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Research Article

TESTING BALLISTIC DISPERSION WHEN THE AIM POINT IS UNKNOWN

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Abstract: In this paper, we point out a fundamental error in the procedures used to measure ballistic dispersion. We argue that there are two components to a proper measure of dispersion: i) dispersion arising from the 'unknown' aim point and ii) dispersion caused by the cloud, which can be measured directly from the sample mean point of impact. The procedures currently in use only take into account the cloud dispersion, and thus, underestimate the actual (total) dispersion. These procedures fail to recognize the existence of the fixed "aim point". As a result, incorrect conclusions may be drawn. An elementary correction to the underlying statistics corrects this error when the fall of shot follows a circular normal distribution.

Keywords: Quality Assurance, dispersion, aim point.

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1. INTRODUCTION

Military and law enforcement organizations regularly conduct quality assurance (QA) testing on random samples of ammunition rounds before accepting large lots from manufacturers of the various types of ammunition that they require. One important component

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of the complete QA program involves testing for acceptable levels of ballistic dispersion. This particular test is carried out under tightly controlled conditions in order to determine the dispersion due strictly to the ammunition itself. Meanwhile, measuring ballistic dispersion may also be of interest in other settings. For example, shooting competitions are widely popular sporting events in many countries. The 2024 Olympic Games counts 15 separate events where athletes compete based on precision and speed. These events involve the use of rifles and pistols. However, unlike QA tests for ammunition, which are used to accept or reject supplies, dispersion measurements can be obtained from target practice within a training program, and would be useful for analyzing and improving the performance of individual athletes. The underlying statistics are the same. Other competitive sporting events such as archery, even the game of darts, which involve precision and stamina, are amenable to similar approaches.

In practice, QA tests for ballistic dispersion require that a random sample of rounds from a lot be fired and their locations on a target recorded. Using this sample data, a sample mean point of impact and component sample standard deviations, s_x and s_y, are calculated. Both the Canadian Forces (CF) and US Army Ordnance Corp (AOC) use these component standard deviations when testing for dispersion. The critical region for these tests takes the form

$$\max(s_x, s_y) \geq \kappa_0 \tag{1}$$

That is, if one of the component standard deviations is too high, the lot is rejected. But in the case where the fall of shot follows a circular normal distribution, Hurley (2008) has shown that a uniformly most powerful test results in a critical region of the form

$$\frac{s_x^2 + s_y^2}{2} \geq \tau_0. \tag{2}$$

That is, the lot is rejected if the average of the component sample variances is high enough. Equivalently, the lot is rejected if the sum of the component sample variances is sufficiently high:

$$s_x^2 + s_y^2 \geq \tau_0'. \tag{3}$$

However, the derivation of the test statistic assumes implicitly that the sample mean point of impact is used in place of the point of aim, which is generally unknown (see, for example, Hurley (2008), Rabbath and Corriveau (2017), and Hurley, et al. (2017)). As a result, the true ballistic dispersion is underestimated. This in turn may lead to acceptance of a lot of ammunition when the correct test would reject it. In this paper, we identify the unknown aim point as a significant component of ballistic dispersion, and show for the case of a circular normal distribution that an elementary correction of the test statistic is required to account for it.

2. MEASURING BALLISTIC DISPERSION

In Canada, the Department of National Defence buys 25mm rounds in lot sizes of 5,000 and 10,000 units. To test a lot for ballistic dispersion, a random sample of rounds is first drawn and then fired from a fixed-mount Mann barrel at a target 300 meters away. The fixed-mount barrel is used to remove, as much as possible, the various sources of error that a gun would be subjected to in an operational setting. The locations of these rounds on the target are recorded electronically. Figure 1 shows a sample "cloud." Note that two axes have been drawn by first inserting a y-axis through the sample point furthest to the left of the cloud and then an orthogonal x-axis through the point furthest down. Let the set of point locations be

$$\mathscr{L} = \{(x_1,y_1),(x_2,y_2),...,(x_n,y_n)\}$$

The Sample Mean Point of Impact ($sMPI$) is calculated by taking the averages of the x and y components of the round locations:

$$\bar{x} = \frac{1}{n}\sum x_i, \quad \bar{y} = \frac{1}{n}\sum y_i. \tag{4}$$

The sMPI has some nice properties. For instance, if the rounds follow a circular normal distribution

$$f_C(x,y) = \frac{1}{2\pi\theta}\exp\left[-\frac{(x-\mu_x)^2+(y-\mu_y)^2}{2\theta}\right], \tag{5}$$

the maximum likelihood estimators for the location parameters μ_x and μ_y are the coordinates of the sMPI, $\bar{x}$ and $\bar{y}$. The CF and AOC measures of dispersion are

$$s_x = \sqrt{\frac{1}{n}\sum(x_i-\bar{x})^2} \quad and \quad s_y = \sqrt{\frac{1}{n}\sum(y_i-\bar{y})^2}. \tag{6}$$

These are just the sample standard deviations in each component direction.

3. THE AIM POINT

Generally speaking, the aim point is a hypothetical point somewhere on the xy-plane shown in Figure 1. Its coordinates, μ_x and μ_y, given in (5) when the fall of the shot follows a circular normal distribution are unknown. We may visualize the aim point as the convergence of $(\bar{x},\bar{y})$ as the sample size n becomes arbitrary large under constant test conditions. However, there is no way of determining this point precisely due to inherent measurement errors. So we use the sMPI, $(\bar{x},\bar{y})$, as argued above.

Now consider the function

$$s^2(x,y) = \frac{1}{n}\sum_{i=1}^{n}\left[(x_i-x)^2+(y_i-y)^2\right], \tag{7}$$

which gives the sum of average component squared deviations of the data set relative to any point (x,y). If $(x,y)=(\bar{x},\bar{y})$, we get the sum of the component sample variances:

$$s^2(\bar{x},\bar{y}) = \frac{1}{n}\sum_{i=1}^{n}\left[(x_i-\bar{x})^2+(y_i-\bar{y})^2\right] = s_x^2+s_y^2. \tag{8}$$

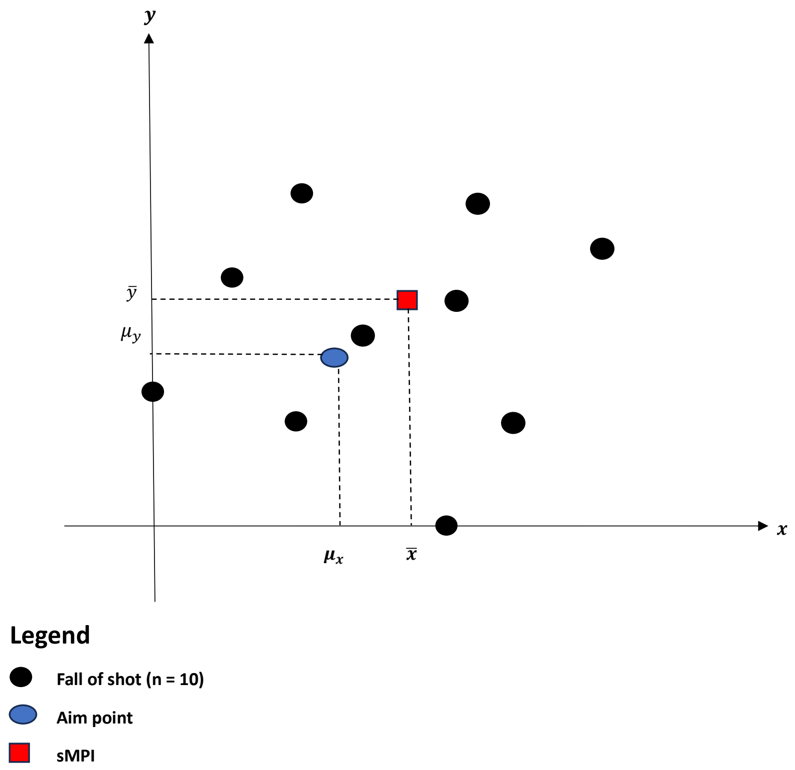

Figure 1: The cloud, sample mean point of impact (sMPI), and aim point

We can also interpret $s^2(x,y)$ as the average squared Euclidean distance between the sample points in $\mathscr{L}$ and the point (x,y). This function has nice properties: (i) it is strictly convex in (x,y); (ii) it is minimized at the center of gravity of the n points in $\mathscr{L}$, which is exactly the sMPI, $(\bar{x},\bar{y})$; and (iii) the iso-contours of $s^2(x,y)$ are circles centered at the sMPI $(\bar{x},\bar{y})$ (see Love et al., 1988).

It follows that the sample estimate of dispersion should be based on the true point of aim (if we know it),

$$s^2(\mu_x,\mu_y) = \frac{1}{n}\sum_{i=1}^{n}\left[(x_i-\mu_x)^2+(y_i-\mu_y)^2\right] > s^2(\bar{x},\bar{y}). \tag{9}$$

Furthermore, the magnitude of the difference between the two estimates increases as the distance between the two points increases. It can be shown that

$$s^2(\mu_x,\mu_y) = s^2(\bar{x},\bar{y}) + r^2, \tag{10}$$

where r is the Euclidean distance between (μ_x, μ_y) and $(\bar{x}, \bar{y})$.

A similar argument holds if we square the measures of dispersion in component directions x and y given in (6), and consider each measure separately.

In summary, the current practice of using the sMPI results in an underestimate of ballistic dispersion. In testing for dispersion, this may lead to erroneous conclusions, especially for smaller sample sizes. In the next section, we derive a simple fix for the case of the circular normal distribution.

4. QUALITY ASSURANCE IN THE CASE OF A CIRCULAR NORMAL DISTRIBUTION

We suppose again that the fall of the shot is consistent with a circular normal distribution as in (5) where the aim point is (μ_x, μ_y) and the variance is θ. Consider a random sample of size n from this population. For a given aim point, the joint probability density function of the sample is

$$J(\theta;\mathscr{L}) = \left(\frac{1}{2\pi\theta}\right)^n \exp\left[-\frac{1}{2\theta}\left(\sum_{i=1}^{n}\left[(x_i-\mu_x)^2+(y_i-\mu_y)^2\right]\right)\right], \tag{11}$$

where $-\infty < x_i, y_i < \infty, \quad i = 1, \ldots, n.$

Following the derivation in Hurley (2008), we consider testing

$$H_0 : \theta = \theta^\star \tag{12}$$

against the simple alternative

$$H_1 : \theta = \theta^{\star\star} \tag{13}$$

where $\theta^{\star\star}$ can be any value greater than $\theta^\star$. By the Neyman-Pearson lemma, a best critical region (or rejection region) for this simple parameter test is obtained by solving

$$\frac{J(\theta^\star;\mathscr{L})}{J(\theta^{\star\star};\mathscr{L})} \leq a \tag{14}$$

for some $a > 0$. Taking the ln of both sides of (14) and simplifying gives:

$$\sum_{i=1}^{n}(x_i-\mu_x)^2 + \sum_{i=1}^{n}(y_i-\mu_y)^2 \geq b \tag{15}$$

where

$$b = \frac{2\theta^{\star\star}\theta^\star}{\theta^{\star\star}-\theta^\star}\left[n\ln\left(\frac{\theta^{\star\star}}{\theta^\star}\right)-\ln(a)\right]. \tag{16}$$

If we substitute the sMPI for the aim point as done in Hurley (2008), and as in current standard practice, the critical region in (15) becomes:

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 + \sum_{i=1}^{n}(y_i-\bar{y})^2 = ns_x^2 + ns_y^2 \geq b \tag{17}$$

But this substitution underestimates the true dispersion as noted in the previous section. In fact, we have

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\mu_x)^2 &= \sum_{i=1}^{n}[(x_i-\bar{x})+(\bar{x}-\mu_x)]^2\\ &= \sum_{i=1}^{n}[(x_i-\bar{x})^2+2(\bar{x}-\mu_x)\sum_{i=1}^{n}(x_i-\bar{x})]+n(\bar{x}-\mu_x)^2\\ &= ns_x^2+n(\bar{x}-\mu_x)^2.\end{aligned}$$

and similarly,

$$\sum_{i=1}^{n}(y_i-\mu_y)^2 = ns_y^2+n(\bar{y}-\mu_y)^2$$

Thus, instead of (17), the critical region should be:

$$s_x^2+s_y^2+(\bar{x}-\mu_x)^2+(\bar{y}-\mu_y)^2 \geq \frac{b}{n}. \tag{18}$$

Comparing (17) with (18), it is clear that rejection of the null hypothesis in the original test automatically leads to rejection in the new one, but not so for the converse. Thus, retaining the point of aim results in a more conservative test. Also, note that there are two components of dispersion in the left-hand side of (18): the sample variance defined by $s_x^2+s_y^2$, and the "positional variance" of the cloud defined by $(\bar{x}-\mu_x)^2+(\bar{y}-\mu_y)^2$. Standard testing neglects this second component.

Let us consider next how to set the constant b. Since the marginal densities in the x and y directions are normal and given respectively by

$$f(x)=\frac{1}{\sqrt{2\pi\theta}}exp\left(-\frac{(x-\mu_x)^2}{2\theta}\right), \quad -\infty<x<\infty \tag{19}$$

$$g(y)=\frac{1}{\sqrt{2\pi\theta}}exp\left(-\frac{(y-\mu_y)^2}{2\theta}\right), \quad -\infty<y<\infty \tag{20}$$

(e.g. Rao (1973), pages 158-159), it follows that

$$Z_i=\frac{X_i-\mu_x}{\sqrt{\theta}}, \qquad Z_{n+i}=\frac{Y_i-\mu_y}{\sqrt{\theta}}, \quad i=1....,n,$$

are all mutually independent standard normal random variables. Therefore

$$\sum_{i=1}^{2n}Z_i^2=\frac{1}{\theta}\left(\sum_{i=1}^{n}(X_i-\mu_x)^2+\sum_{i=1}^{n}(Y_i-\mu_y)^2\right)=\frac{n}{\theta}[S_x^2+S_y^2+(\bar{X}-\mu_x)^2+(\bar{Y}-\mu_y)^2] \tag{21}$$

follows a chi-square distribution with $2n$ degrees of freedom. Thus, for a given level of significance α, we can look up a critical value $\chi_\alpha^{(2n)}$ such that

$$\alpha=Pr\left(\frac{n}{\theta}[S_x^2+S_y^2+(\bar{X}-\mu_x)^2+(\bar{Y}-\mu_y)^2]\geq\chi_\alpha^{(2n)};H_0\right). \tag{22}$$

Hence, the critical region of size α becomes

$$s_x^2 + s_y^2 + (\bar{x} - \mu_x)^2 + (\bar{y} - \mu_y)^2 \geq \frac{\chi_\alpha^{(2n)} \theta^\star}{n}.$$

In other words,

$$b = \chi_\alpha^{(2n)} \theta^\star.$$

Finally, we need to take care of the unknown location parameters μ_x and μ_y. As noted above, standard practice neglects the positional variance of the cloud. Thus Hurley (2008) obtains the critical region

$$s_x^2 + s_y^2 \geq \frac{\chi_\alpha^{(2n)} \theta^\star}{n}, \tag{23}$$

which effectively attributes a level of significance $\bar{\alpha} < \alpha$ to the test. Indeed $\frac{nS_x^2}{\theta}$ and $\frac{nS_y^2}{\theta}$ are both chi-square with $n-1$ degrees of freedom (see Hogg and Craig, 1978). We can also observe this from (22), where two degrees of freedom are lost to the positional variance of the cloud. Hence $\frac{n(S_x^2+S_y^2)}{\theta}$ follows a chi-square distribution with $2n-2$ degrees of freedom, and a correct test at level α is given by the adjustment

$$s_x^2 + s_y^2 \geq \frac{\chi_\alpha^{(2n-2)} \theta^\star}{n}. \tag{24}$$

5. SOME EXAMPLES

Example 1. Consider a small sample, say $n = 3$, of rounds fired in a test. Small sample sizes would typically apply in situations where individual rounds are very expensive. The number of degrees of freedom in the adjusted test given in (24) is $2 \times 3 - 2 = 4$, and for a level of significance $\alpha = 0.05$, we obtain $\chi_{0.05}^{(4)} = 9.49$.

The original test (see (23)) would use $\chi_{0.05}^{(6)} = 12.6$, resulting in a realized level of significance of $\alpha' = 0.013$. Thus, if the observed value of $n(s_x^2 + s_y^2)/\theta^\star$ falls in the interval $(9.49, 12.6)$, we will fail to reject the null hypothesis under the current test (23), while the correct test (24) shows that we should reject it.

Example 2. Suppose a sample size $n = 10$ is used in another test of less expensive ammunition. We get $\chi_{0.05}^{(2\times10-2)} = \chi_{0.05}^{(18)} = 28.9$ for the adjusted test in (24), and $\chi_{0.05}^{(20)} = 31.4$ for the original test in (23) resulting in $\alpha' = 0.026$. We see even for $n = 10$ there is an appreciable difference between the desired level of significance, α, and the realized level α' when the original test is used.

Example 3. Finally, consider a large sample size, say $n = 50$. We get $\chi_{0.05}^{(98)} = 122.1$, and $\chi_{0.05}^{(100)} = 124.3$ giving a realized level of significance $\alpha' = 0.037$. Thus convergence to the desired level of 0.05 appears to be very slow.

We conclude that the positional variance of the cloud, which is ignored in current practice, is a significant component of ballistic dispersion even for moderate to large sample sizes. Thus, current practice may lead to poor decisions, where ammunition is accepted when testing of ballistic dispersion should recommend rejection.

6. CONCLUSIONS

Standard tests for ballistic dispersion use the sample mean point of impact in place of the true point of aim of the test weapon as a matter of convenience since the latter point is generally impossible to determine precisely. We show that the current practice tends to underestimate total ballistic dispersion since it only considers the variance within the sample (or cloud), and does not include a second component referred to here as the positional variance of the cloud. An adjustment to the test is proposed for the case where the fall of the shot follows a circular normal distribution. We show that the accuracy of the test is improved significantly as a result even for moderate to large sample sizes.

Future research will study the impact of the point of aim on more general distributions.

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