w_1$ be higher-order pseudo-invex at u_1 with respect to η_1 and g_1,
(ii)-$f_1(x_1,.) + (.)^T z_1$ be higheror-der pseudo-invex at y_1 with respect to η_2 and -h_1,
(iii) $f_2(.,v_2) + (.)^T w_2$ be higher-order invex at u_2 with respect to η_3 and g_2,
(iv) -$f_2(x_2,.) + (.)^T z_2$ be higher-order invex at y_2 with respect to η_4 and -h_2,
(v) $\eta_1(x_1,u_1) + u_1 + r_1 \in C_3$,
(vi) $\eta_2(v_1,y_1) + y_1 + p_1 \in C_4$,
(vii) $\eta_3(x_2,u_2) + u_2 + r_2 \in C_1$,
(viii) $\eta_4(v_2,y_2) + y_2 + p_2 \in C_2$.

Then,

$$L(x_1,x_2,y_1,y_2,z_1,p_1,p_2) \not\leq M(u_1,u_2,v_1,v_2,w_1,r_1,r_2). \tag{15}$$

Proof: By hypotheses (iii) and (iv), we get

$$f_2(x_2,v_2) + x_2^T w_2 - f_2(u_2,v_2) - (u_2)^T w_2 - g_2(u_2,v_2,p_2) + (r_2)^T \nabla_{r_2} g_2(u_2,v_2,p_2)$$
$$\geqq \eta_3(x_2,u_2)[\nabla_{x_2} f_2(u_2,v_2) + w_2 + \nabla_{r_2} g_2(u_2,v_2,r_2)], \tag{16}$$

and

$$f_2(x_2, y_2) - (y_2)^T z_2 - f_2(x_2, v_2) + (v_2)^T z_2 - h_2(x_2, v_2, p_2) + p_2^T \nabla_{p_2} h_2(x_2, v_2, p_2)$$
$$\geqq \eta_4(v_2, y_2)[-\nabla_{y_2} f_2(x_2, y_2) + z_2 - \nabla_{p_2} h_2(x_2, y_2, p_2)]. \quad (17)$$

Using hypotheses (vii), $(viii)$ and the dual constraints (2) and (9), we have

$$(\eta_3(x_2, u_2) + u_2 + r_2)[\nabla_{x_2} f_2(u_2, v_2) + w_2 + \nabla_{r_2} g_2(u_2, v_2, r_2)] \geqq 0,$$

and

$$(\eta_4(v_2, y_2) + y_2 + p_2)[-\nabla_{y_2} f_2(x_2, y_2) + z_2 - \nabla_{p_2} h_2(x_2, y_2, p_2)] \geqq 0.$$

Above inequalities follows that:

$$\eta_3(x_2, u_2)[\nabla_{x_2} f_2(u_2, v_2) + w_2 + \nabla_{r_2} g_2(u_2, v_2, r_2)] + u_2[\nabla_{x_2} f_2(u_2, v_2) + w_2$$
$$+ \nabla_{r_2} g_2(u_2, v_2, r_2)] \geqq -r_2[\nabla_{x_2} f_2(u_2, v_2) + w_2 + \nabla_{r_2} g_2(u_2, v_2, r_2)],$$

and

$$\eta_4(v_2, y_2)[-\nabla_{y_2} f_2(x_2, y_2) + z_2 - \nabla_{p_2} h_2(x_2, y_2, p_2)] + y_2[-\nabla_{y_2} f(x_2, y_2) + z_2$$
$$- \nabla_{p_2} h_2(x_2, y_2, p_2)] \geqq p_2[\nabla_{y_2} f_2(x_2, y_2) - z_2 + \nabla_{p_2} h_2(x_2, y_2, p_2)]. \quad (18)$$

Using inequalities (5) and (12) gives that

$$\eta_3(x_2, u_2)[\nabla_{x_2} f_2(u_2, v_2) + w_2 + \nabla_{r_2} g_2(u_2, v_2, r_2)]$$
$$\geqq -u_2[\nabla_{x_2} f_2(u_2, v_2) + w_2 + \nabla_{r_2} g_2(u_2, v_2, r_2)], \quad (19)$$
$$\eta_4(v_2, y_2)[-\nabla_{y_2} f_2(x_2, y_2) + z_2 - \nabla_{p_2} h_2(x_2, y_2, p_2)]$$
$$\geqq y_2[\nabla_{y_2} f_2(x_2, y_2) + z_2 + \nabla_{p_2} h_2(x_2, y_2, p_2)]. \quad (20)$$

Further, from inequalities (16) and (17), we obtain

$$f_2(x_2, v_2) + x_2^T w_2 - f_2(u_2, v_2) - (u_2)^T w_2 - g_2(u_2, v_2, p_2) + (r_2)^T \nabla_{r_2} g_2(u_2, v_2, p_2)$$
$$\geqq -u_2[\nabla_{x_2} f_2(u_2, v_2) + w_2 + \nabla_{r_2} g_2(u_2, v_2, r_2)], \quad (21)$$
$$f_2(x_2, y_2) - (y_2)^T z_2 - f_2(x_2, v_2) + (v_2)^T z_2 - h_2(x_2, v_2, p_2) + p_2^T \nabla_{p_2} h_2(x_2, v_2, p_2)$$
$$\geqq y_2[\nabla_{y_2} f_2(x_2, y_2) + z_2 + \nabla_{p_2} h_2(x_2, y_2, p_2)]. \quad (22)$$

Adding the above inequalities, we have

$$f_2(x_2, y_2) + x_2^T w_2 - (y_2)^T z_2 + h_2(x_2, y_2, p_2) - p_2^T \nabla_{p_2} h_2(x_2, y_2, p_2)$$
$$-y_2[\nabla_{y_2} f_2(x_2, y_2) - z_2 + \nabla_{p_2} h_2(x_2, y_2, p_2)]$$

$$\geqq f_2(u_2,v_2)+(u_2)^T w_2-(v_2)^T z_2+g_2(u_2,v_2,p_2)-(r_2)^T\nabla_{r_2}g_2(u_2,v_2,p_2)$$
$$-u_2[\nabla_{x_2}f_2(u_2,v_2)+w_2+\nabla_{r_2}g_2(u_2,v_2,r_2)].$$

Now, using $x_2^T w_2 \leqq s(x_2|E_2)$ and $(v_2)^T z_2 \leqq s(v_2|D_2)$, we obtain

$$f_2(x_2,y_2)+s(x_2|E_2)+h_2(x_2,y_2,p_2)-p_2^T\nabla_{p_2}h_2(x_2,y_2,p_2)$$
$$-y_2[\nabla_{y_2}f(x_2,y_2)+\nabla_{p_2}h_2(x_2,y_2,p_2)]$$
$$\geqq f_2(u_2,v_2)-s(v_2|D_2)+g_2(u_2,v_2,p_2)-(r_2)^T\nabla_{r_2}g_2(u_2,v_2,p_2)$$
$$-u_2[\nabla_{x_2}f_2(u_2,v_2)+\nabla_{r_2}g_2(u_2,v_2,r_2)]. \tag{23}$$

Similarly, using hypotheses $(v),(vi)$, primal-dual constraints and the fact that $(x_1)^T w_1 \leqq s(x_1|E_1)$ and $v_1^T z_1 \leqq s(v_1|D_1)$, we get

$$f_1(x_1,y_1)-y_1^T z_1+s(x_1|E_1)+h_1(x_1,y_1,p_1)-p_1^T\nabla_{p_1}h_1(x_1,y_1,p_1)$$
$$\geqq f_1(u_1,v_1)+v_1^T w_1-s(v_1|D_1)+g_1(u_1,v_1,p_1)-r_1^T\nabla_{r_1}g_1(u_1,v_1,p_1). \tag{24}$$

Combining inequalities (23) and (24), we obtain

$$L(x_1,x_2,y_1,y_2,z_1,p_1,p_2)\geqq M(u_1,u_2,v_1,v_2,w_1,r_1,r_2).$$

This completes the proof.

Theorem 3.2 (Weak Duality). Let $(x_1,x_2,y_1,y_2,z_1,z_2,p_1,p_2)\in P^0$ and $(u_1,u_2,v_1,v_2,w_1,w_2,r_1,r_2)\in Q^0$. Let

(i) $f_1(.,v_1)+(.)^T w_1$ be higher-order C-pseudo-convex at u_1 with respect to g_1,
(ii) -$f_1(x_1,.)+(.)^T z_1$ be higher-order C- pseudo-convex at y_1 with respect to $-h_1$,
(iii) $f_2(.,v_2)+(.)^T w_2$ be higher-order C-convex at u_2 with respect to g_2,
(iv) -$f_2(x_2,.)+(.)^T z_2$ be higher-order C- invex at y_2 with respect to η_4 and $-h_2$,
(v) $C_{x_1,u_1}(\nabla_{u_1}f_1(u_1,v_1)+w_1+\nabla_{r_1}h_1(u_1,v_1,r_1))+u_1^T(\nabla_{u_1}f_1(u_1,v_1)+w_1+\nabla_{r_1}h_1(u_1,v_1,r_1))+r_1^T(\nabla_{u_1}f_1(u_1,v_1)+w_1+\nabla_{r_1}h_1(u_1,v_1,r_1))\in C_3,\ \forall x_1,u_1\in C_3,$ $(\nabla_{u_1}f_1(u_1,v_1)+w_1+\nabla_{r_1}h_1(u_1,v_1,r_1))\in C_3^*,$
(vi) $C_{v_1,y_1}(-\nabla_{y_1}f_1(x_1,y_1)-z_1+\nabla_{p_1}g_1(x_1,y_1,p_1))+y_1^T(-\nabla_{y_1}f_1(x_1,y_1)-z_1+\nabla_{p_1}g_1(x_1,y_1,p_1))+p_1^T(-\nabla_{y_1}f_1(x_1,y_1)-z_1+\nabla_{p_1}g_1(x_1,y_1,p_1))\in C_1,$ $\forall v_1,y_1\in C_1,(-\nabla_{y_1}f_1(x_1,y_1)-z_1+\nabla_{p_1}g_1(x_1,y_1,p_1))\in C_1^*,$
(vii) $C_{x_2,u_2}(\nabla_{u_2}f_2(u_2,v_2)+w_2+\nabla_{r_2}h_2(u_2,v_2,r_2))+u_1^T(\nabla_{u_2}f_2(u_2,v_2)+w_2+\nabla_{r_2}h_2(u_2,v_2,r_2))+r_2^T(\nabla_{u_2}f_2(u_2,v_2)+w_2+\nabla_{r_2}h_2(u_2,v_2,r_2))\in C_4,$ $\forall x_2,u_2\in C_4,(\nabla_{u_2}f_2(u_2,v_2)+w_2+\nabla_{r_2}h_2(u_2,v_2,r_2))\in C_4^*,$
(viii) $C_{v_2,y_2}(-\nabla_{y_2}f_2(x_2,y_2)-z_2+\nabla_{p_2}g_2(x_2,y_2,p_2))+y_2^T(-\nabla_{y_2}f_2(x_2,y_2)$

$$- z_2 + \nabla_{p_2} g_2(x_2, y_2, p_2)) + p_2^T(-\nabla_{y_2} f_2(x_2, y_2) - z_2 + \nabla_{p_2} g_2(x_2, y_2, p_2)) \in C_2,$$
$$\forall v_2, y_2 \in C_2, (-\nabla_{y_2} f_1(x_2, y_2) - z_2 + \nabla_{p_2} g_2(x_2, y_2, p_2)) \in C_2^*.$$

Then,

$$L(x_1, x_2, y_1, y_2, z_1, p_1, p_2) \not\leq M(u_1, u_2, v_1, v_2, w_1, r_1, r_2).$$

Proof: The proof follows on the lines of Theorem 3.1.

Theorem 3.3 (Strong Duality). Let $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2, \bar{p}_1, \bar{p}_2)$ be an optimal solution of (MNHP). Suppose that

(*i*) $\nabla_{p_1p_1} h_1(\bar{x}_1, \bar{y}_1, \bar{p}_1)$ is positive or negative definite and $\nabla_{p_2p_2} h_2(\bar{x}_2, \bar{y}_2, \bar{p}_2)$ is negative definite,
(ii) $\nabla_{y_1} f_1(\bar{x}_1, \bar{y}_1) - \bar{z}_1 + \nabla_{p_1} h_1(\bar{x}_1, \bar{y}_1, \bar{p}_1) \neq 0$ and
$\nabla_{y_2} f_2(\bar{x}_2, \bar{y}_2) - \bar{z}_2 + \nabla_{p_2} h_2(\bar{x}_2, \bar{y}_2, \bar{p}_2) \neq 0$,
(iii) $(\bar{p}_1)^T[\nabla_{y_1} f_1(\bar{x}_1, \bar{y}_1) - \bar{z}_1 + \nabla_{p_1} h_1(\bar{x}_1, \bar{y}_1, \bar{p}_1)] = 0 \Rightarrow \bar{p}_1 = 0$ and
$y_2[\nabla_{y_2} h_2(\bar{x}_2, \bar{y}_2, \bar{p}_2) - \nabla_{p_2} h_2(\bar{x}_2, \bar{y}_2, \bar{p}_2) + \nabla_{y_2y_2} f_2(\bar{x}_2, \bar{y}_2)\bar{p}_2] = 0 \Rightarrow \bar{p}_2 = 0$,
(iv) $h_1(\bar{x}_1, \bar{y}_1, 0) = g_1(\bar{x}_1, \bar{y}_1, 0)$, $\nabla_{x_1} h_1(\bar{x}_1, \bar{y}_1, 0) = \nabla_{r_1} g_1(\bar{x}_1, \bar{y}_1, 0)$,
$\nabla_{y_1} h_1(\bar{x}_1, \bar{y}_1, 0) = \nabla_{p_1} h_1(\bar{x}_1, \bar{y}_1, 0)$ and $h_2(\bar{x}_2, \bar{y}_2, 0) = g_2(\bar{x}_2, \bar{y}_2, 0)$,
$\nabla_{x_2} h_2(\bar{x}_2, \bar{y}_2, 0) = \nabla_{r_2} g_2(\bar{x}_2, \bar{y}_2, 0)$.

Then,

(I) $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{w}_1, \bar{w}_2, \bar{r}_1 = 0, \bar{r}_2 = 0)$ is feasible for $(MNHD)$ and
(II) $L(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2, \bar{p}_1, \bar{p}_2) = M(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{w}_1, \bar{w}_2, \bar{r}_1, \bar{r}_2)$.

Furthermore, if the assumptions of Theorem 3.1 or 3.2 are satisfied $\forall$ P^0 and Q^0 , then $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{w}_1, \bar{w}_2, \bar{r}_1 = 0, \bar{r}_2 = 0)$ is an optimal solution for $(MNHD)$.

Proof:
Since $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2, \bar{p}_1, \bar{p}_2)$ is an optimal solution of $(MNHP)$, by the Fritz John necessary optimality conditions [16], there exist $\alpha, \gamma \in R_+, \delta_1 \in R, \delta_2 \in R$, $\beta_1 \in R^{|K_1|}$, $\beta_2, \zeta \in R^{|K_2|}$, $\xi_1 \in R^{|J_1|}$, $\xi_2 \in R^{|J_2|}$ such that the following conditions are satisfied at $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{\bar{z}}_1, \bar{z}_2, \bar{p}_1, \bar{p}_2)$:

$$(x_1 - \bar{x}_1)^T \Bigg(\alpha[\nabla_{x_1} f_1(\bar{x}_1, \bar{y}_1) + \xi_1 + \nabla_{x_1} h_1(\bar{x}_1, \bar{y}_1, \bar{p}_1) - \nabla_{p_1x_1} h_1(\bar{x}_1, \bar{y}_1, \bar{p}_1)\bar{p}_1]$$
$$+ [\nabla_{y_1x_1} f_1(\bar{x}_1, \bar{y}_1) + \nabla_{p_1x_1} h_1(\bar{x}_1, \bar{y}_1, \bar{p}_1)](\beta_1 - \gamma\bar{y}_1 - \delta_1\bar{p}_1) \Bigg) \geqq 0, \ \forall\, x_1 \in C_3, \quad (25)$$

$$(x_2 - \bar{x}_2)^T \Bigg(\alpha[\nabla_{x_2} f_2(\bar{x}_2, \bar{y}_2) + \xi_2 + \nabla_{x_2} h_2(\bar{x}_2, \bar{y}_2, \bar{p}_2)] + \{\nabla_{p_2x_2} h_2(\bar{x}_2, \bar{y}_2, \bar{p}_2)\}$$
$$(\beta_2 - \alpha\bar{y}_2 - \alpha\bar{p}_2 - \delta_2\bar{p}_2) + \{\nabla_{y_2x_2} f_2(\bar{x}_2, \bar{y}_2)\}(\beta_2 - \alpha\bar{y}_2 - \delta_2\bar{p}_2) \geqq 0, \ \forall\, x_2 \in C_4, \quad (26)$$

$$\alpha[\nabla_{y_1} f_1(\bar{x}_1,\bar{y}_1) - \bar{z}_1 + \nabla_{y_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1) - \nabla_{p_1 y_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1)\bar{p}_1] + (\nabla_{y_1 y_1} f_1(\bar{x}_1,\bar{y}_1)$$

$$+\nabla_{p_1 y_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1))(\beta_1 - \gamma\bar{y}_1 - \delta_1\bar{p}_1) - \gamma[\nabla_{y_1} f_1(\bar{x}_1,\bar{y}_1) - \bar{z}_1 + \nabla_{p_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1)] = 0, \quad (27)$$

$$\{\nabla_{p_2 y_2} h_2(\bar{x}_2,\bar{y}_2,\bar{p}_2)\}(\beta_2 - \alpha\bar{y}_2 - \alpha\bar{p}_2 - \delta_2\bar{p}_2) + \alpha[\nabla_{y_2} h_2(\bar{x}_2,\bar{y}_2,\bar{p}_2)$$
$$-\nabla_{p_2} h_2(\bar{x}_2,\bar{y}_2,\bar{p}_2)] + \{\nabla_{y_2 y_2} f_2(\bar{x}_2,\bar{y}_2)\}(\beta_2 - \alpha\bar{y}_2 - \delta_2\bar{p}_2) - \zeta = 0, \quad (28)$$

$$\{\nabla_{p_1 p_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1)\}(\beta_1 - \alpha\bar{p}_1 - \gamma\bar{y}_1 - \delta_1\bar{p}_1)$$
$$-\delta_1[\nabla_{y_1} f_1(\bar{x}_1,\bar{y}_1) - \bar{z}_1 + \nabla_{p_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1)] = 0, \quad (29)$$

$$\{\nabla_{p_2 p_2} h_2(\bar{x}_2,\bar{y}_2,\bar{p}_2)\}(\beta_2 - \alpha\bar{y}_2 - \alpha\bar{p}_2 - \delta_2\bar{p}_2)$$
$$-\delta_2[\nabla_{y_2} f_2(\bar{x}_2,\bar{y}_2) - \bar{z}_2 + \nabla_{p_2} h_2(\bar{x}_2,\bar{y}_2,\bar{p}_2)] = 0, \quad (30)$$

$$\beta_1[\nabla_{y_1} f_1(\bar{x}_1,\bar{y}_1) - \bar{z}_1 + \nabla_{p_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1)] = 0, \quad (31)$$
$$\beta_2[\nabla_{y_2} f_2(\bar{x}_2,\bar{y}_2) - \bar{z}_2 + \nabla_{p_2} h_2(\bar{x}_2,\bar{y}_2,\bar{p}_2)] = 0, \quad (32)$$
$$\gamma\bar{y}_1[\nabla_{y_1} f_1(\bar{x}_1,\bar{y}_1) - \bar{z}_1 + \nabla_{p_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1)] = 0, \quad (33)$$
$$\delta_1\bar{p}_1[\nabla_{y_1} f_1(\bar{x}_1,\bar{y}_1) - \bar{z}_1 + \nabla_{p_1} h_1(\bar{x}_1,\bar{y}_1,\bar{p}_1)] = 0, \quad (34)$$
$$\delta_2\bar{p}_2[\nabla_{y_2} f_2(\bar{x}_2,\bar{y}_2) - \bar{z}_2 + \nabla_{p_2} h_2(\bar{x}_2,\bar{y}_2,\bar{p}_2)] = 0, \quad (35)$$
$$(\alpha - \gamma^1)y_1 + \beta_1 - \delta_1 p_1 \in N_{D_1}(\bar{z}_1), \quad (36)$$
$$\beta_2 - \delta_2 p_2 \in N_{D_2}(\bar{z}_2), \quad (37)$$
$${\xi_1}^T\bar{x}_1 = s(x_1|E_1), \xi_1 \in E_1, \quad (38)$$
$${\xi_2}^T\bar{x}_2 = s(x_2|E_2), \xi_2 \in E_2, \quad (39)$$
$$\mu_1\bar{x}_1 = 0, \quad (40)$$
$$\mu_2\bar{x}_2 = 0, \quad (41)$$
$$\zeta\bar{y}_2 = 0, \quad (42)$$
$$(\alpha,\beta_1,\beta_2,\gamma,\delta_1,\delta_2,\zeta) \neq 0, \quad (43)$$
$$(\alpha,\beta_1,\beta_2,\gamma,\delta_1,\delta_2,\zeta) \geqq 0, \quad (44)$$

Remaining proof of Theorem 3.3 follows on the lines of [19]

Theorem 3.4 (Converse Duality). Let $(\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2,\bar{w}_1,\bar{w}_2,\bar{r}_1,\bar{r}_2)$ be an optimal solution of $(MNHD)$. Suppose that

(i) $\nabla_{r_1 r_1} g_1(\bar{u}_1,\bar{v}_1,\bar{r}_1)$ is positive or negative definite and $\nabla_{r_2 r_2} g_2(\bar{u}_2,\bar{v}_2,\bar{r}_2)$ is negative definite,
(ii) $\nabla_{u_1} f_1(\bar{u}_1,\bar{v}_1) + \bar{w}_1 + \nabla_{r_1} g_1(\bar{u}_1,\bar{v}_1,\bar{r}_1) \neq 0$ and
$v_2[\nabla_{y_2} f_2(\bar{x}_2,\bar{y}_2) + \bar{w}_2 + \nabla_{p_2} h_2(\bar{x}_2,\bar{y}_2,\bar{p}_2)] \neq 0$,

(iii) $(\bar{r}_1)^T[\nabla_{u_1} f_1(\bar{u}_1, \bar{v}_1) + \bar{w}_1 + \nabla_{r_1} g_1(\bar{u}_1, \bar{v}_1, \bar{r}_1)] = 0 \Rightarrow \bar{r}_1 = 0$ and
$u_2[\nabla_{u_2} g_2(\bar{u}_2, \bar{v}_2, \bar{r}_2) - \nabla_{r_2} g_2(\bar{u}_2, \bar{v}_2, \bar{r}_2) + \nabla_{u_2 u_2} f_2(\bar{u}_2, \bar{v}_2)\bar{r}_2] = 0 \Rightarrow \bar{r}_2 = 0,$
(iv) $g_1(\bar{u}_1, \bar{v}_1, 0) = g_1(\bar{u}_1, \bar{v}_1, 0)$, $\nabla_{u_1} g_1(\bar{u}_1, \bar{v}_1, 0) = \nabla_{r_1} g_1(\bar{u}_1, \bar{v}_1, 0)$,
$\nabla_{v_1} g_1(\bar{u}_1, \bar{v}_1, 0) = \nabla_{r_1} g_1(\bar{u}_1, \bar{v}_1, 0)$ and $g_2(\bar{u}_2, \bar{v}_2, 0) = h_2(\bar{u}_2, \bar{v}_2, 0)$,
$\nabla_{u_2} g_2(\bar{u}_2, \bar{v}_2, 0) = \nabla_{r_2} h_2(\bar{u}_2, \bar{v}_2, 0)$.

Then,

(I) $(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2, \bar{z}_1, \bar{z}_2, \bar{p}_1 = 0, \bar{p}_2 = 0)$ is feasible for $(MNHP)$ and
(II) $L(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{z}_1, \bar{z}_2, \bar{v}_2, \bar{p}_1, \bar{p}_2) = M(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2, \bar{w}_1, \bar{w}_2, \bar{r}_1, \bar{r}_2)$.

Furthermore, if the hypotheses of Theorem 3.1 or 3.2 are satisfied for all feasible solutions of $(MNHP)$ and $(MNHD)$, then $(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2, \bar{z}_1, \bar{z}_2, \bar{p}_1 = 0, \bar{p}_2 = 0)$ is an optimal solution for $(MNHP)$.

Proof. Follows on the line of Theorem 3.3.

4. CONCLUSIONS

A pair of non-differentiable mixed type symmetric dual programs has been formulated over arbitrary cone constraints by considering the optimization under the assumptions of η-pseudoinvexity/η-invexity/C-pseudoconvexity/C-convexity. For future prospectus, the primal-dual problems may be extended for mixed symmetric duality in integer for scalar and multi-objective programming over cone constraints. The present work can also be extended to nondifferentiable higher order symmetric fractional programming over arbitrary cones. This will orient the future task of the authors.

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