Yugoslav Journal of Operations Research
xx (20xx), Number xx, xxx–xxx
DOI: https://doi.org/10.2298/YJOR250415044G

Research Article

NONCONVEX OPTIMIZATION APPROACH TO BILEVEL PROBLEM OF COAL PRODUCTION AND PURCHASE WITH STORAGE

Tatiana V. GRUZDEVA*
Matrosov Institute for System Dynamics and Control Theory of SB RAS
gruzdeva@icc.ru, ORCID: 0000-0002-4579-3927

Anton V. USHAKOV
Matrosov Institute for System Dynamics and Control Theory of SB RAS
aushakov@icc.ru, ORCID: 0000-0001-8289-6266

Received: April 2025 / Accepted: December 2025

Abstract: The paper addresses a new mathematical model of interaction between coal manufacturers and coal importers in a country coal market as a bilevel optimization problem. The goal of the upper level which is decided on coal purchases and storage, is to satisfy the demand for coal in each time period, minimizing the costs of purchase and storage. At the lower level, each supplier company maximizes its profit by varying the volume of coal production based on the purchase prices. The presence of binary variables adds complexity to the model. We transform the bilevel problem into a single-level nonconvex problem and apply the Global Search Theory to develop solution approaches for the resulting problem. The advantages and disadvantages of the proposed approaches are discussed.

Keywords: bilevel optimization modeling, KKT-approach, reverse-convex programming, convex maximization, Global Search Theory, local search, global search scheme.

MSC: 90C11, 90C26, 90C90.

*Corresponding author

1. INTRODUCTION

Coal is one of the major source for total energy supply in the world. Rapid changes in the coal market and the intensification of global competition compel coal manufacturers and coal traders to re-evaluate their business and corporate strategies [1]. In order to hedge against business risks, many companies are planning to rely on operations research methods or already use them.

Despite the green transition, the international and internal coal trade has constantly been growing, For instance, according to analytical reports, China (a country with one of the fastest growing economics in the world) saw the highest imports of coking coal in 2024 at 121.7 million tons. It rose by 19.7% compared to 2023. At the same time, Mongolia and Russia became one of the key coal exporters to China in 2023 supplying 53.9 million tons (+5.3% to previous year) and 30.2 million tons (+7.7%), respectively.

Following the mentioned recent trends, we introduce a new bilevel optimization model to govern interaction between a coal importer (for instance, China) and coal manufacturers-exporters (Mongolia, Russia, and other countries) in the importer's coal market. The goal of the upper level player is to satisfy the demand for coal in each time period, minimizing the purchase and storage costs. At the lower level, each supplier company aims to maximize its profit by varying the volume of coal production depending on the current purchase prices.

It is well-known that bilevel optimization is a conventional tool for modeling hierarchical conflicts involving players with different decision-making powers [2, 3]. This is appealing for modeling real-world problems, but it also makes the optimization models hard to solve. It is also known that bilevel optimization problems involving integer variables become even more complicated [3]. Therefore, the development of efficient and effective numerical methods for solving diverse bilevel problems remains a challenge in modern mathematical optimization theory [4]. In this paper, we propose a new bilevel model that establish competitions between coal manufacturers and coal exporters who are supposed to have balancing stockpile facilities. We proposed a solution approach for the introduced bilevel problem resting upon the following two-stage approach:

- Considering the equivalence between bilevel and nonconvex optimization problems (in the sense of equivalence of their global optimal solutions), one can reduce a bilevel problem to a single-level one [2, 4].
- Applying the Global Search Theory (GST) developed by A.S. Strekalovsky, one can solve the resulting nonconvex problem by efficient numerical methods relying on modern convex optimization methods and software [5, 6].

The paper has the following structure. Section 2 describes the problem formulation, which is reduced to quadratically constrained nonconvex single-level problem in Section 3. In Section 4, we propose and discuss two different approaches to solving the problem: first one is based on the reverse-convex programming [5, 7, 8, 9], whereas the second one relies on the theory and methods of convex maximization [5, 10]. We discuss the advantages and disadvantages of the proposed approaches in Section 5.

2. PROBLEM STATEMENT

We consider a bilevel optimization problem where the upper level player is a importer company (country), whereas the lower level players are coal manufacturers.

Let T be a set of planning periods, $T = \{1,2,\dots,m\}$, I be a set of coal manufacturers, $I = \{1,2,\dots,n\}$. Let us introduce the following variables:

- p_t is the unit purchase cost (price) in period t set by the upper level player for manufacturers;
- x_{it} is the amount of coal offered by manufacturer $i \in I$ (a lower level player) at period $t \in T$;
- s_t are inventory variable defining the stock of the importer (the upper level player) at the end of period $t \in T$, $s_0 = 0$;
- y_t is a binary variable that takes 1 if the upper level player imports coal in period $t \in T$, and 0 otherwise.

It should be noted that the importer sets the same price p_t, $t \in T$ for all the manufacturers. The upper level parameters are unit storage cost $h_t \geq 0$, import cost $c_t \geq 0$ (e.g. think of currency exchange rate fluctuations over period T), and demand in coal $d_t > 0$ in period $t \in T$. Let us denote the total demand (starting from period t until the end of the planning horizon m) as $D_t = \sum_{j=t}^{m} d_j$. We also suppose that purchase initiation is a quite complicated process governed by the local laws, hence the number of purchases over the planning horizon T may not exceed N.

Note that if the price p_t is not competitive, a manufacturer may refuse offering coal at this period. Otherwise, the amount of coal offered by each manufacturer at each period is bounded by $\bar{x}_i$. We suppose that the production cost of manufacturer $i \in I$ is defined by convex quadratic function $\phi(\cdot)$.

With these notations, the aforementioned competition may be written as the following mixed integer bilevel optimization problem:

$$\sum_{t\in T}\left(p_t\sum_{i\in I}x_{it} + h_t s_t + c_t y_t\right) \downarrow \min_{p,s,y,x}, \tag{1}$$

$$s_{t-1} + \sum_{i\in I}x_{it} = d_t + s_t,\ \ s_0 = 0, \qquad t \in T, \tag{2}$$

$$\sum_{i\in I}x_{it} \leq D_{tm}y_t, \qquad t \in T, \tag{3}$$

$$\sum_{t\in T}y_t \leq N, \tag{4}$$

$$p_t \geq 0, \quad s_t \geq 0, \quad y_t \in \{0,1\}, \qquad t \in T, \tag{5}$$

$$x_{it} \in \mathrm{Sol}(\mathscr{F}_i(p)), \qquad i \in I,\ \ t \in T, \tag{6}$$

i.e. x_{it} is the optimal solution of the following lower level problem:

$$\left.\begin{array}{l} \sum_{t\in T}[p_t x_{it} - \phi_i(x_{it})] \uparrow \max_x, \\ \quad 0 \leq x_{it} \leq \bar{x}_i,\ \ t \in T, \end{array}\right\} \tag{$\mathscr{F}_i(p)$}$$

where $\phi_i(x) = a_i x^2 + b_i x$ defines quadratic costs ($a_i > 0$, $i \in I$). Let us denote the introduced bilevel problem as $(\mathscr{BP})$.

The goal of the importer (upper level player) is to satisfy the demand in each time period, minimizing the overall costs related to purchasing, storing, and importing coal. The upper level problem can be viewed as a variation of the uncapacitated lot-sizing problem [11], where equations (2) are balance constraints for every time period and the inequalities (3) define the upper bound constraints to amount of imported coal. The constraint (4) limits the number of coal purchases.

The manufacturers aim to maximize their own profit over all time periods by varying their amounts of offered coal. The latter depends on the purchase prices defined by the upper level player. Note that the lower-level decision do not involve multiple criteria optimization, since each manufacturer operates independently, trying to maximize its own profit. Also, one can easily see that each lower level problem has a unique optimal solution, so there is no need to distinguish between optimistic or pessimistic solutions [2] in the bilevel problem $(\mathscr{BP})$.

For every given p_t, $t \in T$, the lower level problem $(\mathscr{F}_i(p))$, $i \in I$, tuns out to be a concave quadratic maximization which is a convex problem. One can also note that the upper level objective function is bilinear, and the lower level has a general quadratic structure with a bilinear term in the objective function. All constraints at both levels are linear, but the upper level problem turns out to be mixed integer programming problem due to the binary variables y_t. This fact adds additional difficulty to the problem $(\mathscr{BP})$.

3. REDUCTION TO SINGLE-LEVEL NONCONVEX CONTINUOUS PROBLEM

Usually, the transformation of a bilevel optimization problem into a single-level one is implemented by applying the traditional KKT-approach, i.e. the lower level convex problem is replaced with its optimality conditions [2, 12, 13, 14, 15]. In our case, it can be done straightforwardly as the optimization problems at the lower level depend only of their own variables $x^i = (x_{i1}, \dots, x_{it}, \dots, x_{im})$ for every given upper level variables $p = (p_1, \dots, p_t, \dots, p_m)$. Thus, it is possible to obtain an explicit solution to the problems $(\mathscr{F}_i(p))$, $i \in I$, analytically.

Let us fix the variables p_t for all $t \in T$ $(p_t := \hat{p}_t)$ and consider the problem $(\mathscr{F}_i(\hat{p}))$, such that $\psi(x_{it}) = \hat{p}_t x_{it} - a_i x_{it}^2 - b_i x_{it}$.

Proposition 1. *Let $\hat{x}^i$ be the solution to the problem $(\mathscr{F}_i(\hat{p}))$. Then only one of the following alternatives takes place for all $i \in I$, $t \in T$:*

(a) $\hat{x}_{it} = 0$; *(b)* $\hat{x}_{it} = \dfrac{\hat{p}_t - b_i}{2a_i}$; *(c)* $\hat{x}_{it} = \overline{x}_i$.

Proof. Since $a_i > 0$, $i \in I$, the objective function $\sum\limits_{t\in T}\sum\limits_{i\in I}\left[-a_i x_{it}^2 + (\hat{p}_t - b_i)x_{it}\right]$ of the problem $(\mathscr{F}_i(\hat{p}))$ is concave, and hence $(\mathscr{F}_i(\hat{p}))$ is a convex optimization problem.

The first-order optimality conditions imply $\hat{p}_t - 2a_i x_{it} - b_i = 0$. Therefore, $x_{it} = \dfrac{\hat{p}_t - b_i}{2a_i}$, $i \in I$, $t \in T$. As the produced amount x_{it} lies within $[0, \overline{x}_i]$ $\forall i \in I$, we can get the optimal solution to $(\mathscr{F}_i(\hat{p}))$ as

$$\hat{x}_{it} = \begin{cases} 0, & \text{if } x_{it} < 0, \\ x_{it}, & \text{if } 0 \le x_{it} \le \overline{x}_i, \\ \overline{x}_i, & \text{if } x_{it} > \overline{x}_i, \end{cases} \tag{7}$$

where the upper bounds $\bar{x}_i$ remain unchanged for any period $t \in T$. $\square$

Remark 2. *As a consequence of Proposition 1, we can refine the bounds for the upper level unit price variables* p_t, $t \in T$*:*

$$\underline{p}_t := b_i \le p_t \le 2a_i\bar{x}_i + b_i =: \overline{p}_t, \tag{8}$$

Furthermore, the dependence between the unit purchase price and the amount of coal offered can be defined as

$$p_t = 2a_i x_{it} + b_i. \tag{9}$$

Substituting the variables p_t in the objective function (1) of the upper level problem, according to the equalities (9), we can cast the following single-level optimization problem equivalent to the bilevel problem $(\mathscr{BP})$:

$$f(s,y,x) = \sum_{t\in T}\sum_{i\in I}\left[2q_i x_{it}^2 + c_i x_{it}\right] + \sum_{t\in T} h_t s_t + \sum_{t\in T} c_t y_t \downarrow \min_{s,y,x}, \tag{10}$$

$$s_{t-1} + \sum_{i\in I} x_{it} = d_t + s_t, \;\; s_0 = 0, \;\; s_t \ge 0, \qquad t \in T, \tag{11}$$

$$\sum_{i\in I} x_{it} \le D_{tm} y_t, \qquad t \in T, \tag{12}$$

$$\sum_{t\in T} y_t \le N, \tag{13}$$

$$0 \le x_{it} \le \bar{x}_i, \qquad i \in I, \; t \in T, \tag{14}$$

$$y_t \in \{0,1\}, \qquad t \in T. \tag{15}$$

Despite the objective function of the problem (10)–(15) is a convex ($a_i > 0$) quadratic function, the resulted optimization problem is non-convex (mixed integer program).

Let us reduce the integer program to a continuous non-convex problem by replacing the binary constraints (15) for decision variables y with the following equivalent constraints [7]:

$$y_t^2 - y_t \ge 0, \qquad t \in T, \tag{16}$$

$$0 \le y_t \le 1, \qquad t \in T. \tag{17}$$

Summing up all the constraints (16) by index t, we obtain

$$0 \le \sum_{t=1}^{m}\left(y_t^2 - y_t\right) = \sum_{t=1}^{m}\left(\left[y_t^2 - y_t + \frac{1}{4}\right] - \frac{1}{4}\right) = \left\| y - \frac{e}{2}\right\|^2 - \frac{m}{4}, \tag{18}$$

where $e = (1,\ldots,1)^T \in \mathbb{R}^m$. It can readily be seen that the inequality (18) holds iff all the variables y_t take integer values 0 or 1.

Therefore, the problem (10)–(15) is reduced to the problem of minimizing a convex objective function over a nonconvex feasible set, i.e.

$$\left.\begin{aligned} f(s,y,x) \downarrow \min_{s,y,x}, \ \ (s,y,x) \in \mathscr{S}, \\ h(y) = \left\| y - \frac{e}{2} \right\|^2 - \frac{m}{4} \geq 0, \end{aligned}\right\} \tag{19}$$

where $\mathscr{S} \subset I\!R^{(2+n)m}$ is a convex set defined by constraints (11)–(14), (17). One can see that the function $h(\cdot)$ in the problem (19) is also convex function, and due to $h(y) \geq 0$, such problem is often called as the reverse-convex program [5, 8, 9].

Thus, the problem (19) can be considered as a particular case of a general DC optimization problem [16, 17], and we can apply the Global Search Theory-based approach to develop a solution algorithm.

Once the solution of the problem (19) is obtained, it is possible to determine the optimal prices for the corresponding amounts of offered coal for all periods t using formula (9).

4. SOLUTION APPROACHES

The core of Global Search Theory (GST) for DC optimization problems is the Global Optimality Conditions (GOCs) developed by A.S. Strekalovsky [5, 6, 16, 17]. They allow producing a minimizing sequence for any DC optimization problems [12, 13, 14, 15, 18, 19, 20, 21, 22, 23], since they bear the so-called constructive (algorithmic) property. It induces the procedures of escaping local, stationary, or critical points. In other words, when a point under study is not a global solution, there exists a possibility to violate the GOCs, and, as a consequence, one can improve the point in question with respect to the objective function value. The solution approach based on the GST consists of the two main components:

1. Special local search, which provides an approximate local or critical point and is based on solving the sequence of linearized problems.
2. Procedure of escaping critical points which is based on the GOCs.

We suppose to address the problem (19) folowing two directions: the first one is based on viewing it as the reverse-convex program (19), while the second one is to consider it as a convex maximization program. The latter is followed from the corresponding reduction theorem [8].

4.1. Reverse-convex programming

The so-called basic nonconvexity of the problem (19) follows from the fact that its feasible set is non-convex. To implement the GST-based solution approach, we at first have to design a local search method (LSM) for the reverse-convex problem [5, 24, 25, 26] to find critical points of the problem (19).

One proper variant of the LSM for the reverse-convex problem is the modification (LSM-RCP) [24] of the method proposed by Rosen [27]. In this local search procedure, the linearization of the basic nonconvex terms in the constraints is combined with a free descent procedure. Our computational experience [24] shows that LSM-RCP performs better on the problems with linear objective function and quadratic reverse-convex constraint. It is exactly the problem we want to solve here.

The LSM-RCP is based on solving the following linearized (at some $\hat{y}$: $(\hat{s},\hat{y},\hat{x}) \in \mathscr{S}$) problem:

$$\left.\begin{array}{l} f(s,y,x) \downarrow \min\limits_{s,y,x}, \;\; (s,y,x) \in \mathscr{S}, \\ \langle \nabla h(\hat{y}), y-\hat{y}\rangle + h(\hat{y}) \geq 0, \end{array}\right\} \tag{20}$$

where $\nabla_t\, h(y) = 2y_t - 1,\ t \in T$. Note that if $\hat{y}_t \in \{0,1\}\ \forall t \in T$, then $h(\hat{y}) = 0$. The problem (20) is obviously a convex problem and can be solved by any convex optimization method [28, 29] or an appropriate optimization solver.

Proposition 3. *In any solution* (s,y,x) *to the linearized problem* (20)*, y turns out to be binary.*

Proof. Due to the convexity of function $h(\cdot)$ defined by (18), the following chain holds:

$$0 \leq \langle \nabla h(\hat{y}), y-\hat{y}\rangle + h(\hat{y}) \leq h(y) - h(\hat{y}) + h(\hat{y}) = h(y).$$

Since $y \in [0,1]^m$, the inequality $h(y) \geq 0$ implies $y \in \{0,1\}^m$. □

Remark 4. *Note that the problem* (20) *has the so-called non-full-dimensional nonconvexity, hence only the variable* y^0 *is required to construct the first linearized problem. Nevertheless, it should be selected so that triple* (s^0,y^0,x^0) *is feasible and*

$$\mathscr{S} \cap \{y \in I\!R^m \mid \langle \nabla h(y^0), y-y^0\rangle + h(y^0) \geq 0\} \neq \emptyset.$$

In terms of the problem (19), the LSM-RCP consists of the following two procedures:

Procedure 1 begins with the point $(\hat{s},\hat{y},\hat{x}) \in S,\ h(\hat{y}) \geq 0$, and constructs a point $\tilde{z} = (\tilde{s},\tilde{y},\tilde{x})$ such that $\tilde{z} \in S:\ h(\tilde{y}) = 0,\ f(\tilde{s},\tilde{y},\tilde{x}) \leq f(\hat{s},\hat{y},\hat{x})$.

Procedure 2 starts from a feasible point $\tilde{z} \in S,\ h(\tilde{y}) = 0$, and produces a sequence $\{u^l\}$ such that

$$u^l = (s^l,y^l,x^l) \in S:\ \ h(y^l) \geq 0,\ \ f(\tilde{s},\tilde{y},\tilde{x}) \leq f(\hat{s},\hat{y},\hat{x}),\ \ l = 0,1,2,\ldots, \tag{21}$$

where $u^0 := \tilde{z}$. The sequence $\{u^l\}$ is formed according to the following rule. If the point u^l, $l \geq 0$, satisfies (21), then the next point u^{l+1} is constructed as an approximate solution of the convex problem (20), so that the following inequality holds

$$f(s^{l+1},y^{l+1},x^{l+1}) + \tau_l \leq \inf_{(s,y,x)} \{f(s,y,x) \mid (s,y,x) \in S, \langle \nabla h(y^l), y-y^l\rangle + h(y^l) \geq 0\},$$

where $\{\tau_l\}$ is a numerical sequence, $\tau_l > 0,\ l = 0,1,2,\ldots,\ \sum\limits_{l=1}^{\infty} \tau_l < +\infty$.

An approximate solution to the linearized (convex) problem (20) defines a critical point of the problem (19) which can be obtained by LSM-RCP with a given accuracy $\tau > 0$ using the following stopping criterion [5, 24, 25]:

$$h(y^l) \leq \tau. \tag{22}$$

To escape from a critical points provided by LSM-RCP, one may design an algorithm based on the following Global Search Scheme (GSS) for solving reverse-convex problems (19) [5, 26, 30].

Suppose that the starting point $(s_0, y_0, x_0) \in \mathscr{S} \subset I\!R^{(2+n)m}$ and the numerical sequences $\{\tau_k\},\ \{\delta_k\}:\ \tau_k, \delta_k > 0,\ k = 1, 2, \ldots;\ \tau_k \downarrow 0,\ \delta_k \downarrow 0\,(k \to \infty)$ are given.

Global search scheme for the reverse-convex problem (19)

Step 0. Set $k := 0$, $\hat{z}^k := z_0 = (s_0, y_0, x_0)$.
Step 1. Start from the point $\hat{z}^k$ and, using the LSM-RCP, find a critical point $z^k := (s^k, y^k, x^k):\ h(y^k) \le \tau_k$. Calculate $\zeta_k := f(s^k, y^k, x^k)$.
Step 2. Construct a finite approximation

$$R_k = \{v^1, \ldots, v^{N_k} \mid v^j \in I\!R^m,\ h(v^j) = 0,\ j = 1, \ldots, N_k\}$$

of the level surface $\{h(y) = 0\}$ of the function $h(\cdot)$.
Step 3. For all $j = 1, \ldots, N_k$ find a δ_k-solution $u^j := (s^j, y^j, x^j) \in I\!R^{(2+n)m}$ to the following linearized problem:

$$\langle \nabla h(v^j), y\rangle \uparrow \max,\ \ (s, y, x) \in \mathscr{S},\ \ f(s, y, x) \le \zeta_k. \tag{23}$$

Step 4. For all $j = 1, \ldots, N_k$ find a δ_k-solution $w^j \in I\!R^m$ to the following level $(h(\cdot) = 0)$ problem:

$$\langle \nabla h(w), y^j - w\rangle \uparrow \max_{w},\ \ h(w) = 0. \tag{24}$$

Step 5. Compute the value

$$\eta_k = \langle \nabla h(w^{j*}), y^{j*} - w^{j*}\rangle = \max_{1 \le j \le N_k} \langle \nabla h(w^j), y^j - w^j\rangle$$

and construct the point $u^{j*} = (s^j, w^{j*}, x^j)$.
Step 6. If $\eta_k \ge 0$, then set $\hat{z}^{k+1} := u^{j*}$, $k := k+1$, and go to Step 1.
Step 7. If $\eta_k < 0$, then set $\hat{z}^{k+1} := z^k$, $k := k+1$, and go to Step 1.
Step 8. If the solution z^k is of sufficient accuracy τ_k, then stop: (s^k, y^k, x^k) is the best solution found.

Remark 5. *The auxiliary problems in Steps 3 and 4 are simple and may easily be solved. For example, in case of the quadratic function $h(\cdot)$, the level problem* (24) *can be analytically solved applying the KKT approach [5, 30].*

Regarding the approximation of the level surface of the convex function $h(\cdot)$ (Step 2), there are many ways and techniques to build it. For instance, the approximation R_k can be constructed using the basis (unit) vectors of the Euclidean space [5, 30]:

$$v^i = y^k + \mu_i e^i,\ \ i = 1, \ldots, N_k. \tag{25}$$

The search of μ_i is simple and, in fact, analytical (that is, it may be reduced to solving the quadratic equation of one variable μ_i): $h(\cdot) = 0$.

Being a key step of the global search, constructing a "good" approximation R_k of the level surface $h(y) = 0$ on Step 2 allows one to escape from any stationary point and finally obtain a point which turns out to be an approximate global solution [12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 30]. Moreover, the concept of ("good") resolving approximation plays a crucial role for the proof of convergence of the GSS. In other words, the sequence $\{z^k\}$, $z^k := (s^k, y^k, x^k)$, generated by the GSS turns out to be minimizing for the problem (19), subject to natural regularity conditions and the assumptions that the linearized problem and level problem can efficiently be solved. (see Steps 3, 4 of the GSS) [5, 26]. Furthermore, every point z_* of the sequence $\{z^k\}$ yields the infimum of $f(\cdot)$ over the feasible set of the reverse-convex problem (19).

An advantage of the presented solution approach is that it is applied to the reverse-convex constraint directly without any transformations of the original problem.

4.2. Convex maximization

One of the fundamental results in the theory of reverse-convex programming is the reduction theorem [8, 31], which establishes a connection between a reverse-convex optimization problem and a convex maximization (concave minimization) problem.

Moreover, a convex maximization problem turns out to be dual (in some sense) to a reverse-convex constraint problem. Thus, according to the reduction theorem [8, 31], instead of the problem (19), one can solve the following convex maximization problem:

$$h(y) \uparrow \max, \ (s,y,x) \in \mathscr{S}, \ f(s,y,x) \leq \zeta, \tag{26}$$

and if the value $V(\zeta) := \max\{h(y) \mid (s,y,x) \in \mathscr{S}, \ f(s,y,x) \leq \zeta\}$ for $\zeta = f(\hat{s},\hat{y},\hat{x})$ is equal to zero, then the point $(\hat{s},\hat{y},\hat{x})$ is a solution of the problem (19).

This resulted in emergence of the methods for solving reverse-convex problems; however, two main points associated with such a reduction should be clarified. First, the basic nonconvexity in a reverse-convex problem does not disappear: it "moves" from the feasible set to the objective function. Thus, even with convex $f(\cdot)$ and $\mathscr{S}$, the problem (26) remains nonconvex. Second, the choice of the parameter ζ in (26) is non trivial.

To find critical points of the convex maximization problem (26), we apply the special local search method (LSM-CMP) from [5, 10, 32], which is based on the following linearization.

Given a numerical sequence $\{\tau_l\}$, $\tau_l > 0$, $l = 0,1,2,\ldots$, and a feasible point $(s,y,x) \in \mathscr{D} := \{(s,y,x) \in \mathscr{S} \mid f(s,y,x) \leq \zeta\}$, one can produce a sequence $\{u^l\}$, $u^l := (s^l, y^l, x^l)$, using the following rule

$$u^{l+1} \in \mathscr{D}: \ \langle \nabla h(y^l), y^{l+1} \rangle + \tau_l \geq \sup_{(s,y,x)} \{\langle \nabla h(y^l), y \rangle : \ (s,y,x) \in \mathscr{D}\}. \tag{27}$$

The function $h(\cdot)$ in (19) is convex, continuously differentiable, and bounded from above on the feasible set $\mathscr{D}$. If $\sum\limits_{l=1}^{\infty} \tau_l < +\infty$, then the sequence $\{u^l\}$ generated by the rule (27) satisfies the condition

$$\lim_{l \to +\infty} \left[\sup_{(s,y,x) \in \mathscr{D}} \langle \nabla h(y^l), y - y^l \rangle \right] = 0,$$

and the classical stationarity condition holds at any limit point of the sequence [5]. We call such a point critical to the problem (26).

Therefore, the LSM-MCP consists in solving a sequence ($l = 0, 1, 2, \ldots$) of the following convex programs:

$$\langle \nabla h(y^l), y \rangle \uparrow \max, \ \ (s, y, x) \in \mathscr{D}. \tag{28}$$

According to [5], one can apply several stopping criteria for LSM-CMP:

$$\| y^{l+1} - y^l \| \leq \frac{\tau}{2 \| \nabla h(y^l) \|},$$
$$h(y^{l+1}) - h(y^l) \leq \frac{\tau}{2},$$
$$\langle \nabla h(y^l), y^{l+1} - y^l \rangle \leq \frac{\tau}{2},$$

where τ is a given accuracy. For $\tau_l \leq \frac{\tau}{2}$ the point (s^l, y^l, x^l) turns out to be τ-critical to the problem (26).

Remark 6. *The critical points provided by LSM-RCP (see Subsection 4.1) and LSM-CMP may differ due to the difference in the linearized problems* (20) *and* (28)*. Moreover, as was proven in Proposition 3, y must be binary in the LSM-RCP solution.*

Since problems (19) and (26) are very similar, due to the reduction theorem [8, 31], the Global Search Schemes are mostly the same for both problems. Therefore, we do not duplicate the GSS from Subsection 4.1, but only point out the main differences induced by solving the problem (26):

- In Step 0, the value $\zeta_k = \zeta_0 = f(s_0, y_0, x_0)$ should also be calculated to define the feasible set $\mathscr{D}$. Furthermore, the value $\gamma_k = \gamma_0 = h(y_0)$ should be calculated to define the current level surface of the function $h(\cdot)$.
- In Step 1 applies the LSM-CMP to yield a τ_k-stationary point.
- For construction of a finite approximation on Step 2, another level surface is used: $\{h(y) = \gamma_k\}$.
- The level problem (24) from Step 4 has $h(w) = \gamma_k$ as the constraint.
- Looping to the next cycle, in Steps 6 and 7, a recalculation of the value ζ_{k+1} is added.

Moreover, an additional stopping criterion should be added, as if the method finds a point $(\hat{s}, \hat{y}, \hat{x})$ such that $h(\hat{y}) = 0$ while $\zeta = f(\hat{s}, \hat{y}, \hat{x})$, then a solution to the problem (19) is found.

The sequence produced by the GSS for solving (26) is proven to be maximizing one if the approximation is resolving and the regularity conditions are satisfied, and the linearized and level problems can be solved efficiently [5, 10]. An advantage of solving the reverse-convex problem via the convex maximization problem consists in the usage of different level surfaces $h(\cdot) = \gamma_k$, $k = 1, 2, \ldots$, in the GSS, which will probably allow more efficient jumping out of critical points obtained by local search.

4.3. Toy example

In the paper, we provide only the first stage of the research, which includes modeling and describing possible solution techniques. Due to the absence of real-world data, in this section, we provide an illustrative small-scaled example to demonstrate the main stages of the developed approach.

Consider 4 planning periods, $T = \{1,2,3,4\}$, and 2 manufacturers at the lower level, $I = \{1,2\}$. Let the upper level parameters be the unit storage cost $h = (1,20,3,7)$, the import cost $c = (10,10,5,5)$, the demand for coal $d = (10,100,150,50)$, and the number of purchases $N = 2$. At the lower level, let the limit of coal production be $\bar{x} = (150,250)$ and the production costs of the manufacturers be defined by quadratic functions with coefficients $a = (0.5,0.7)$, $b = (20,10)$.

First, let us compare the performance between two local search methods: the LSM-RCP based on solving linearized problems (23) (see Subsection 4.1) and the LSM-CMP with linearized problems (28) (see previous subsection).

Thus, we select starting point $(s_0,y_0,x_0) \in S$ such that $s_0 = (140,240,50,40)$, $y_0 = (1,1,0,0)$, $x_0 = (x_{01};x_{02}) = (100,50,0,0;200,0,0,0)$ with objective function value $f_0 = 44640$, and show how local search works in Step 1 of the global search scheme.

The LSM-RCP consists of two procedures, but y_0 is already on the surface of level $h(\cdot) = 0$, so it is not necessary to perform Procedure 1. After applying 2 iterations of Procedure 2, we got the solution (the critical point) $z^I = (s^I,y^I,x^I) \in S$: $s^I = (144.14,200,50,0)$, $y^I = (1,1,0,0)$, $x^I = (85.75,86.75,0,0;68.39,69.11,0,0)$; $\zeta_I := f(z^I) = 23195.82$.

As for another local search method, LSM-CMP, by solving a series of linearized problems, gets as a result the critical point $z^{II} = (s^{II},y^{II},x^{II}) \in S : s^{II} = (269.14,225.44,75.44, 25.44)$, $y^{II} = (1,1,0,0)$, $x^{II} = (69.2,56.3,0,0;209.9,0,0,0)$; $\zeta_{II} := f(z^{II}) = 44639.96$.

We can easily see $\zeta_I < \zeta_{II}$, which emphasizes the LMS-RCP's efficiency over the LSM-CMP regarding the objective function value.

Now let us demonstrate the step-by-step performance of the global search procedures for reverse-convex problem (19) which allow us to escape from the critical point z^I.

Step 2. The following points ($N_1 = 5$) was chosen at the level surface $\{h(\cdot) = 0\}$ for the approximation: $v^1 = (-0.5,0.5,0.5,0.5)$, $v^2 = (0.5,0.5,-0.5,0.5)$, $v^3 = (-0.1,0.5,-0.3,0.5)$, $v^4 = (0.5,-0.1,1,-0.1245)$, $v^5 = (-0.1,0.5,1,-0.1245)$; $h(v^i) = 0$, $i = 1,\dots,5$.

Step 3. By solving the linearized (at v^j) problem (23), the following points $u^j = (s^j,y^j,x^j) \in I\!R^{16}$, $j = 1,\dots,5$, were got:

	s	y	x
u^1 :	0,0,45,0	0.03,0.33,1,0	8.31,100,83.55,0;1.69,0,116.45,0
u^2 :	165.7,165,50,0	0.57,0.33,0.18,0	98.3,99.3,16.2,0;77.38,0,18.72,0
u^3 :	11.6,52.9,50,0	0.07,0.47,0.74,0	0,141.23,81.75,0;21.63,0,65.41,0
u^4 :	100,0,50,0	1,0,1,0	108.39,0,105.52,0;1.61,0,94.48,0
u^5 :	0,0,50,0	0.03,0.33,1,0	1.16,100,82.81,0;8.84,0,117.19,0

Step 4. By solving the level ($h(\cdot) = 0$) problem (24) for all $j = 1,\dots,5$, the following points $w^j \in I\!R^4$ with the corresponding objective values were found:

		$\langle \nabla h(w), y-w \rangle$
w^1 :	$0.02, 0, 1.02, 0$	-0.37
w^2 :	$0.63, -0.03, -0.17, 0$	-0.87
w^3 :	$-0.12, 0, 0.84, 0$	-0.78
w^4 :	$1, 0, 1, 0$	0
w^5 :	$-0.04, 0.31, 1.08, -0.08$	-0.27

Step 5. $\eta_1 = \max\limits_{1 \le j \le 5} \langle \nabla h(w^j), y^j - w^j \rangle = 0$, $u^{4*} = (s^4, w^4, x^4)$.

Step 6. Since $\eta_0 \ge 0$, we set $\hat{z}^1 := u^{4*}$, $k := 1$, and go to Step 1.

Step 1. ($k = 1$) Starting from the point $\hat{z}^1$, LSM-RCP provides the solution
$z^1 := (s^1, y^1, x^1) : \; s^1 = (127, 27, 50, 0),\; y^1 = (1, 0, 1, 0),$
$x^1 = (75.75, 0, 96.75, 0; 61.25, 0, 76.25, 0);\; h(y^1) = 0,\; f(z^1) = 19902.25.$

Next, by performing Steps 2–5, trying to improve the local solution z^1, we fail: all values η calculated turn out to be less than zero. Here, we can conclude that we got the best solution to the problem in question. The SCIP solver [33] also found this solution to the toy example in the formulation (10)–(15), which proved that the solution obtained using the proposed approach is global.

Upon solving the problem under consideration using a convex maximization problem, the global search iterates a different way. Although the level surface approximation at Step 2 of the GSS from Subsection 4.2 consisted of the same points ($k = 0,\ \gamma_k = 0$), the feasible set of the linearized problems was different due to the different value of $\zeta_k = f(z^{II})$, so their solutions were also different. Therefore, more iterations of the algorithm were required to obtain a solution to the problem.

Thus, the overall cost is 19902.25 when purchasing coal in the first and third periods in volumes of 75.75 and 96.75 from the first manufacturer, 61.25 and 76.25 from the second with storage of volumes of 127, 27, and 50 in periods 1, 2, 3, respectively.

5. CONCLUSION

A new mathematical optimization model of interaction between coal manufacturers and a coal importer in the importer's country coal market as a bilevel optimization problem has proposed. It may be useful to describe relationships and trade between China, Russia and Mongolia as key players in the Chinese coal market.

The upper level objective function is bilinear, and the lower level has a general quadratic structure with a bilinear term in the objective function. The presence of binary variables at the upper level makes the problem even more complex.

We transformed the problem into a single-level nonconvex continuous problem and applied Global Search Theory to develop solution approaches for the resulting problem.

We studied single-level nonconvex problem from two viewpoints: as the reverse-convex program and as convex maximization program, and developed two solution methods. An advantage of the first approach is that it handles and considers the reverse-convex constraint directly, without transforming the problem.

The usage of different level surfaces of convex function referred to reverse-convex constraint in the second solution approach might allow more efficient jumping out of critical points obtained by local search. Therefore, it is superior over the approach based on solving the reverse-convex problem via the convex maximization problem.

As for further research, it would be interesting to apply the proposed bilevel optimization model and developed solution approach in different sectors for mineral mining companies.

Funding: The research was supported by RSF (project No 24-41-03004, https://rscf.ru/project/24-41-03004/).

REFERENCES

[1] P. Benalcazar, J. Kaminski, and P. Saluga, "The storage location problem in a coal supply chain: Background and methodological approach," *Gospodarka Surowcami Mineralnymi*, vol. 33, 2017. doi: 10.1515/gospo-2017-0009

[2] S. Dempe, *Foundations of bilevel programming*. Dordrecht: Kluwer Academic Publishers, 2002.

[3] T. Kleinert, M. Labbe, I. Ljubic, and M. Schmidt, "A survey on mixed-integer programming techniques in bilevel optimization," *EURO Journal on Computational Optimization*, vol. 9, p. 100007, 2021. doi: 10.1016/j.ejco.2021.100007

[4] S. Dempe and A. Zemkoho, Eds., *Bilevel Optimization: Advances and Next Challenges*. Cham: Springer International Publishing, 2020.

[5] A. Strekalovsky, *Elements of nonconvex optimization [in Russian]*. Nauka, 2003.

[6] ——, "On solving optimization problems with hidden nonconvex structures," in *Optimization in science and engineering*, C. F. T.M. Rassias and S. Butenko, Eds. Springer, 2014, pp. 465–502.

[7] R. Horst, P. Pardalos, and N. Thoai, *Introduction to Global Optimization. Nonconvex Optimization and Its Applications*. Springer, 2001.

[8] R. Horst and H. Tuy, *Global Optimization. Deterministic Approaches*. Springer-Verlag, 1993.

[9] I. Tseveendorj, "Reverse convex problems: an approach based on optimality conditions," *Journal of Applied Mathematics and Decision Sciences*, vol. 2006, 2006. doi: 10.1155/JAMDS/2006/29023

[10] A. S. Strekalovsky and A. A. Kuznetsova, "Convergence of a global search algorithm in the problem of convex maximization on an admissible set," *Russian Mathematics*, vol. 43, pp. 70–77, 1999.

[11] Y. Pochet and L. A. Wolsey, "Lot-size models with backlogging: Strong reformulations and cutting planes," *Mathematical Programming*, vol. 40, no. 1, pp. 317–335, 1988. doi: 10.1007/BF01580738

[12] T. Gruzdeva and E. Petrova, "Numerical solution of a linear bilevel problem," *Computational Mathematics and Mathematical Physics*, vol. 50, pp. 1631–1641, 2010.

[13] A. Orlov, *The global search theory approach to the bilevel pricing problem in telecommunication networks*. Springer International Publishing AG, 2018, pp. 57–73.

[14] A. Strekalovsky and A. Orlov, *Linear and quadratic-linear problems of bilevel optimization [in Russian]*. SB RAS publishing, 2019.

[15] ——, *Global search for bilevel optimization with quadratic data*. Cham: Springer International Publishing, 2020, vol. 161, pp. 313–334.

[16] A. S. Strekalovsky, "On global optimality conditions for d.c. minimization problems with d.c. constraints," *Journal of Applied and Numerical Optimization*, vol. 3, pp. 175–196, 2021.

[17] ——, "Minimizing sequences in a constrained dc optimization problem," *Proceedings of the Steklov Institute of Mathematics*, vol. 323, pp. S255–S278, 2023.

[18] T. Gruzdeva and A. Strekalovsky, “A d.c. programming approach to fractional problems,” in *Learning and Intelligent Optimization*, ser. Lecture Notes in Computer Science, R. Battiti, D. Kvasov, and S. Ya., Eds., vol. 10556. Cham: Springer, 2017, pp. 331–337.

[19] R. Enkhbat, T. V. Gruzdeva, and M. V. Barkova, “D.c. programming approach for solving an applied ore-processing problem,” *Journal of Industrial and Management Optimization*, vol. 14, no. 2, pp. 613–623, 2018. doi: 10.3934/jimo.2017063

[20] M. Gaudioso, T. V. Gruzdeva, and A. S. Strekalovsky, “On numerical solving the spherical separability problem,” *Journal of Global Optimization*, vol. 66, no. 1, pp. 21–34, 2016. doi: 10.1007/s10898-015-0319-y

[21] T. Gruzdeva and A. Strekalovsky, “On solving the sum-of-ratios problem,” *Applied Mathematics and Computation*, vol. 318, 2017. doi: 10.1016/j.amc.2017.07.074

[22] T. V. Gruzdeva and A. V. Ushakov, “K-means clustering via a nonconvex optimization approach,” in *Mathematical Optimization Theory and Operations Research*, ser. Lecture Notes in Computer Science, P. Pardalos, M. Khachay, and A. Kazakov, Eds., vol. 12755. Cham: Springer, 2021. doi: 10.1007/978-3-030-77876-7_31 pp. 462–476.

[23] A. Strekalovsky and M. Barkova, “On the solution of systems of quadratic equations [in russian],” *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, vol. 30, pp. 173–187, 2024. doi: 10.21538/0134-4889-2024-30-2-173-187

[24] T. Gruzdeva and A. Strekalovsky, “Local search in problems with nonconvex constraints,” *Computational Mathematics and Mathematical Physics*, vol. 47, pp. 381–396, 2007. doi: 10.1134/S0965542507030049

[25] A. S. Strekalovsky, “On local search in d.c. optimization problems,” *Applied Mathematics and Computation*, vol. 255, no. 0, pp. 73–83, 2015.

[26] A. Strekalovsky, “On convergence of a global search strategy for reverse covex problems,” *Journal of Applied Mathematics and Decision Sciences*, vol. 2005, pp. 149–164, 01 2005. doi: 10.1155/JAMDS.2005.149

[27] J. B. Rosen, “Iterative solution of nonlinear optimal control problems,” *SIAM Journal on Control*, vol. 4, no. 1, pp. 223–244, 1966. doi: 10.1137/0304021

[28] J. Bonnans, J. C. Gilbert, C. Lemaréchal, and C. Sagastizabal, *Numerical Optimization – Theoretical and Practical Aspects*. Springer, 2006.

[29] J. Nocedal and S. J. Wright, *Numerical Optimization*, 2nd ed., ser. Operations Research and Financial Engineering. New York: Springer, 2006.

[30] A. Strekalovsky and I. Tsevendorj, “Testing the r-strategy for a reverse convex problem,” *Journal of Global Optimization*, vol. 13, pp. 61–74, 1998. doi: 10.1023/A:1008260227877

[31] H. Tuy, “The complementary convex structure in global optimization,” *Journal of Global Optimization*, vol. 2, no. 1, pp. 21–40, Mar 1992. doi: 10.1007/BF00121300

[32] A. Kuznetsova, “On a quadratic problem of convex maximization,” *Computational Mathematics and Mathematical Physics*, vol. 42, pp. 1268–1275, 2002.

[33] S. Bolusani, M. Besançon, K. Bestuzheva, A. Chmiela, J. Dionísio, T. Donkiewicz, J. van Doornmalen, L. Eifler, M. Ghannam, A. Gleixner, C. Graczyk, K. Halbig, I. Hedtke, A. Hoen, C. Hojny, R. van der Hulst, D. Kamp, T. Koch, K. Kofler, J. Lentz, J. Manns, G. Mexi, E. Mühmer, M. E. Pfetsch, F. Schlösser, F. Serrano, Y. Shinano, M. Turner, S. Vigerske, D. Weninger, and L. Xu, “The SCIP Optimization Suite 9.0,” Zuse Institute Berlin, ZIB-Report 24-02-29, February 2024. [Online]. Available: https://nbn-resolving.org/urn:nbn:de:0297-zib-95528