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Research Article

EFFICIENT OPTIMAL CONTROL PROBLEM SOLUTION FOR CAR-LIKE MOBILE ROBOTS WITH THE PICARD-SHOOTING METHOD

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Abstract: This article investigates the optimal control problem of a Car-Like mobile robot, with our primary objective being the identification of the optimal control strategy

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that facilitates the transition from an initial point to an endpoint in the shortest possible time. To address this challenge, we employ Pontryagin's Maximum Principle, coupling it with the shooting method to determine the initial condition of the adjoint state (p_0), which is then integrated through the iterative Picard method. The validation of our approach is demonstrated through a numerical example utilizing real-world data, and we provide a comparative analysis of our results against those from previous studies for comprehensive assessment.

Keywords: Optimization, robotics, non-linear optimal control, Pontryagin's maximum principle, dynamic system, boundary value problems..

MSC: 34K10, 34K35, 65Kxx, 90C30, 93C10, 93C20, 93C95.

1. INTRODUCTION

Robots have made significant inroads across various domains in recent years, including industrial, agricultural, military, aeronautical, and medical sectors. Furthermore, they are increasingly becoming integral parts of our daily lives. The adoption of robots is driven by their capacity to enhance human existence by tackling tasks that are either beyond human capabilities or by simplifying those that would otherwise be arduous to accomplish.

Robotics and optimization are intricately intertwined fields, often collaboratively employed to tackle intricate challenges. Optimization, in essence, revolves around the quest for the most optimal solution to a given problem, achieved through the application of specialized algorithms. This discipline finds application across a broad spectrum of domains, spanning engineering, finance, and beyond.

Optimization algorithms serve as invaluable tools in the quest to identify the most precise parameters for governing and fine-tuning complex systems. Robotics, on the other hand, constitutes a branch of engineering that revolves around the conception, fabrication, and operation of robots. These multifaceted machines are proficient in executing a diverse array of tasks.

Effectively controlling a robot necessitates the utilization of optimization algorithms to ascertain the aptest parameters governing the robot's motion and behaviour. One pertinent issue in the realm of optimization involves determining how to maximize or minimize the robot's performance, trajectory, or energy consumption, often predicated upon predefined criteria.

Typically, such optimization challenges manifest in the framework of optimal control problems. The quest is to unearth the most effective control strategies, allowing for the accomplishment of predefined objectives set by the robot's operator, all the while adhering to specific constraints governing the system's behaviour. Optimal control theory serves as the enabling mechanism for achieving this, involving the mathematical modelling of system dynamics and the delineation of performance criteria.

Notably, in the 1960s and 1970s, researchers harnessed optimal control principles for both the design and orchestration of robotic systems. This pivotal convergence laid the foundation for the development of advanced robotic systems endowed with the capability to execute intricate tasks across diverse environments. In the contemporary era, optimal control is ubiquitous in robotics, facilitating the design of control systems that optimize performance, curtail energy consumption, and enhance precision. In essence, the fusion of optimal control and robotics has propelled the creation of formidable and highly sophisticated robotic systems.

Addressing optimization problems in the realm of robotics is by no means a novel undertaking. A significant body of work has been dedicated to this domain, each with its unique characteristics varying across different types of robots [1, 2, 3, 4], spanning aquatic, airborne, underwater, and more. However, the majority of optimal control problems in this field find their solutions through numerical methods, with only a handful being amenable to analytical resolutions.

Indeed, when faced with a specific problem, there exists a plethora of techniques that can be employed. It's crucial to recognize that numerical methods carry both merits and demerits, including issues pertaining to stability, sensitivity to initial conditions, precision, convergence rate, and computational complexity [5, 6]. There's no panacea that can proficiently address all problem types. The ultimate results and precision levels are contingent upon the choices made during the problem-solving process. Mathematicians often find themselves in the position of having to explore various techniques, subsequently comparing the outcomes of each resolution algorithm. This process assists in determining the optimal method, considering factors like precision, convergence time, etc.

This paper addresses the optimization problem concerning a car-like robot, a model previously explored in [4, 7]. Herein, we introduce a novel approach for solving this problem, regardless of the initial and final states, by leveraging a combination of Picard's method and the shooting method. Picard's method, a numerical technique used to solve differential equations, stands out for its intuitiveness, akin to Euler's method. It finds its niche in the realm of nonlinear differential equations that defy analytical solutions. The method involves the decomposition of the problem into a series of more manageable subproblems, each simpler to solve than the original conundrum. In the context of optimal control problems, Picard's method can be employed to iteratively tackle the underlying differential equations. During each iteration, the control input can be updated based on the current system state and the desired performance index. This iterative process continues until the performance index is optimized or a satisfactory solution is attained.

This study entails a comparative analysis between our findings and those presented in [7]. The primary objective of this research is to determine an optimal control strategy for reaching a specified destination from a given initial state in the shortest time possible.

Our approach commences with the mathematical modelling of the problem, expounded in Section 2. This entails the computation of the system's Hamiltonian and Euler-Lagrange equations, culminating in the application of Pontryagin's Maximum Principle (PMP) to derive the optimal control. Additionally, we elucidate the transversality condition imposed on the system and provide an in-depth exploration of E. Picard's iterative method, which we subsequently apply to this control problem in Section 3. Section 4 delves into the shooting method employed to resolve the boundary-value problems (BVP) aimed at determining the initial condition (p_0) of the differential system [1, 7, 8, 9, 10, 11, 12, 13]. The results of our numerical application are presented in Section 5, with further discussions and analysis featured in Section 6. We conclude with a summary and outline of potential future avenues in Section 7.

Table 1: Nomenclature

Symbol	Usage
$\mathbb{R}^n$	real n-dimensional space
$x(t)$	state(position) of a robot at time t, $x(t) \in \mathbb{R}^3$
$p(t)$	adjoint state, $p(t) \in \mathbb{R}^3$
$\Longrightarrow$	implies
$\in$	element inclusion
$\subset$	set inclusion
t_0	initial time
t_f	final time
H	Hamiltonian of the system
$\dot{f}$	the first derivative of f at t
$\frac{\partial f}{\partial x}$	the partial derivative with respect to x
$x^{(k)}$	the iteration k of x
max	maximum
min	minimum
$\lvert.\rvert$	absolute value

2. STATEMENT OF THE PROBLEM

According to [1], [3], [4] and [7], the modeling of the problem is (Figure 1):

$$J = t_f \to min \tag{1}$$

$$\begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta}_r \end{bmatrix} = \begin{bmatrix} \cos\theta_r & 0 \\ \sin\theta_r & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}. \tag{2}$$

With the boundary conditions:

$$x(0) = \begin{bmatrix} x_{r0} \\ y_{r0} \\ \theta_{r0} \end{bmatrix}, \; x(t_f) = \begin{bmatrix} x_{rf} \\ y_{rf} \\ \theta_{rf} \end{bmatrix} \tag{3}$$

The constraint on the control $U = (v, w)$ is: $U \in \Omega$ such as Ω is a compact with:

$$\Omega = \{(v,w)/-1 \leq v \leq +1, -1 \leq w \leq +1\}$$

The equality (1) can be written as follows:

$$J = t_f = \int_0^{t_f} dt \to min \tag{4}$$

The Hamiltonian of the system is given by:

$$H(t, x(t), p(t), p^0, U(t)) = p_1(t) v \cos\theta_r + p_2(t) v \sin\theta_r + p_3(t) w + p^0, \tag{5}$$

$x(t) = \begin{pmatrix} x_r(t) \\ y_r(t) \\ \theta_r(t) \end{pmatrix}$ the state of the system at time t, $p(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix}$ the adjoint state, $p^0 \leq 0$.

The condition of maximization (according to [14], [15], [16], [17],) is:

$$H(t, x(t), p(t), p^0, U(t)) = \max_{\omega \in \Omega} H(t, x(t), p(t), p^0, \omega), \tag{6}$$

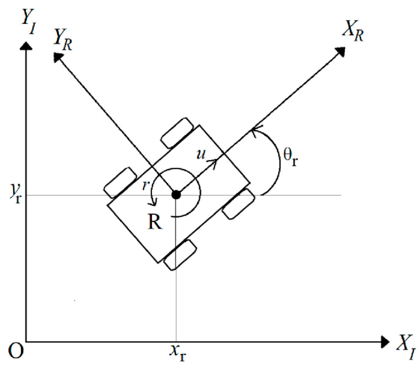

Figure 1: Kinematics of a robot.

with:

$$\dot{x}(t) = \frac{\partial H}{\partial p}(t, x(t), p(t), p^0, U(t)), \tag{7}$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(t, x(t), p(t), p^0, U(t)), \tag{8}$$

$p(t)$ nontrivial adjoint vector $(p(.), p^0) \neq (0,0)$.

According to the equations (7) and (8) the system is:

$$\begin{cases} \dot{x}_r = v\cos\theta_r \\ \dot{y}_r = v\sin\theta_r \\ \dot{\theta}_r = w \\ \dot{p_1} = 0 \\ \dot{p_2} = 0 \\ \dot{p_3} = p_1 v\sin\theta_r - p_2 v\cos\theta_r \end{cases} \tag{9}$$

Equation (6) is equivalent to:

$$H(t, x(t), p(t), p^0, U(t)) = \max_{|w|\leq 1, |v|\leq 1} [p_1(t)\cos\theta_r + p_2(t)\sin\theta_r]\, v + p_3(t)w - 1.$$

Let $\psi(t)$ such as:

$$\psi(t) = p_1(t)\cos\theta_r + p_2(t)\sin\theta_r,$$

$$H(t, x(t), p(t), p^0, U(t)) = \max_{|w|\leq 1, |v|\leq 1} \psi(t)v + p_3(t)w - 1,$$

we deduce:

$$v = sign(\psi(t)), w = sign(p_3(t)). \tag{10}$$

2.1. Transversality condition

The condition of transversality on the Hamiltonian is defined by the following equality:

$$H(t_f, x(t_f), p(t_f), p^0, U(t_f)) = 0. \tag{11}$$

This implies that the Hamiltonian terminates at the final time. For an in-depth exploration of the transversality condition, refer to [15] and [16].

2.2. The switching functions

From the equalities (10) we deduce:

$$v = \begin{cases} +1 & \text{if} \quad \psi(t) > 0, \\ -1 & \text{if} \quad \psi(t) < 0, \\ \in [-1, +1] & \text{if} \quad \psi(t) = 0. \end{cases} \tag{12}$$

and

$$w = \begin{cases} +1 & \text{if} \quad p_3(t) > 0, \\ -1 & \text{if} \quad p_3(t) < 0, \\ \in [-1, +1] & \text{if} \quad p_3(t) = 0. \end{cases} \tag{13}$$

$\psi(.)$, $p_3(.)$ are the switching functions, determine where the controls can switch, the last case of control for linear velocity ($v \in [-1, +1]$) and the last case for angular velocity ($w \in [-1, +1]$) indicate when the switches occur, where the control is discontinuous and changes from one trajectory segment to another when $\psi(t) = 0$ for v or $p_3(t) = 0$ for w. These switches occur only in isolated instances and the singular controls do not exist, the demonstration involves the Lie bracket and the Lie algebras; for more details we can refer to [4]. It is important to note the high frequency of these switches can complicate the integration of the Boundary-Value Problems (BVP), thereby posing challenges in the application of the shooting method.

3. PICARD'S METHOD

This method was proposed by E. Picard in 1890 [18], [19], [20].
Let the following differential system:

$$\begin{cases} \dfrac{dy_1}{dt} = f_1(t, y_1, y_2, ..., y_m) \\ \dfrac{dy_2}{dt} = f_2(t, y_1, y_2, ..., y_m) \\ . \\ . \\ . \\ \dfrac{dy_m}{dt} = f_m(t, y_1, y_2, ..., y_m) \end{cases} \tag{14}$$

We assume that the functions (f_i) are real continuous in the neighbourhood of t_0, $y_i^0 = y_i(t_0)$ for $i = 1, ...m$. The functions (f_i) are defined in $t \in [t_0 - \alpha, t_0 + \alpha]$ and $y_i \in [y_i^0 - \beta, y_i^0 + \beta]$, $\alpha, \beta > 0$. Let $M > 0$: $|f_i(t, y_1, y_2, ..., y_m)| \leqslant M$ and we assume that there exist positive constants: $\gamma_1, \gamma_2, ..., \gamma_m \in \mathbb{R}_+$ such as $|f_i(t, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_m) - f_i(t, y_1, y_2, ..., y_m)| \leqslant \gamma_1|\tilde{y}_1 - y_1| + \gamma_2|\tilde{y}_2 - y_2| + ... + \gamma_m|\tilde{y}_m - y_m|$.

Let a sequence $y_i^{(n+1)}$ $(i = 1, ...m)$ defined as follows:

$$\begin{cases} \dfrac{dy_i^{(n+1)}(t)}{dt} = f_i(t, y_1^{(n)}, y_2^{(n)}, ..., y_i^{(n)} ..., y_m^{(n)}) \\ y_i^0 = y_i(t_0) \end{cases} \tag{15}$$

The sequence (15) uniformly converges to the analytical solution $y_i(t)$ of:
$\dfrac{dy_i(t)}{dt} = f_i(t, y_1, y_2, ...y_i..., y_m)$ and:

$$\lim_{n \to +\infty} y_i^{(n)}(t) = y_i(t)$$

3.1. Application of the method to the problem

The functions cos, sin are Lipschitzian, in fact:
$|\cos t - \cos \tilde{t}| = |(-\sin \zeta)(t - \tilde{t})| = |-\sin \zeta||(t - \tilde{t})| \leqslant |t - \tilde{t}|$,
$|\sin t - \sin \tilde{t}| = |(\cos \eta)(t - \tilde{t})| = |\cos \eta||(t - \tilde{t})| \leqslant |t - \tilde{t}|$.
$\zeta, \eta \in]t, \tilde{t}[$ are given by the mean value theorem of the functions cos and sin, so any linear combination is still lipschitzian.

From (2.2) we deduce:

$$U = \{(+1, +1), (+1, -1), (-1, +1), (-1, -1)\}.$$

To address the control problem described in Equation (9), we identify this various cases:

1. $(v, w) = (+1, +1)$ i.e. $(p_1(t) \cos \theta_r + p_2(t) \sin \theta_r > 0)$ and ($p_3(t) > 0$).
   Then the system (9) becomes:

$$\begin{cases} \dot{x}_r = \cos \theta_r \\ \dot{y}_r = \sin \theta_r \\ \dot{\theta}_r = 1 \\ \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = p_1 \sin \theta_r - p_2 \cos \theta_r \end{cases} \tag{16}$$

   Applying Picard's successive approximation method with the initial condition: $x_r^{(0)} = x_r(t_0) = x_{r0}$, $y_r^{(0)} = y_r(t_0) = y_{r0}$, $\theta_r^{(0)} = \theta_r(t_0) = \theta_{r0}$, $p_1^{(0)} = p_1(t_0)$, $p_2^{(0)} = p_2(t_0)$, $p_3^{(0)} = p_3(t_0)$.
   The iteration (1) is :
   $x_r^{(1)} = x_r^{(0)} + \int_{t_0}^{t} \cos \theta_r^{(0)} ds = x_r^{(0)} + \left[\cos \theta_r^{(0)} s\right]_{s=t_0}^{s=t} = x_{r0} + (t - t_0) \cos \theta_{r0}$,
   $y_r^{(1)} = y_r^{(0)} + \int_{t_0}^{t} \sin \theta_r^{(0)} ds = y_r^{(0)} + \left[\sin \theta_r^{(0)} s\right]_{s=t_0}^{s=t} = y_{r0} + (t - t_0) \sin \theta_{r0}$,

$\theta_r^{(1)} = \theta_r^{(0)} + \int_{t_0}^{t} 1 ds = \theta_r^{(0)} + [s]_{s=t_0}^{s=t} = \theta_{r0} + (t - t_0)$,
$p_1^{(1)} = p_1(t_0) = c_1$,
$p_2^{(1)} = p_2(t_0) = c_2$,
$p_3^{(1)} = p_3(t_0) + c_1 \int_{t_0}^{t} \sin\theta_r^{(0)} ds - c_2 \int_{t_0}^{t} \cos\theta_r^{(0)} ds = p_3(t_0) + (c_1 \sin\theta_{r0} - c_2 \cos\theta_{r0})(t - t_0)$.
So the iteration $(n+1)$, $(n \geqslant 1)$is:
$x_r^{(n+1)} = x_r^{(0)} + \int_{t_0}^{t} \cos\theta_r^{(n)} ds = x_r^{(0)} + \int_{t_0}^{t} \cos(\theta_r^{(0)} + (s - t_0)) ds = x_{r0} + \sin(\theta_{r0} + (t - t_0)) - \sin\theta_{r0}$,
$y_r^{(n+1)} = y_r^{(0)} + \int_{t_0}^{t} \sin\theta_r^{(n)} ds = y_r^{(0)} + \int_{t_0}^{t} \sin(\theta_r^{(0)} + (s - t_0)) ds = y_{r0} - \cos(\theta_{r0} + (t - t_0)) + \cos\theta_{r0}$,
$\theta_r^{(n+1)} = \theta_r^{(0)} + \int_{t_0}^{t} 1 ds = \theta_{r0} + (t - t_0)$,
$p_1^{(n+1)} = c_1$,
$p_2^{(n+1)} = c_2$,
$p_3^{(n+1)} = p_3(t_0) + c_1 \int_{t_0}^{t} \sin\theta_r^{(n)} ds - c_2 \int_{t_0}^{t} \cos\theta_r^{(n)} ds = p_3(t_0) + c_1[-\cos(\theta_{r0} + (t - t_0)) + \cos(\theta_{r0})] - c_2[\sin(\theta_{r0} + (t - t_0)) - \sin\theta_{r0}]$,
this control is valid if $(p_1(t)\cos\theta_r + p_2(t)\sin\theta_r > 0)$ and $(p_3(t) > 0)$, so we determine t such that both of these conditions are satisfied.

2. $(v, w) = (+1, -1)$ i.e. $(p_1(t)\cos\theta_r + p_2(t)\sin\theta_r > 0)$ and $(p_3(t) < 0)$.
The system (9) becomes:

$$\begin{cases} \dot{x}_r = \cos\theta_r \\ \dot{y}_r = \sin\theta_r \\ \dot{\theta} = -1 \\ \dot{p_1} = 0 \\ \dot{p_2} = 0 \\ \dot{p_3} = p_1 \sin\theta_r - p_2 \cos\theta_r \end{cases} \tag{17}$$

The iteration $(n+1)$, $(n \geqslant 0)$ is given by:
$x_r^{(n+1)} = x_r^{(0)} + \int_{t_0}^{t} \cos\theta_r^{(n)} ds = x_r^{(0)} + \int_{t_0}^{t} \cos(\theta_r^{(0)} - (s - t_0)) ds = x_{r0} - \sin(\theta_{r0} - (t - t_0)) + \sin\theta_{r0}$,
$y_r^{(n+1)} = y_r^{(0)} + \int_{t_0}^{t} \sin\theta^{(n)} ds = y_r^{(0)} + \int_{t_0}^{t} \sin(\theta_r^{(0)} - (s - t_0)) ds = y_{r0} + \cos(\theta_{r0} - (t - t_0)) - \cos\theta_{r0}$,
$\theta_r^{(n+1)} = \theta_r^{(0)} + \int_{t_0}^{t} -1 ds = \theta_{r0} - (t - t_0)$,
$p_1^{(n+1)} = p_1(t_0) = c_1$,
$p_2^{(n+1)} = p_2(t_0) = c_2$,
$p_3^{(n+1)} = p_3(t_0) + c_1 \int_{t_0}^{t} \sin\theta_r^{(n)} ds - c_2 \int_{t_0}^{t} \cos\theta_r^{(n)} ds = p_3(t_0) + c_1[\cos(\theta_{r0} - (t - t_0)) - \cos\theta_{r0}] + c_2[\sin(\theta_{r0} - (t - t_0)) - \sin\theta_{r0}]$.
In this case, we also determine t such that the condition mentioned earlier is satisfied.

3. $(v, w) = (-1, +1)$ i.e. $(p_1(t)\cos\theta_r + p_2(t)\sin\theta_r < 0)$ and $(p_3(t) > 0)$.

The system (9) becomes:

$$\begin{cases} \dot{x}_r = -\cos\theta_r \\ \dot{y}_r = -\sin\theta_r \\ \dot{\theta}_r = +1 \\ \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -p_1\sin\theta_r + p_2\cos\theta_r \end{cases} \tag{18}$$

The $(n+1)^{th}$ iteration ($n \geqslant 0$) is:
$x_r^{(n+1)} = x_r^{(0)} - \int_{t_0}^{t} \cos\theta_r^{(n)} ds = x_r^{(0)} - \int_{t_0}^{t} \cos(\theta_r^{(0)} + (s-t_0))ds = x_{r0} - \sin(\theta_{r0} + (t-t_0)) + \sin\theta_{r0}$,
$y_r^{(n+1)} = y_r^{(0)} - \int_{t_0}^{t} \sin\theta_r^{(n)} ds = y_r^{(0)} - \int_{t_0}^{t} \sin(\theta_r^{(0)} + (s-t_0))ds = y_{r0} + \cos(\theta_{r0} + (t-t_0)) - \cos(\theta_{r0})$,
$\theta_r^{(n+1)} = \theta_r^{(0)} + \int_{t_0}^{t} 1 ds = \theta_{r0} + (t-t_0)$,
$p_1^{(n+1)} = p_1(t_0) = c_1$,
$p_2^{(n+1)} = p_2(t_0) = c_2$,
$p_3^{(n+1)} = p_3(t_0) - c_1\int_{t_0}^{t} \sin\theta_r^{(n)} ds + c_2\int_{t_0}^{t} \cos\theta_r^{(n)} ds = p_3(t_0) + c_1[\cos(\theta_{r0} + (t-t_0)) - \cos\theta_{r0}] + c_2[\sin(\theta_{r0} + (t-t_0)) - \sin\theta_{r0}]$.
We determine t such that the condition ($p_1(t)\cos\theta_r + p_2(t)\sin\theta_r < 0$) and ($p_3(t) > 0$) is verified.

4. $(v,w) = (-1,-1)$ i.e. ($p_1(t)\cos\theta_r + p_2(t)\sin\theta_r < 0$) and ($p_3(t) < 0$)
In this case, the system (9) is:

$$\begin{cases} \dot{x}_r = -\cos\theta_r \\ \dot{y}_r = -\sin\theta_r \\ \dot{\theta}_r = -1 \\ \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -p_1\sin\theta_r + p_2\cos\theta_r \end{cases} \tag{19}$$

The $(n+1)^{th}$ iteration ($n \geqslant 0$) is:
$x_r^{(n+1)} = x_r^{(0)} - \int_{t_0}^{t} \cos\theta_r^{(n)} ds = x_r^{(0)} - \int_{t_0}^{t} \cos(\theta_r^{(0)} - (s-t_0))ds = x_{r0} + \sin(\theta_{r0} - (t-t_0)) - \sin\theta_{r0}$,
$y_r^{(n+1)} = y_r^{(0)} - \int_{t_0}^{t} \sin\theta_r^{(n)} ds = y_r^{(0)} - \int_{t_0}^{t} \sin(\theta_r^{(0)} - (s-t_0))ds = y_{r0} - \cos(\theta_{r0} - (t-t_0)) + \cos(\theta_{r0})$,
$\theta_r^{(n+1)} = \theta_r^{(0)} + \int_{t_0}^{t} -1 ds = \theta_{r0} - (t-t_0)$,
$p_1^{(n+1)} = p_1(t_0) = c_1$,
$p_2^{(n+1)} = p_2(t_0) = c_2$,
$p_3^{(n+1)} = p_3(t_0) - c_1\int_{t_0}^{t} \sin\theta_r^{(n)} ds + c_2\int_{t_0}^{t} \cos\theta_r^{(n)} ds = p_3(t_0) - c_1[\cos(\theta_{r0} - (t-t_0)) - \cos(\theta_{r0})] + c_2[-\sin(\theta_{r0} - (t-t_0)) + \sin\theta_{r0}]$.
In this context, we also determine t such that the condition mentioned earlier is satisfied.

Based on the analysis provided above, we deduce that the robot's optimal trajectory consists of a series of circles. In fact, for the command $(+1,+1)$, we have:

$$\begin{cases} x_r^{(n+1)} = x_r^{(0)} + \sin\left(\theta_r^{(0)} + (t-t_0)\right) - \sin\theta_r^{(0)} \\ y_r^{(n+1)} = y_r^{(0)} - \cos\left(\theta_r^{(0)} + (t-t_0)\right) + \cos\theta_r^{(0)} \end{cases}$$

$$\Longrightarrow \begin{cases} x_r^{(n+1)} - x_r^{(0)} + \sin\theta_r^{(0)} = \sin\left(\theta_r^{(0)} + (t-t_0)\right) \\ y_r^{(n+1)} - y_r^{(0)} - \cos\theta_r^{(0)} = -\cos\left(\theta_r^{(0)} + (t-t_0)\right) \end{cases}$$

so: $$\begin{cases} \left[x_r^{(n+1)} - (x_r^{(0)} - \sin\theta_r^{(0)})\right]^2 = \left[\sin(\theta_r^{(0)} + (t-t_0))\right]^2 \\ \left[y_r^{(n+1)} - (y_r^{(0)} + \cos\theta_r^{(0)})\right]^2 = \left[-\cos(\theta_r^{(0)} + (t-t_0))\right]^2 \end{cases}$$

$\left(x_r^{(n+1)} - (x_r^{(0)} - \sin\theta_r^{(0)})\right)^2 + \left(y_r^{(n+1)} - (y_r^{(0)} + \cos\theta_r^{(0)})\right)^2 = 1$, and this is an equation of a circle with radius 1 and center O, $O(x_{r0} - \sin\theta_{r0}\ ;\ y_{r0} + \cos\theta_{r0})$. We proceed in the same way as the other commands.

4. SHOOTING METHOD

To utilize the outcomes of the preceding section, we employ the shooting method for solving boundary-value problems. This technique enables us to determine the initial condition of the adjoint state, denoted as $p(t_0)$. For a comprehensive resource on the shooting method, you can refer to the classic book [21].
The boundary-value problems (BVP) are:

$$\begin{cases} \dot{x}_r = v\cos\theta_r \\ \dot{y}_r = v\sin\theta_r \\ \dot{\theta} = w \\ \dot{p_1} = 0 \\ \dot{p_2} = 0 \\ \dot{p_3} = p_1 v\sin\theta_r - p_2 v\cos\theta_r \\ v = sign(p_1\cos\theta_r + p_2\sin\theta_r) \\ w = sign(p_3) \\ x_0 = x(0),\ x(t_f) = x_f \end{cases} \tag{20}$$

We define the shooting function S as follows:
let $p_0, x_0, x_f \in \mathbb{R}^3$ such as :

$$S : \left| \begin{array}{lll} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ p_0 & \longmapsto & S(p_0) = x(t_f, x_0, p_0) - x_f,\ p_0 = p(0),\ x_0 = x(0),\ x(t_f) = x_f. \end{array} \right.$$

$$S(p_0) = \begin{pmatrix} x_r(t_f, x_0, p_0) - x_{rf} \\ y_r(t_f, x_0, p_0) - y_{rf} \\ \theta_r(t_f, x_0, p_0) - \theta_{rf} \end{pmatrix} \tag{21}$$

To solve problem (20), we must find p_0 such as :

$$S(p_0) = 0, \ p_0 \in \mathbb{R}^3 \tag{22}$$

The solution of equation (22) is the initial state of the adjoint vector $p(t)$ to reach the desired final state $x(t_f)$. We apply Newton's method to solve this equation, Newton's methods are known for their sensitivity to the initial conditions. The challenge lies in providing an appropriate initial value of p_0, denoted as $p_0^{(0)}$, to guarantee the convergence of the method.

Let $J_S(p_0)$ represent the Jacobian matrix associated with the shooting function S:

$$\Big(J_S(p_0)\Big)_{ij} = \left(\frac{\partial S_i}{\partial p_{0j}}\right), \ \ i,j = \overline{1,3}.$$

To solve the equation (22), we follow this procedure:
We start with an initial value $p_0^{(0)} \in \mathbb{R}^3$ for the variable p_0 and then iterate for $k = 0, 1, \ldots$ until convergence is achieved.
Find $\delta p_0^{(k)}$ solution of the linear equation:

$$J_S(p_0^{(k)})\delta p_0^{(k)} = -S(p_0^{(k)}),$$

set:

$$p_0^{(k+1)} = p_0^{(k)} + \delta p_0^{(k)}.$$

At each iteration k, we are required to solve a linear system involving the Jacobian matrix $J_S(p_0^{(k)})$.

The steps to solve the boundary-value problems (BVP) (20) are:

- Establish the initial and final states of the robot;
- The differential system (20) is integrated using the Picard method;
- Solve equation (22) by a Newton-type method.

5. NUMERICAL APPLICATION

In our numerical application, we will provide specific examples by specifying the initial state and the final state of the system. We will then employ the previously outlined algorithm to compute the optimal control, the trajectory, and the final time, and generate the associated graphical representations.

- Example 1: $x(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $x(t_f) = \begin{pmatrix} 4.87 \\ 0.85 \\ 0.30 \end{pmatrix}$, the trajectory consists of a sequence of interconnected circles and the final time is set to $t_f = 7.33s$.
  The robot's trajectory and switching times are detailed in Table 2, and you can refer to Figures (2), (3), (4), (5), (6), and (7) for graphical representations.

Table 2: Example 1

x_r	y_r	θ_r	switching time t
0	0	0	0
1	1	1.571	1.571
2	2	0	3.131
3	3	01.571	4.71
4	4	0	6.28
4.87	4.0001	1.0552	7.33

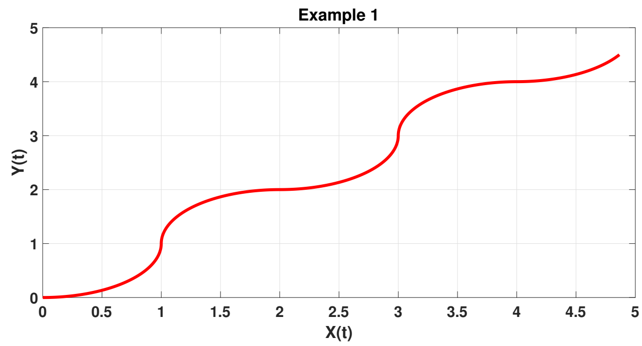

Figure 2: Trajectory optimization (example1)

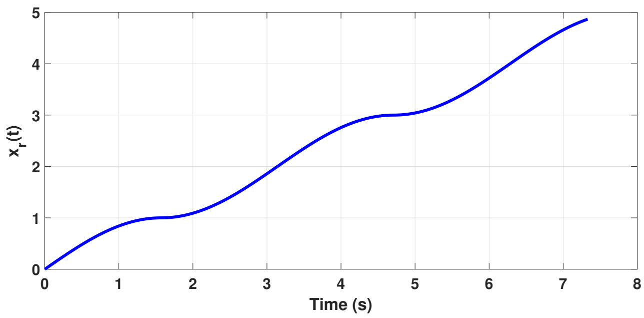

Figure 3: State evolution: x_r analysis

- Example 2: $x(142.9) = \begin{pmatrix} -3.85 \\ 0.74 \\ 5.28 \end{pmatrix}$, $x(t_f) = \begin{pmatrix} -0.38 \\ 0.26 \\ 6.25 \end{pmatrix}$.

  The trajectory consists of a sequence of interconnected circles, and the final time is defined as $t_f = 148.7s$. You can find detailed information in Table 3 and refer to Figure 8 for a visual representation.

Table 3: Example 2

x_r	y_r	θ_r	switching time t
−3.85	0.74	5.28	142.9
−2.0095	1.2037	7.7800	145.4
−2.0122	1.1338	7.8500	145.47
−2.0212	1.0042	07.7200	145.6
−0.0345	0.7783	4.62	148.7

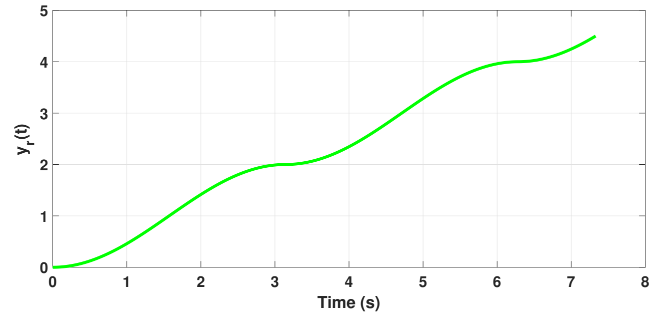

Figure 4: State evolution: y_r analysis

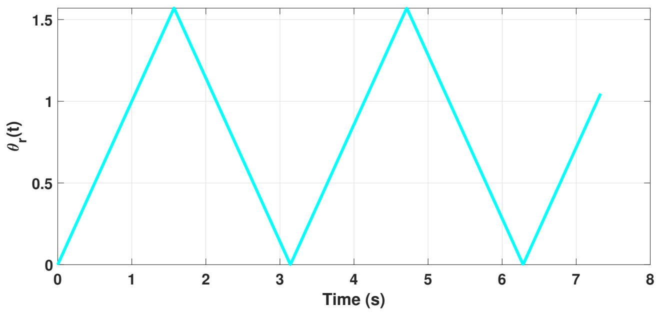

Figure 5: State evolution: θ_r analysis

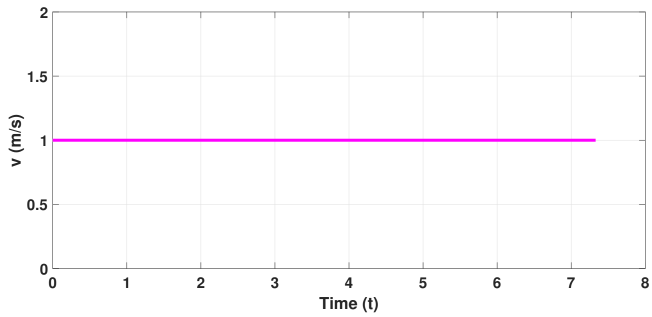

Figure 6: Optimal control: linear velocity v analysis

6. DISCUSSION

Our article addresses the optimal control problem for a car-like mobile robot. Our study demonstrates superior results when compared to the research conducted in [7]. Specifically, in Example 1, our approach achieves a significantly reduced time of $7.33s$ compared to the previous result of $14.2s$, while in Example 2, the final time is reduced to $148.7s$ from the previous value of $158.6s$. Additionally, our robot's trajectory closely approximates the analytical (exact) solution. This outcome is attributed to the uniform convergence of the Picard method employed in our study.

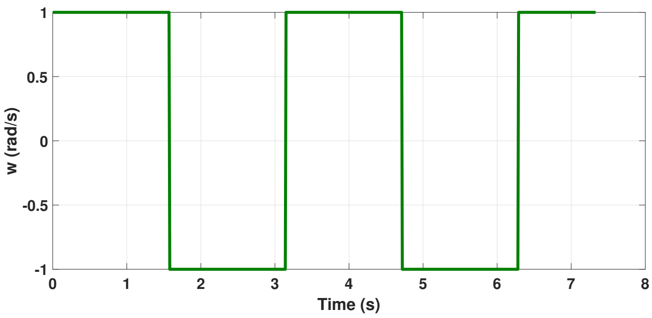

Figure 7: Optimal control: angular velocity w analysis

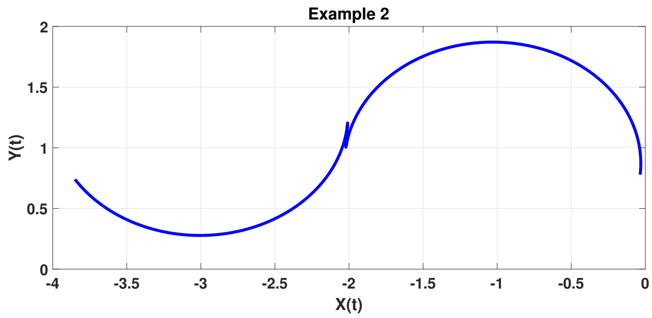

Figure 8: Trajectory optimization (example 2)

This study presents two significant challenges. The first is associated with the Newton method, which is notably sensitive to the initial conditions. The primary concern revolves around selecting the appropriate initial value of p_0, denoted as $p_0^{(0)}$, $p_0^{(0)}$ must be sufficiently close to the unknown solution p_0 to ensure convergence of the Newton method, this means that this method requires prior knowledge of the robot's optimal trajectory. A poor choice of $p_0^{(0)}$ runs the risk of divergence, meaning the solution may deviate from the final state of the system. In our examples, we managed to achieve convergence of the method, but we were unable to precisely reach the final state, although we came very close to it (as observed in example 1 and example 2). This limitation represents a notable drawback of the Newton method. The second challenge pertains to the number of switchings. A high number of switches complicates the application of the method, leading to increased complexity, particularly in terms of execution time.

It's noticed that this study does not account for constraints on the state of the system, denoted as $x(t)$, i.e. in this study we did not impose a path to follow by the robot to avoid, for example, obstacles, disturbances during its trajectory, which we propose as future directions in order to generalize this optimization problem, mathematically we add the constraint $x(t) \in X \subset \mathbb{R}^3$ to the problem, in this case we can distinguish two types of constraints: holonomic and nonholonomic. Adding constraints on state $x(t)$ is our current research focus.

This article is an investigation conducted under the absence of state constraints. The key distinction between this optimal control problem and one with state constraints is that the presence of state constraints can lead to discontinuities in the adjoint vector. To address this, junction points are introduced [17, 22, 23]. Despite these challenges and limitations, our approach yields superior results when compared to the study in [7]. Furthermore, the results obtained in our study closely approximate the exact solutions of the boundary-value problems (BVP).

7. CONCLUSION AND PERSPECTIVES

The primary objective of this work is to address the challenge of achieving optimal control for a Car-Like mobile robot, enabling it to transition from any initial state to any final state within the minimum time. To accomplish this, we applied Pontryagin's Maximum Principle (PMP) to derive the optimal control strategy. Our proposed approach, incorporating Picard's method and Newton's method to tackle the associated differential problem, outperforms the methods presented in [7], despite the challenges we encountered.

It is important to note that this study does not consider constraints on the state of the system, denoted as $x(t)$. The potential future research directions include:

- The extension of this study by introducing state constraints ($x(t) \in X \subset \mathbb{R}^3$), these constraints can be holonomic or nonholonomic, we investigate the applicability and efficiency of the proposed method on this type of constraints, and when the robot is confronted during its trajectory to obstacles and disturbances. Integration of state constraints improves the versatility of the method.
- Applying the proposed method to other robots, such as underwater robots (ROVs, AUVs), flying robots, multi-robot systems to extend the applicability of the method in the field of robotics.
- Exploration for the practical implementation of the proposed control method on a real car-like mobile robot, this leads to conducting experiments in a real controlled environment to test and validate the effectiveness of the approach.
- Combining optimal control theory with path planning algorithms.
- Improve the precision of trajectory tracking by minimizing the difference between the followed trajectory and the desired trajectory especially when the robot is subject to environmental disturbances.

These avenues pave the way for further advancements in the field of optimal control for Car-Like mobile robots and related domains in robotics and control theory.

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REFERENCES

[1] P. Corke, *Robotics, Vision and Control*. Springer, 2016. doi: 10.1007/978-3-319-54413-7

[2] B. Siciliano and O. Khatib, *Springer Handbook of Robotics*. Springer, 2016. doi: 10.1007/978-3-319-32552-1

[3] G. Cook, *Mobile Robots: Navigation, Control and Remote Sensing*. John Wiley & sons, 2011. doi: 10.1002/9781118026403

[4] H. J. Sussmann and G. Tang, *Shortest paths for the Reeds-Shepp car: a worked out example of the use of geometric techniques in nonlinear optimal control.* Rutgers Center for Systems and Control Technical Report, 1991.

[5] A. Quarteroni, R. Sacco, and F. Saleri, *Numerical Mathematics*. Springer, 2000. doi: 10.1007/b98885

[6] A. Fortin, *Analyse Numérique pour Ingénieurs*. Presses Internationales Polytechnique Montéreal, 2001.

[7] F. Demim, A. Nemra, K. Louadj, and M. Hamerlain, "Simultaneous localization, mapping and path planning for unmanned vehicle using optimal control," *Advances in Mechanical Engineering Journal*, vol. 10, no. 1, pp. 1–25, 2018. doi: 10.1177/1687814017736653

[8] S. Titouche, P. Spiteri, F. Messine, and M. Aidene, "Optimal control of a large thermic process," *Journal of Process Control*, vol. 25, pp. 50–58, 2015. doi: 10.1016/j.jprocont.2014.09.015

[9] F. Demim, A. Nemra, K. Louadj, Z. Mehal, M. Hamerlain, and A. Bazoula, "Simultaneous localization and mapping algorithm for unmanned ground vehicle with svsf filter," in *8th International Conference on Modelling, Identification and Control*. IEEE, 2016, p. 155–162.

[10] F. Demim, K. Louadj, M. Aidene, and A. Nemra, "Solution of an optimal control problem with vector control using relaxation method," *Journal of Automatic Control and System Engineering*, vol. 16, no. 2, pp. 29–41, 2016.

[11] K. Louadj, P. Spitéri, M. Aidene, and F. Messine, "An optimal control problem with free final time for aircraft flight with wind," in *Colloque sur l'Optimisation et les Systèmes d'Information*, 2014, pp. 239–250.

[12] ——, "Optimal command for the control of the air navigation of an aircraft," in *The Second International Conference on Electrical Engineering and Control Applications*. IEEE, 2014, p. 19.

[13] F. Kara, P. Spiteri, F. Messine, and M. Aidene, "A numerical optimal control method for solving a large thermic process," *RAIRO - Operations Research*, vol. 50, no. 2, pp. 297–314, 2016. doi: 10.1051/ro/2015018

[14] J. B. Thomas and F. Benjamin, *Hybrid Systems, Optimal Control and Hybrid Vehicles*. Springer, 2017. doi: 10.1007/978-3-319-51317-1

[15] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory*. John Wiley & sons, 1967.

[16] E. Trélat, *Contrôle Optimal: Théorie et Applications*. Vuibert, 2005.

[17] L. Pontryagin, V. Boltyanski, R. Gamkrelidze, and E. Mishchenko, *The Mathematical Theory of Optimal Processes*. John Wiley & sons, 1962.

[18] E. Picard, *Traité d'Analyse Tome II*. Gauthier-Villars et Fils, 1893.

[19] ——, *Traité d'Analyse Tome III*. Gauthier-Villars et Fils, 1896.

[20] ——, "Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives," *Journal de Mathématiques Pures et Appliquées*, vol. 4, no. 6, pp. 145–210, 1890.

[21] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*. Springer, 1980. doi: 10.1007/978-1-4757-5592-3

[22] H. Maurer, "On optimal control problems with bounded state variables and control appearing linearly," *SIAM Journal on Control and Optimization*, vol. 15, no. 3, p. 345–362, 1977. doi: 10.1137/0315023

[23] R. F. Hartl, S. P. Sethi, and R. G. Vickson, "A survey of the maximum principles for optimal control problems with state constraints," *SIAM Review*, vol. 37, no. 2, p. 181–218, 1995. doi: 10.1137/1037043