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Research article

BASIC PRINCIPLES AND TECHNIQUES OF NORMALIZATION OF MULTIDIMENSIONAL DATA

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Abstract: The paper presents the main approaches and methods of multivariate data normalization in the context of MCDM problems. A technique for normalizing multivariate data is presented that results in: 1) equal "upper" bound of the value's interval for all attributes (Up.Norm); 2) equal "upper" and "lower" bounds (IZ.Norm); 3) equal "mean" values (Mean.Norm); 4) equal "mean" and "upper" (or "lower") bound (Dist.Norm); 5) equal "mean" and "range" (MS.Norm). The entire pool of solutions is aimed at preventing the hidden priority of the contribution of individual attributes to the integral indicator that determines the ranking of alternatives. The interpretation of normalized values for different methods made it possible to expand the range of methods based on the arithmetic mean and to identify normalization methods based on harmonic, geometric, counterharmonic, quadratic, and median means. A generalization of the methodology for nonlinear normalization of multivariate data is presented, along with an algorithm for normalizing target criteria consistent with linear normalization methods. Numerical experiments demonstrate the existence of MCDC problems that are highly sensitive to the choice of normalization method, which creates not only uncertainty but also the impossibility of multicriteria selection using ranking methods.

Keywords*:* Normalization of multivariate data, MCDM, scale coordination, linear transformations of normalized values, data normalization principles, the rank reversal phenomenon.

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1. INTRODUCTION

Normalization of multivariate data is used in multi-criteria decision making (MCDM), multivariate classification, and other areas where there are several competing goals and many alternatives (objects) whose attributes are specified in terms of the selected criteria. One of the main approaches to multi-criteria selection is the definition of an integral indicator by aggregating partial indicators [1]. Given the different nature of the features, it is necessary to preliminarily transform the values of all features so that they fall into of comparable intervals. This procedure of reducing the values measured in different scales to a conditionally common scale is defined as normalization of multivariate data.

The main provisions of multivariate data normalization in the context of MCDM and multivariate classification problems can be found in studies [2-16] and the recently published monograph [17]. Among them, we note the review by Jahan & Edwars [7], which presents a wide range of available methods for normalizing multidimensional data (linear and nonlinear). The study by Aytekin [10] to some extent duplicates the review [7], providing the same available list of normalization methods and stating the binding of the normalization method to the data structure. Using examples, an attempt was made to highlight the positive and negative aspects of various normalization methods.

Summarizing the research, we can state, firstly, the influence of the normalization method on the ranking of alternatives; secondly, this influence is not general and is determined by a specific task. Therefore, a wide range of different normalization methods is used in the actual absence of selection criteria. Comparative analysis and the success of the application are the main criteria for choosing a normalization method.

An analysis of current research shows that the range of different normalization methods for solving MCDM problems has remained unchanged compared to the review [7]. However, the specific features of existing methods remain largely unexplained. In this study, we examine whether certain normalization methods meet the requirements of two fundamental principles of multidimensional normalization. For this, firstly, a meaningful interpretation of normalized values is used. Secondly, the main properties of linear transformations for multidimensional data are used. And, thirdly, we check how the choice of different normalization options for two basic MCDM models, such as the Weighted Sum Model (WSM) and the Technique for Order of Preference by Similarity to the Ideal Solution (TOPSIS) [1], affects the ratings.

The peculiarity of normalization of multidimensional data is that each of the features has its own normalization scale. In this case, the normalizations are not “isotropic”, i.e. they compress the data cloud in some directions more and in others less. However, despite some violation of the data structure (mutual distances), this approach is considered generally accepted. The normalized values of the attributes of alternatives represent the share of the feature in their (different) scales. One of the reasons why the same normalization method is applied to different attributes is the same interpretation of the normalized values of different attributes. This allows to subsequently aggregate values of the same order and not add up the shares of different values. These shares can in some cases differ significantly, and it is possible that the contribution of one of the attributes will dominate in the performance indicator of the alternative. Therefore, the different range of normalized attribute values and the shift of the areas of normalized values relative to each other leads to the priority of the contribution of individual criteria to the performance indicator of alternatives. For linear normalization methods, this is formally due to the fact that the compression coefficients depend on the measurement scale and on the range of natural values of attributes.

To control the ratio of data compression ratios in different directions, this paper uses the terms "domain" of the jth attribute, including all values from the smallest to the largest ($[x_j^{min}, x_j^{max}]$), and "range", which determines the length of this domain ($range_j = x_j^{max} - x_j^{min}$).

Scaling (geometry) is a linear transformation that increases or decreases the size of objects. In the multidimensional case, scaling is performed independently for each attribute. For linear normalization methods, this is formally due to the fact that compression ratios depend on the measurement scale and the range of natural attribute values. Accordingly, linear normalization of multidimensional data is scaling for each attribute according to a specific rule. Consequently, the domains of different attributes may differ in both range and may be shifted relative to each other. The shift of domains is relative, since it is determined by two parameters — the endpoints of the interval. The limiting situation that determines a significant shift in the domains of two attributes j and k is determined by the inequality: $x_j^{min} > x_k^{max}$.

The basis for conducting the research and the publication of this article were two serious problems that arise when normalizing multidimensional data. The first is the shift in the domains of normalized values of different attributes relative to each other. When aggregating normalized values of different features, if the best value (or average) of one of the attributes exceeds the best (or average) value of another attribute, then the contribution of this attribute to the integral indicator will be higher. This means the presence of a hidden priority of one of the attributes before assigning weights (significance) of attributes. Why hidden priority? Because in many cases, recognizing the situation is difficult due to the presence of conflicting criteria in the task (the best and worst values are distributed differently among different attributes). But this situation does occur, and it's desirable to eliminate it. This situation is also typical if the normalization uses the goal inversion of the type ($1-x$) [6-10], which is called in most studies the normalization of cost criteria (see, for example, reviews [7, 10]).

If the normalized values x_{ij} are shifted to 1, for example for the Max normalization method, then ($1-x_{ij}$) will be shifted to 0, and the contribution of cost criteria to the integral indicator will obviously be lower than the contribution of benefit criteria.

Another problem is the unreasonable use of nonlinear normalization. Any nonlinear transformation violates the structure of the original data, namely the mutual distances between the attribute values change. After such a transformation, a problem similar to the original one is solved. Obviously, it is necessary to answer the question about the degree of approximation. For example, logarithmic normalization [18, 19], normalization, inversed of the squares of the values [7], sigmoid [20] are used quite often. A hidden similar error occurs if a nonlinear inversion of the goal of the type $1/x$ [7, 10] is used during normalization (incorrectly called as consistent normalization of cost criteria), while the author of this article proposed a universal Reverse Sorting algorithm (ReS) of the inversion of the goal back in 2019, which is consistent with all normalization methods [17].

This paper shows in a compact form the ways of solving the above-mentioned problems for multi-criteria decision-making tasks. In the absence of criteria, the author states the choice of the normalization method is based on the principles of preserving the information contained in the initial data of the task. The correspondence between the normalized scales of various attributes is also regulated based on the principle of the absence of priority of individual features.

The paper is structured as follows. Section 2 presents an overview and current state of the problem of selecting a normalization method for multivariate data in multi-criteria optimization problems on a discrete set of objects. The significance of normalization in the structure of the MCDM rank model is defined. The main linear methods of multivariate normalization are presented, and a meaningful interpretation of normalized scales is given. The main properties of linear normalization are defined, and the fundamental principles of multivariate data normalization are formulated.

Section 3 presents approaches to comparing various MCDM rank models and demonstrates the invariance of ratings under linear transformation of normalized values. Based on the property of rating invariance under linear transformations, the identical nature of the Value Measurement Methods and the Goal Level Model is demonstrated.

Section 4 presents an innovative technique for aligning normalized value scales, which allows eliminating data structure violations or mutual distances that arise during normalization. A technique for normalizing multivariate data is presented, the result of which is 1) equal "upper" attribute values (Up.Norm); 2) equal upper and lower domain boundaries for all attributes (IZ.Norm); 3) equal "mean" values (Mean.Norm); 4) equal "mean" and upper domain boundaries (Dist.Norm); and 5) equal "mean" and "range" (MS.Norm). All approaches, including the inversion of the objective of cost or benefit criteria (ReS algorithm), use linear transformations of the domain of normalized values, and as a result, the choice of normalizations is multivariante. The entire pool of solutions is aimed at preventing hidden priority of the contribution of individual attributes to the integral indicator determining the ranking of alternatives.

Correct interpretation of the normalized values of the Sum and Vec normalization methods allowed us to expand the range of methods based on the arithmetic mean and define normalization methods based on the harmonic, geometric, contra-harmonic, quadratic, and median means.

A literature review reveals that most studies traditionally employ linear normalization methods. Unfortunately, Z-score normalization has been largely overlooked. In this study, the author addresses this gap. The MS transformation technique is presented as a series of five normalization methods, analogous to Z-score normalization (Section 4.5). All of these methods have a remarkable property for multivariate data: they produce normalized values with equal means and equal deviations from the mean for all attributes.

Section 4.6 provides a rationale for using nonlinear normalization in solving MCDM problems to attenuate (or amplify) the contribution of values to the integral indicator of alternatives. Two standardized techniques for nonlinear normalization of multivariate data are presented using a two-step normalization procedure. In the first case, Max-Min standardization is used in the first step. The second technique uses the Z-score transformation.

Section 4.7 presents a generalized normalization algorithm for target criteria. Its advantage over similar algorithms is that the rate of data change (increment) is maintained over the entire range of values, which is important when using linear normalization methods for target criteria and cost-benefit criteria together.

Section 5 provides practical recommendations for choosing a normalization method. The essence of these recommendations is as follows: incorporate the five basic linear multivariate normalization methods listed above into the problem solution. These methods ensure:

- a variety of basic normalization scales,

- satisfying the basic principles of multivariate data normalization,
- having a rational interpretation of normalized values,
- ensuring consistency between the normalized values of different attributes.

In Section 6, using a numerical experiment as an example, we demonstrate the existence of MCDM problems that are highly sensitive to the choice of normalization method.

This creates not only uncertainty but also the impossibility of multi-criteria selection for such problems using rank-based methods.

The conclusion presents the main findings, recommendations, and directions for further research.

2. BASIC PRINCIPLES OF NORMALIZATION OF MULTIDIMENSIONAL DATA

2.1. Normalization in MCDM problems. Modern trends: Literature review

Comparing modern studies with the review [7], it should be noted that the available set of different normalization methods for solving MCDM problems has not changed. Most authors traditionally use linear normalization methods Max, Sum, Vec, dSum, Max-Min in their studies. At the same time, the features of existing methods are not actually disclosed. At the present stage, there is an understanding of the fact that in decision-making problems, the aggregation method and the normalization method are independent procedures. However, there are many publications in which acceptable combinations of them are selected. Existing attempts at harmonization are based on various similarity metrics. For example, in early studies, Chakraborty and Yeh [21, 22] discussed the suitability of the Max, Sum, Vec and Max-Min normalization methods, and assessed the best method using the ranking consistency index (RCI). Çelen [5] used consistency conditions to analyze the effects of three normalization methods (Max, Sum and Max-Min) and recommended the most suitable one for the TOPSIS method. Jahan and Edwards [7] proposed a target range-based normalization method and compared the performance of three types of normalization methods (non-monotonic, complex and target) using statistical measures. Various combinations of them are also used. For example, Vafaei et al. presented a three-level metrics framework to select the best normalization method for decision models [23]. Among the modern studies, the continuing trend of these approaches can also be noted. Ersoy [24] selecting the best normalization technique for Range of Value Method. Trung & Nguyen [25] investigated 12 normalization options for the Preference Selection Index method. Kacprzak [26] also investigated several normalization options for the Combined Compromise Solution (CoCoSo) method. Similarly, the study by Trung et al. [27, 28] aims to select a normalization technique for combination with the Measurement of Alternatives and Ranking according to Compromise Solution (MARCOS) and Ranking of Alternatives with Weights of Criterion (RAWEC) methods. Baydaş et al. [29] additionally note that the choice of the best normalization method depending on the data structure should be assessed from a dynamic point of view. This list can be continued.

Existing attempts at harmonization are based on various similarity metrics [26]. Krishnan's review [30] identifies 11 different similarity metrics. Among them are: Pearson's correlation coefficient (PCC), the cosine similarity measure (CSM), Spearman's rank correlation coefficient (SRCC), the rank similarity index (RSI), the ranking consistency index (RCI), and the standard deviation metric.

Despite the presence of measures, the comparison criteria are subjective and based on comparative analysis and the success of application and are based on the majority principle. Namely, if several different normalization methods in combination with the attribute aggregation method have close metrics based on the rating or rank of alternatives, then this group of methods is considered consistent. Consistency does not mean truth. If some normalization method satisfies the basic principles but is not consistent (in terms of metrics) with a group of others, this does not mean that the method is unacceptable and the rating (rank) results are incorrect. Rather, it is better to treat this as a possible solution, designate the problem as sensitive to the choice of normalization method (see Section 4), and entrust the decision making to the decision maker. Another trend is associated with the concept of the multi method MCDM model, which is a synthesis and unity of the aggregation method, the weighting method, and the normalization method (3M model) [31]. The methods are expanded by including acceptable normalization methods in the model. Thus, in the study [32], a synthesis of a solution was performed for combinations of 7 aggregation methods, 4 weighting methods, and 8 normalization methods. In contrast to the previous approach, when only consistent normalization methods (within the given metrics) are selected for the aggregation method, the study [32] uses available methods that do not contradict the basic principles of multidimensional normalization (see Section 2.8). This extends the MCDM decision sheet. Subsequent analysis and synthesis of the solutions is the basis for increasing the reliability of the results. This approach can be traced in modern studies. For example, in the study of Nguyen [16], the Mixed Aggregation by COmprehensive Normalization Technique (MACONT) is used. In this study, 6 data normalization methods are combined and studied, which lead to 11 different data normalization combination strategies. In the study of Liao and Wu [33], the integrated utilities of the alternatives are derived using two normalizations, and then the weighted Euclidean distance formula is used to integrate the collective ranking.

Thus, one of the main questions regarding how the choice of different normalization options affects the ranking of alternatives and the specific features of existing methods remains largely unexplored. In this study, we examine whether certain normalization methods satisfy the two fundamental principles of multivariate normalization. This is accomplished by, first, using a meaningful interpretation of normalized values. Second, by exploiting the fundamental properties of linear transformations for multivariate data.

2.2. The significance of normalization in the structure of the MCDM rank model

The MCDM rank model is a multi-step procedure for ordering a set of m alternatives (or objects) A_i, each of which is characterized by a set of n features (a_{i1}, a_{i2}, …, a_{in}) relative to the selected criteria C_j.

The design of the model includes the selection of a set of alternatives and a set of criteria, the assessment of the values of the attributes of the alternatives in the context of each criterion — determining the decision matrix $D=(a_{ij})$, choosing a method for normalizing the decision matrix, assessing the weight or priority of the criteria (w) based on one of the available methods, choosing an aggregation method (including various parameters of the method), choosing metrics for calculating distances in the n-dimensional space of criteria (for Goal or Reference Level Models) and other parameters in the context of the problem.

The ordering is performed based on the integral performance indicator Q_i of each alternative:

$$Q_i = F(x_{ij}, w_j),\ i = 1,...,m, \tag{2.1}$$

Where, F is one of the methods or functions (including multi-step) of data aggregation. F transforms the normalized values of the features x_{ij}, taking into account the significance of the criteria w_j, to the numerical value Q_i — the rating of the alternative (or the performance indicator, or the assessment score). For example, one of the simplest and most widely used aggregation methods is a weighted sum of the normalized values of the features of each alternative.

Then, each alternative is assigned a rank based on the values of the performance indicator (rating) and taking into account the established distinguishability rules. If there are no distinguishability rules, then for the goal "a larger (or smaller) value is better" the dominance rules have a simple form:

$$\text{if } Q_p < Q_q \text{ then } A_p \prec A_q,\ p,q \in \{1,\ 2,\ \ldots,m\},\ p \neq q, \tag{2.2}$$

Where, the sign "≺" means that the alternative A_q is preferable to A_p.

As noted above, aggregation of private attribute values requires bringing the natural values a_{ij} of attributes to a single dimensionless scale by normalization:

$$x_{ij} = Norm\left(a_{ij}\right), \tag{2.3}$$

Where, *Norm* — a data normalization method or matrix function (including multi-step) that returns normalized values of the attributes of alternatives x_{ij}. The various normalization procedures are the subject of this study and are presented below.

In studies [3-17] and others it is shown that for rank MCDM models the choice of normalization is not unambiguous. Moreover, the choice of the normalization method in MCDM models is not formalized. This determines the multi-variance of the alternatives ranking.

Additive (or multiplicative) aggregation functions F assume the consistency of the direction of improvement of each feature. Three types of goal are possible for each feature: larger the best (LTB), smaller the best (STB) and target the best (TTB). For models based on value measurement (Value Measurement Methods, such as WSM), the coordination of goals is achieved by data inversion, which means rearranging the sorted list in reverse order. Inversion methods are also multi-variant (see section 2.8, 2.9).

The assessment of the weight or priority of criteria w_j is also multi-variant. There are two groups of prioritization methods:

1) based on subjective judgments,

2) based on objective information obtained from the decision matrix.

The group of subjective methods uses comparisons in various forms (expert assessments, paired comparisons AHP/PC, comparisons of the best-worst alternatives (Best Worst Method), etc. to assess the significance of criteria. Each of the methods is a multi-step procedure for forming and processing subjective judgments or comparisons. The number of options for subjective methods taking into account various options for processing comparisons is more than 10.

For objective methods, the weight is determined based on objective data from the decision matrix using the measure defined by Zeleny [34] as the intensity of the criterion contrast. As a quantitative assessment of the intensity of contrast, the standard deviation (SD method), entropy measure of importance (EW method), and other options (CRITIC, MEREC, CILOS methods, etc. [35, 36, 37]) were proposed. For example, the SD method determines weights proportionally to the standard deviation of the *j*th feature, determined by the values of all alternatives (columns of the normalized decision matrix).

If an objective weighting method is used, the weights directly depend on the normalization of the decision matrix:

$$w_j = W(x_{ij}), \tag{2.4}$$

Where, *W* is one of the admissible methods or functions (including multi-step) for weighting criteria from the class of objective methods. At the first step, for each criterion, its contrast is determined using the feature values of all alternatives. Then, based on the obtained contrast, the weights of the criteria are derived. Due to this, the left-hand side of expression (4) does not depend on index *i*.

If one of the objective methods for assessing the weight of criteria is used in MCDM, then the effect of normalization on the rating of alternatives, in accordance with formula (1), is twofold. On the one hand, the choice of the normalization method determines the normalized values of features, on the other hand, the weight coefficients of the criteria depend on the choice of the normalization method in accordance with Eq. (2.4).

Thus, the performance of alternatives Q_i and the ranking of alternatives depends on the aggregation method (including various parameters of the method), on data normalization, on the weights of the criteria. The choice of the trio of methods (*F*, *Norm*, *W*) is not formalized and is carried out on the basis of some general principles [19]. Below is a far from complete list of MCDM methods:

F={WSM, WPM, WASPAS, CODAS, TOPSIS, VIKOR, …}[†], (2.5)

W={SD, CRITIC, EWM, MEREC, SECA, …}[‡],. (2.6)

using different normalization methods:

Norm={Max, Sum, Vec, dSum, Max-Min, Z, … }, (2.7)

discussed in detail below in the next section 2.4.

Considering that the normalized values of the alternative attributes are explicitly included in the argument of the aggregation function, it is clear that in such MCDM models, the ratings of alternatives and the ranking result depend on normalization. Therefore, the normalization of the indicators of alternatives plays a significant role in the structure of the MCDM ranking model, clearly affects the ratings of alternatives and in some cases determines their ranking.

2.3. WSM and TOPSIS is two most popular MCDM models

One common approach to solving MCDM problems involves transforming the feature vector of each alternative into a scalar feature — the rating of the alternative. The alternative's rating is determined using an aggregation function whose arguments are normalized feature values and criterion weights. A specially constructed aggregation function defines the MCDM method.

† WSM – Weighted Sum Model; WPM – Weighted Product Model; WASPAS – Weighted Aggregated Sum Product Assessment; TOPSIS – Technique for Order Performance by Similarity to Ideal Solution; VIKOR – VIse Kriterijumska Optimizacija kompromisno Resenje, in Serbian

‡ SD – Standard Deviation; EWM – Entropy Weighting Method; CRITIC – CRiteria Importance Through Inter-criteria Correlation; MEREC – MEthod based on the Removal Effects of Criteria; SECA – Simultaneous Evaluation of Criteria and Alternatives.

By the method of constructing the aggregation function, MCDM methods can be divided into three groups:

(1) the Value Measurement Methods, such as WSM, which operate with normalized values of attributes of the alternative;

(2) the Goal Level Model , such as TOPSIS, which operate with the distance of values of the from the ideal (or anti-ideal);

(3) outranking methods, such as PROMETHEE (Preference Ranking Organization METHod for Enrichment of Evaluations), which operate with a quantitative measure of preference for pairs of objects based on selected rules.

The Weighted Sum Model (WSM) uses a weighted sum algorithm of normalized values x_{ij} and has the following simple formula:

$$Q_i = \sum_{j=1}^{n} w_j \cdot x_{ij} \ , \tag{2.8}$$

Where, w_j weighting factors determining the priority of criteria. WSM is the basic aggregation method for the Value Measurement Methods.

Another main method is the Technique for order of Order of Preference by Similarity to the Ideal Solution (TOPSIS) [1], which use a *homogeneous* function of aggregation:

$$Q_i = \frac{S_i^-}{S_i^+ + S_i^-} \ , \tag{2.9}$$

Where,

$$v_{ij} = x_{ij} \cdot w_j ,\ S_i^+ = d(v_{ij}, v_j^+),\ S_i^- = d(v_{ij}, v_j^-) \ . \tag{2.10}$$

$$v_j^+ = \begin{cases} \max\limits_i v_{ij},\ if\ j \in C_j^+ \\ \min\limits_i v_{ij},\ if\ j \in C_j^- \end{cases} ,$$

$$v_j^- = \begin{cases} \min\limits_i v_{ij},\ if\ j \in C_j^+ \\ \max\limits_i v_{ij},\ if\ j \in C_j^- \end{cases} .$$

S_i^+ and S_i^- are the distances d, respectively, between the ideal and anti-ideal objects, and the alternative A_i in the n-dimensional attribute space, defined in one of the L_p-metric; v_{ij} – weighted normalized values of the jth attribute of the ith alternative, $v_j^+=\{v_1^+, v_2^+,\ldots, v_n^+\}$ – the best (ideal) values of the jth column of the matrix v_{ij}, $v_j^-=\{v_1^-, v_2^-,\ldots, v_n^-\}$ – the worst (anti-ideal) values of the jth column of the matrix v_{ij}; C_j^+– benefit criteria, C_j^- – cost criteria.

The value Q_i is always between 0 and 1, and the alternatives which have values closer to 1 are better.

TOPSIS is the basic aggregation method for the Goal or Reference Level Models. These models do not require data inversion. In fact, the matching of criteria when aggregating attribute values is performed according to the goal of maximizing the distance from the anti-ideal, or, conversely, by minimizing the distance from the ideal.

2.4. Basic linear methods of multivariate normalization

Multivariate normalization transforms data independently on each attribute. The general formula for linear normalization is:

$$x_{ij} = Norm(a_{ij}) = \frac{a_{ij} - a_j^*}{k_j},\ i = 1,...,m,\ j = 1,...,n\ . \tag{2.11}$$

The peculiarity of multidimensional normalization is that the normalization of the values of the attributes of alternatives is carried out for each criterion separately, i.e. the shear coefficients a_j^* and tension-compression coefficients k_j are different for each criterion. This is due to the fact that the attributes of objects and the ranges of their values can differ greatly from each other. Therefore, for each of the attributes, its own normalization scale is applied and the normalized values depend on the measurement scale and on the range of natural values of the attributes.

The main linear normalization methods are presented in Table 1.

Table 1: Basic linear methods for decision matrix normalization

non displacement (a_j^*=0): $x_{ij} = a_{ij} / k_j$			*with displacement:* $x_{ij} = (a_{ij} - a_j^*) / k_j$		
Max	Sum	Vec	dSum[a)]	Max-Min	Z[0,1]
$k_j = a_j^{\max}$	$k_j = \sum_{i=1}^{m} a_{ij}$	$k_j = \sqrt{\sum_{i=1}^{m} a_{ij}^2}$	$k_j = \sum_{i=1}^{m} (a_j^{\max} - a_{ij})$ $a_j^* = a_j^{\max} - k_j$ $a_j^{\max} = \max_i(a_{ij})$	$k_j = a_j^{\max} - a_j^{\min}$ $a_j^* = a_j^{\min}$ $a_j^{\min} = \min_i(a_{ij})$	Transform Z-score to [0,1] (see part 4.5)

a) dSum normalization (displacement sum) also referred to in the literature [7] as the Enhanced Accuracy method (this name, in essence, has nothing to do with accuracy) was first used in the study [38]. Another form of this normalization is x_{ij}=1-($a_j^{\max}$ -a_{ij}) / k_j. Therefore, the smaller the deviation of a feature from the maximum value, the more preferable the alternative for this feature. One of the remarkable properties of this method (as for the Sum method) is that the means of the normalized values for all criteria are the same (excluding inversion): $\bar{x}_j$=mean$_i$ (x_{ij})=x_0, $\forall j$. Due to the fulfillment of this property, as for the Sum method, the name and abbreviation dSum are appropriate.

The methods presented above transform natural values into dimensionless data and map them to the interval [0, 1], where the value 1 means the best value (for the purpose, larger is better), and the value 0 means the worst. Fig. 1 shows a graphical illustration of the normalized values for the decision matrix D_0, representing the problem of choosing from 8 alternatives according to 5 criteria, of which the 3rd and 5th criteria are cost criteria:

$$D_0 = \begin{pmatrix} 71 & 4500 & 150 & 1056 & 478 \\ 85 & 5800 & 145 & 2680 & 564 \\ 76 & 5600 & 135 & 1230 & 620 \\ 74 & 4200 & 160 & 1480 & 448 \\ 82 & 6200 & 183 & 1350 & 615 \\ 81 & 6000 & 173 & 1565 & 580 \\ 80 & 5900 & 160 & 1650 & 610 \\ 85 & 4700 & 140 & 1750 & 667 \end{pmatrix}. \tag{2.12}$$

It seems that all that remains is to choose one of the options and the issue of multidimensional normalization is resolved.

However, the normalized data for different attributes in some methods have different ranges, in other cases the domains of normalized values (for one method) are shifted relative to each other. Based on Figure 1, it can be concluded that for the Max and Max-

Min normalization methods, the contribution of criterion C_4 to the integral rating will be significantly underestimated compared to the contribution of criterion C_1 due to domain shift. For the Sum, Vec, and Z[0,1] normalization methods, one of the values of criterion C_4 can be identified as an outlier with a large contribution to the integral rating. Therefore, the integral characteristics of objects, according to formula (2.1), will be determined by the choice of the normalization method. According to the dominance rules by Eq. (2.2), the ranks of alternatives may also change.

In this study, available normalization methods are grouped into five groups based on how they align domains of normalized values of different features (by how they align the normalized scales of different features). Domain alignment allows us to reduce the hidden priority of the contribution of some features to the integral indicator that determines the ranking of alternatives.

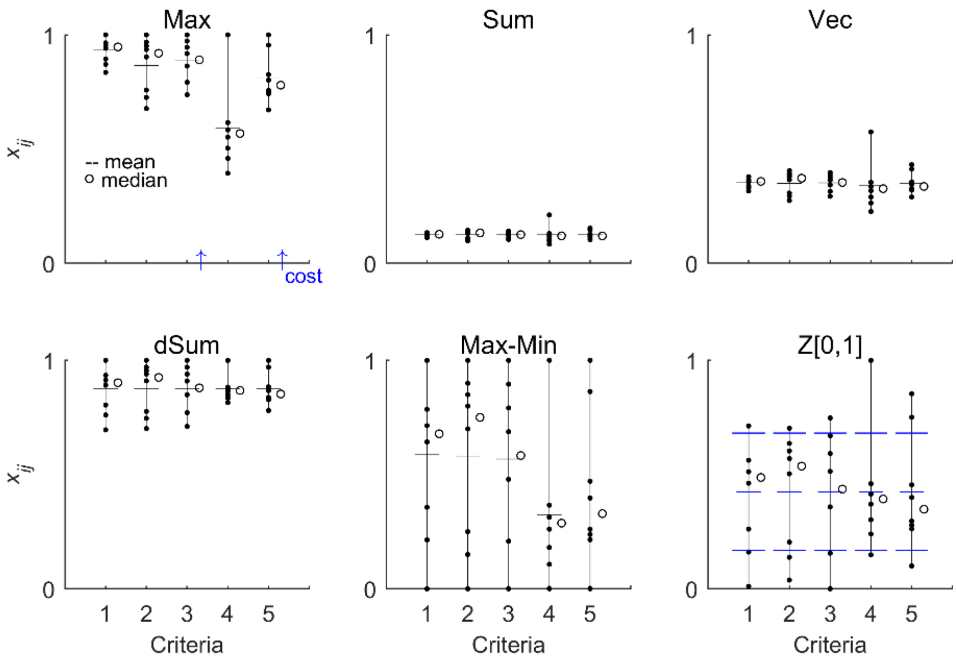

Figure 1: Domains of normalized values and their position on the interval [0, 1] for the main linear methods. Decision matrix D_0 by Eq. (2.12)

In many situations, the priorities for comparing domains of different attributes are insufficient to select a normalization method, and with different normalization methods, the rankings of alternatives can vary significantly, even to the point of changing ranks. Below, in Section 6, we present an example of two decision matrices with the same range of values for each attribute, for which all first-rank alternatives obtained in the WSM (TOPSIS) model using the normalization methods described above are different. This means that if domain shift occurs when applying a method, that method is not recommended. Therefore, the domain shift criterion is necessary, but not sufficient, for selecting a normalization method.

2.5. Substantive interpretation of the normalization scales

Multidimensional normalization transforms data independently for each attribute. However, with multidimensional normalization, the same normalization method is applied to all attributes. This is so that when obtaining an integral indicator of objects, various dimensionless shares are not aggregated. But even after reduction to a dimensionless form, the content of the natural feature in the data is preserved. What happens, for example, when adding shares of the maximum weight and shares of the maximum cost. Of course, there is no meaningful interpretation, except for the case when the best is added to the best.

The rational choice of normalization parameters is based, among other things, on the meaningful interpretation of the normalized scales. The normalized values of x_{ij} means the proportion of a natural feature in relation to its characteristic scale k_j :

Max: proportion of the attribute of the *i*th alternative relative to the highest value of the attribute, or the degree of approximation to the best value of 1,

Sum: the share of the attribute of the *i*th alternative relative to the arithmetic mean (see section 4.3) or the contrast of the *i*th alternative by the *j*th criterion, $\sum_i x_{ij}=1$,

Vec: the share of the attribute of the *i*th alternative relative to the root mean square value (see section 4.3),

Max-Min: proportion equal to the ratio of the deviation of the attribute of the *i*th alternative from the smallest value to the range of values of all alternatives according to the *j*th criterion,

dSum: inverted contrast of maximizing attribute values of the *i*th alternative by the *j*th criterion,

Z-score: standardized deviation of the attribute of the *i*th alternative from the average of all alternatives, defined in units of multiples of the standard deviation.

The Max method has a better interpretation. The alternative with the best values is preferable for the decision maker. However, the result will depend on the distribution of values in the domain. If the features are closer to the minimum value and the domains of different attributes are shifted (see Fig. 1, 4th attribute), then such normalization can lead to the priority of the contribution to the integral indicator of individual attributes.

2.6. Basic properties of linear normalization

Linear normalization is not “isotropic”. Normalization compresses the data cloud more in some directions and less in others. The number of degrees of freedom of the normalized data is n (for normalization without bias) and $2n$ (for normalization with bias). The proportions between values are preserved for each attribute, but are not consistent across attributes. In the univariate case, linear normalization methods are linear combinations of each other. In the one-dimensional case, linear normalization methods are linear combinations of each other.

Table 2 presents the properties (**P**) of statistics for the natural a_{ij} and normalized x_{ij} values that are useful when transforming data.

Table 2: Relationships for statistics of natural and normalized values, $x_{ij}=(a_{ij}-a_j^*)/k_j$

statistic	formula	property [a)]
The arithmetic mean $\bar{x}_j$ and median $md_j(x)$ are calculated using the normalization formulas through the mean $\bar{a}_j$ and median $md_j(a)$ of natural values [b)]	$\bar{x}_j = \frac{1}{m}\cdot\sum_{i=1}^{m} x_{ij},\ \bar{a}_j = \frac{1}{m}\cdot\sum_{i=1}^{m} a_{ij}$	**P.1** $\bar{x}_j = \frac{\bar{a}_j - a_j^*}{k_j}$ $md_j(x) = \frac{md_j(a) - a_j^*}{k_j}$
Standard deviation is scaled	$s_j(x) = \sqrt{\frac{1}{m}\cdot\sum_{i=1}^{m}(x_{ij}-\bar{x}_j)^2}$	**P.2** $s_j(x) = \frac{1}{k_j}\cdot s_j(a)$
Pearson correlation coefficient for the columns of the decision matrix — *invariant*	$r_{jk}(x) = \frac{\sum_{i=1}^{m}(x_{ij}-\bar{x}_j)\cdot(x_{ik}-\bar{x}_k)}{\sqrt{\sum_{i=1}^{m}(x_{ij}-\bar{x}_j)^2\sum_{i=1}^{m}(x_{ik}-\bar{x}_k)^2}}$	**P.3** $r_{jk}(x) = r_{jk}(a)$
Skewness of the values of the *j*th attribute — *invariant*	$\gamma_j(x) = \frac{\mu_3}{s_j^{3/2}} = \frac{\frac{1}{m}\cdot\sum_{i=1}^{m}(x_{ij}-\bar{x}_j)^3}{\left(\frac{1}{m}\cdot\sum_{i=1}^{m}(x_{ij}-\bar{x}_j)^2\right)^{3/2}}$	**P.4** $\gamma_j(x) = \gamma_j(a)$
Disposition [c)] of the values of the *j*th attribute for A_p and A_q alternatives — *invariant*	$\Delta x_{pq}^{(j)} = \frac{\lvert x_{pj} - x_{qj}\rvert}{x_j^{\max} - x_j^{\min}}$	**P.5** $\Delta x_{pq}^{(j)} = \Delta a_{pq}^{(j)}$ $\forall p, q = 1,...,m, \forall j$
The disposition dQ of values of the rating A_p and A_q of alternatives is *invariant* with respect to the linear transformation: $u_{ij}=c\cdot x_{ij}+h_j$, $\forall$ c, $h_j \in \mathbb{R}$ if a linear or homogeneous aggregation function is applied in formula (2.1)	$dQ_{pq}(x) = \frac{Q_p(x) - Q_q(x)}{Q_{\max}(x) - Q_{\min}(x)}$,	**P.6** Let $u_{ij}=c\cdot x_{ij}+h_j$, $\forall$ c, $h_j \in \mathbb{R}$, then $dQ_{pq}(u) = dQ_{pq}(x)$ $\forall p, q = 1,...,m$ for linear or homogeneous aggregation functions F by Eq. (2.1)

[a)] $s_j(x)$ and other analogs in this table should be understood as follows: s statistics of the *j*th attribute for the normalized matrix (x_{ij}). All statistics for the original matrix (a_{ij}) (3rd column of the table) are calculated similarly to those for the normalized matrix in the second column.. It is enough to replace x with a in the formulas.

[b)] property P.1 is also true for point statistics: harmonic mean (m_{HM}) geometric mean (m_{GM}), root mean square (m_{RMS}) provided that non displacement normalization is applied (a_j^*=0).

[c)] The disposition is the relative (reduced to the range) distance between the values of the *j*th attribute for the alternatives A_p and A_q or in the case of a sorted list, has the form:

$$\Delta x_i^{(j)} = \frac{x_{i+1,j} - x_{ij}}{x_{mj} - x_{1j}}, i = 1,...,m-1, \forall j$$

The term "disposition" is a concept used in the military affairs. In the military field, real objects are transferred to maps of varying scales while maintaining distances in two dimensions. Disposition determines how objects are located relative to each other on different measurement scales (j) when the scale changes. Disposition $\Delta x_{pq}^{(j)}$ is scale invariant, i.e., it does not change when scaled.

The main property **P.5** of linear normalization: with a fixed j (fixed criterion), for all linear normalization methods the disposition between natural and normalized values is preserved.

A visual illustration of the property of maintaining proportions when scaling using the scale technique 'YAxisLocation' of the MatLab programming and numeric computing platform is presented in Figure 2.

Property **P.5** is illustrated for three normalization methods Max, Sum and Max-Min respectively for attributes C_1, C_1 and C_5 of matrix D_0 according Eq. (2.12). Dispositions of normalized values fully correspond to dispositions of natural values. If normalized values are subjected to linear transformation, then dispositions are preserved (invariant with respect to linear transformations). However, in case of normalization of multidimensional data, the scaling factor should be the same for all attributes.

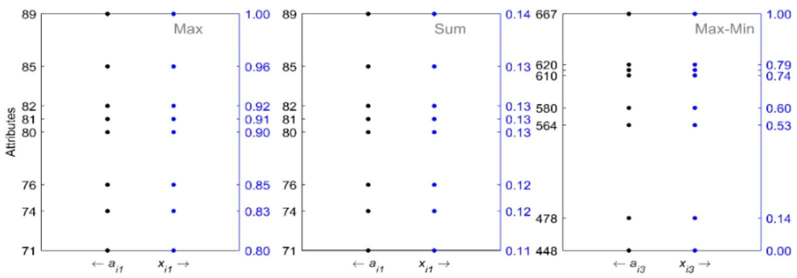

Figure 2: Correspondence of dispositions of natural (a_{ij}) and normalized values (x_{ij}) of attributes for the Max (C_1), Sum (C_1) and Vec (C_5) normalization methods

2.7. Basic principles of multidimensional data normalization

Preserving the information content of data after conversion includes:

i. Preserving the information content of data in the context of each attribute

It's simple. For all linear normalization and all linear transformation of data the dispositions of natural values for each attribute are preserved [17] (property P.5).

ii. Coordination of scales of normalized values

It is not simple. The range of values for different attributes is different. Additionally, there is a shift in the domains of normalized values of various features relative to each other (Figure 1). Each attribute has its own normalization scale, which is not consistent with the normalization scale of other attributes.

For multi-objective problems, linear methods produce anisotropic scaling when at least one of the scaling factors is different from the others. There is a deformation of the multidimensional cloud of features (Figure 3).

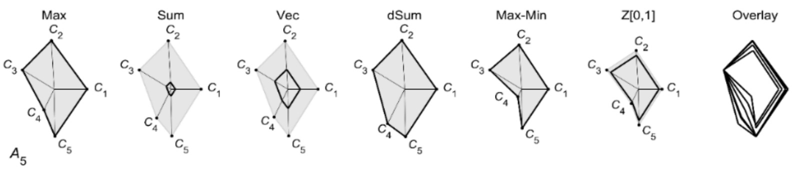

Figure 3: Range of normalized feature values for linear methods. 5th criteria of D_0. Radar chart

Figure 3 illustrates the deformation of the feature cloud for the 5th alternative with different normalization methods. Such deformation can alter the alternative's ranking by changing the priority of the criteria's contributions. All this leads to the priority of the contribution of individual attributes to the performance indicator of alternatives. The degree of difference depends on the decision matrix *D* (on the problem), so the choice of data normalization method is also related to the problem.

To eliminate priority, it is necessary to harmonize the scales of various characteristics. For each alternative, the scale is individual (k_j are different) and depends on the normalization method and on the range and values of the selected alternatives. Additional consistency is achieved by transforming the normalized values.

2.8. The goal inversion for the Value Measurement Methods

Aggregation of attributes involves matching the goals of the criteria if the goal type of the different criteria is different. For models based on value measurement methods, such as WSM, matching the goals is achieved by data inversion, which is understood as rearranging the sorted list in reverse order.

For MCDM models with a common goal of LTB, the cost criteria are inverted, and with an STB goal, the benefit criteria are inverted. For criteria with a specified target value, with a common goal of the LTB task, the inversion is performed for all values greater than the target, and with an STB goal, the inversion is performed for all values less than the target [17].

By review Jahan [7] and other studies, the term inversion is not used, but separate consistent normalization of benefit criteria and cost criteria is used. In most studies, the normalization of cost criteria is performed using a nonlinear inversion of the $1/a$ type.

For example, the Max normalization method is combined with the *i*Max nonlinear inversion:

$$x_{ij} = \mathrm{Max}(a_{ij}) = \frac{a_{ij}}{a_j^{\max}},\ \text{for } j \in C_j^+, \quad x_{ij*} = i\mathrm{Max}(a_{ij}) = \frac{a_j^{\min}}{a_{ij}},\ \text{for } j^* \in C_j^-, \tag{2.13}$$

It is clear that the monotone strictly decreasing function $1/a$ will preserve the sequence of values in the list and invert the data, so that larger values become smaller and vice versa. In the study [] it is shown that such an inversion is incorrect. The nonlinear transformation changes the data, or rather changes the distances between the values. This means that the normalized data are fundamentally distorted compared to the original (natural) values.

As an alternative, a linear inversion method is proposed — the Reverse Sorting (ReS) algorithm, which eliminates the above problems [17]. The ReS algorithm is a linear transformation, preserves the dispositions of values, preserves the domains of normalized values, and can be applied to both the original (2.14) and normalized (2.15) sets:

$$ReS\left(a_{ij}\right) = -a_{ij} + a_j^{min} + a_j^{max},\ \text{for } j \in C^- - \text{cost criteria}, \tag{2.14}$$

$$ReS\left(x_{ij}\right) = -x_{ij} + x_j^{min} + x_j^{max},\ \text{for } j \in C^- - \text{cost criteria}. \tag{2.15}$$

The ReS algorithm works equally well for the case of the “smaller the better” objective, transforming all criteria to the STB type. Formulas (2.14)-(2.15) have the same form and are applied to the benefit criteria: $j \in C^+$ — benefit criteria.

Dozens of works on MCDM use the nonlinear inversion *i*Max=$x_j^{\min}/x_{ij}$ (and analogs *i*Sum, *i*Vec), violating the first principle of normalization and changing the information contained in the decision matrix. This also applies to objective weighing methods (Entropy, CRITIC, SILOS, SECA, etc. [35, 36, 37]), which use the inversion of the objective.

Violation of the original data can be easily demonstrated if we use dispositions for a sorted list of attribute values (natural or normalized):

$$\Delta x_i = \frac{x_{i+1} - x_i}{x_m - x_1}, \ i = 1,...,m-1. \tag{2.16}$$

Dispositions show the relative position of the option regardless of the measurement scale. According to property P.5, Δx_i is an invariant under linear transformations, i.e. the dispositions of the natural values of the attribute are equal to the dispositions of the normalized values under the linear normalization method.

Figure 4 shows the domains of normalized values for the normalization of Max and the inversion using a nonlinear transformation of the type $1/a$ and the inversion using the linear transformation ReS (5th criterion of the D_0 matrix).

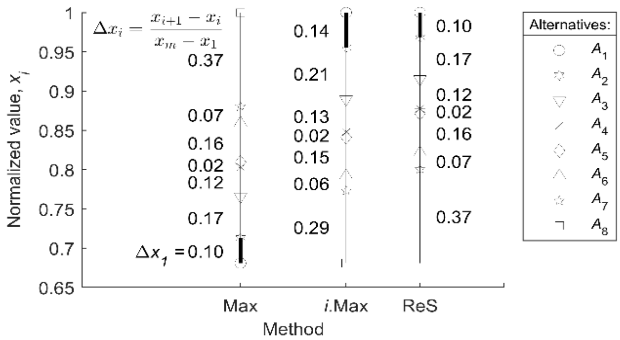

Figure 4: Distortion of dispositions during nonlinear inversion $x_j^{\min} / x_{ij}$

For the inversion of *i*Max, the dispositions do not correspond to the dispositions of the original normalized values Max(*a*). For some points, the dispositions differ by up to 3 times. This is a significant violation of the structure of the original data.

When using the ReS transformation, the data are inverted symmetrically with respect to the middle of the domain (midpoint, $x_j^*=(\max_i x_{ij}+\min_i x_{ij})/2$), and not with respect to the mean value (or median). Therefore, if normalization requires equality of means of different attributes, for example, as in normalization of Z-scores, it is recommended to perform inversion before normalization.

Inversion is part of normalization and is intended to align the goals of different criteria. It is possible to first invert some of the criteria (for natural values), and then normalize all the data. Or vice versa, first normalize all the criteria, and then invert some of the criteria.

The order of applying the inversion/normalization affects the result if the shift coefficients a_j^* or the stretch-compression coefficients k_j change:

Res(Norm(*a*)) ≠ Norm (Res(*a*)).

This is particularly true for the normalization methods Sum, Vec, dSum. For these methods, the greater the asymmetry of the data relative to the mean, the greater the difference in the result.

For the Max and Max-Min normalization methods, the inversion order does not affect the normalization result, since with ReS inversion the maximum and minimum values remain unchanged and, therefore, the normalization parameters of the Max and Max-Min methods (the shift coefficients a_j^* and the stretch-compression coefficients k_j) are preserved:

- Res(Max(a))=Max(Res(a)),
- Res(Max-Min(a))=Max-Min(Res(a)).

For the Z[0, 1] normalization method and its analogues (see section 4.5 below), the inversion/normalization order does not affect the stretch-compression coefficient k_j. Only the shift coefficient a_j^* changes. Therefore, given the normalization property P.6, the inversion/normalization order does not affect the ranking of alternatives, although the normalized values will be shifted.

As a rule, the influence of the order of applying normalization/inversion on the ranking of alternatives is insignificant. There is no formal criterion for the order of applying normalization/inversion, which is another degree of freedom for MCDM methods.

2.9. Linear inversion of normalized values of the form (1–*x*)

Another popular inversion approach combines normalization and inversion. First, normalize all attributes, and then perform a linear inversion of the form (1–x) for the cost attributes. For example, for the Max method, the normalization/inversion is:

$$x_{ij} = \mathrm{Max}(a_{ij}) = \frac{a_{ij}}{a_j^{\max}}, \ \forall j, \tag{2.17}$$

$$x_{ij*} = 1 - x_{ij}, \text{ for } j^* \in C_j^- - \text{cost criteria}. \tag{2.18}$$

In this case, if the normalized values x_{ij} are shifted to 1, as for the Max method, then (1–x_{ij}) will be shifted to 0 and the contribution of the cost criteria to the integral indicator will obviously be lower than the contribution of the benefit criteria. However, since (2.18) is a linear transformation, then in accordance with property P.6 this will not affect the ranking in the case of using the WSM or TOPSIS aggregation methods, including analogues.

Note that the above is also true for the inversion –x:

$$x_{ij*} = -x_{ij}, \text{ for } j^* \in C_j^- - \text{cost criteria}. \tag{2.19}$$

In this case, if normalization transforms values to [0, 1], then when using inversion (2.19) $x_{ij} \in [-1, 1]$.

Inversions (2.18), (2.19) are obviously problematic when using nonlinear methods of aggregating normalized values of different attributes.

3. COMPARISON OF DIFFERENT RANK MODELS OF MCDM

3.1. Comparison of the rating of different MCDM models

The presence of a large number of multi-criteria decision-making methods requires the development of a tool that can be used to compare them with each other and to synthesize solutions. Comparison of the ranks of different methods is simple, but is rather rough, since the ranks do not reveal the degree of superiority of alternatives among themselves. A rating list, in contrast to a ranking list, reflects the "thin" structure of relationships between alternatives [31]. However, the ratings of different methods are defined in different scales determined by the method of aggregating private attribute values.

As a result of applying two different MCDM ranking models, two rating lists are obtained: $Q^{(1)}=\{Q_i^{(1)}\}_{i=1,\dots,m}$ и $Q^{(2)}=\{Q_i^{(2)}\}_{i=1,\dots,m}$. The lower index corresponds to the ith alternative, the upper index corresponds to the model number.

Definition 1. *Two rating lists with the same number of elements are equivalent ($Q^{(1)} \sim Q^{(2)}$), if they are transformed into each other using a linear transformation.*

For comparison, it is rational to order the rating of alternatives and transform both lists to the Relative Performance Indicator (RPI) form [31]:

$$\mathrm{d}Q_i = \frac{Q_{i+1} - Q_i}{Q_m - Q_1} \cdot 100\%,\ \ i = 1,\dots,m-1, \tag{3.1}$$

Where Q_i is the value of the performance indicator corresponding to the alternative of the ith rank in the ordered list.

In this case, the indicator dQ is the relative (given in the range of Q scale) increase or decrease in the performance indicator for the ordered list of alternatives and the property is satisfied:

$$\sum_{i=1}^{m-1} dQ_i = 100, \tag{3.2}$$

Definition 2. *Two ranking lists are equivalent if they have the same RPI:*

$$dQ_i^{(1)} = dQ_i^{(2)},\ \forall i=1,\dots,m-1. \tag{3.3}$$

Property: if two ranking lists are equivalent, then both ranking models are equivalent.

Example: consider two lists:

$Q^{(1)}$={0.2948 0.7995 0.3337 0.4127 0.3956 0.5646 0.4573 0.5299}

$Q^{(2)}$={0.6017 0.7279 0.6114 0.6312 0.6269 0.6692 0.6423 0.6605}

Offhand, it is difficult to establish a relationship between these two sets. Let's sort them in descending order and at the same time set the ranks of the elements (element number in the ordered list):

$Qs^{(1)}$={0.7995 0.5646 0.5299 0.4573 0.4127 0.3956 0.3337 0.2948}

$Qs^{(2)}$={0.7279 0.6692 0.6605 0.6423 0.6312 0.6269 0.6114 0.6017}

$R_i^{(1)}$={2 6 8 7 4 5 3 1}

$R_i^{(2)}$={2 6 8 7 4 5 3 1}

It turned out that $Q^{(1)}$ and $Q^{(2)}$ have the same ranks.

Let us present a graphical illustration for $Q^{(1)}$ and $Q^{(2)}$ regardless of the scale (Figure 5).

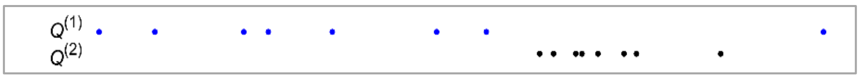

Figure 5: Positions of the rating lists $Q^{(1)}$ and $Q^{(2)}$ after reduction to a common numerical scale

Only with a careful analysis of the figures, can we assume that the distances between adjacent elements for the two lists are proportional. Indeed, using formula (3.1) we obtain:

d$Q^{(1)}$={46.5444 6.8698 14.3956 8.8325 3.3856 12.2668 7.7053} d$Q^{(2)}$={46.5444 6.8698 14.3956 8.8325 3.3856 12.2668 7.7053}

Initially, the first list represents the ratings Q_i of alternatives of the decision-making problem with the matrix D_0 by Eq. (2.12) for the WSM(Max-Min(D_0), w) model with the inversion ReS and weight coefficients w=(⅙ ⅙ $^1/_{12}$ ⅓ ¼). The second list represents the rankings of alternatives of the same problem, for a similar model, but using the normalization proposed by Lai and Hwang method [39]:

$$x_{ij} = \frac{2}{m} \cdot \frac{a_{ij}}{a_i^{\max} - a_i^{\min}}, \tag{3.4}$$

The coefficient 2/m in the previous formula represents a scaling factor, which, as follows from property P.6, does not affect the RPI. The shift coefficient in the formula for Max-Min normalization also does not affect the RPI. Thus, the two rating lists are equivalent and both ranking models are equivalent, as demonstrated in the presented example.

The proposed dQ indicator serves as an important characteristic for comparing MCDM models. In particular, in the study [31] using RPI, identical methods of aggregating private attributes of alternatives were established: WSM, MABAC, TOPSIS(L_1), Ratio System approach (RS), provided that each model uses the same linear normalization method and the same criterion weights. Later, the methods Multi Atributive Ideal-Real Comparative Analysis (MAIRCA) and Ranking of Alternatives with Weights of Criterion (RAWEC) were added to this list.

3.2. Linear transformation of normalized values: Rank invariance

According to property P.6, in the case of using linear (WSM and similar) or homogeneous (TOPSIS and similar) aggregation functions, the linear transformation of normalized values of the form:

$$u_{ij}=C\cdot x_{ij}+h_j, \tag{3.5}$$

does not affect the ranking of alternatives.

Let us denote the ratings of alternatives based on the normalized values of x_{ij} as $Q_i^{(1)}$, and the ratings of alternatives based on the normalized values of u_{ij}, according to formula (3.5), as $Q_i^{(2)}$. Then

$$Q_i^{(1)} = F\left(x_{ij}, w_j\right),\ Q_i^{(2)} = F\left(u_{ij}, w_j\right) \Rightarrow dQ_i^{(1)} = dQ_i^{(2)}. \tag{3.6}$$

Indeed:

$$Q_i^{(2)} = \sum_{j=1}^{n} w_j u_{ij} = \sum_{j=1}^{n} w_j \cdot (C \cdot x_{ij} + h_j) = C \cdot \sum_{j=1}^{n} w_j \cdot x_{ij} + \sum_{j=1}^{n} w_j \cdot h_j = C \cdot Q_i^{(1)} + h_w,$$

$$\mathrm{d}Q_{pq}^{(2)} = \frac{Q_p^{(2)} - Q_q^{(2)}}{Q_{\max}^{(2)} - Q_{\min}^{(2)}} = \frac{C \cdot Q_p^{(1)} + h_w - C \cdot Q_q^{(1)} - h_w}{C \cdot Q_{\max}^{(1)} + h_w - C \cdot Q_{\min}^{(1)} - h_w} = \frac{Q_p^{(1)} - Q_q^{(1)}}{Q_{\max}^{(1)} - Q_{\min}^{(1)}} = \mathrm{d}Q_{pq}^{(1)}.$$

Similarly, for the TOPSIS method.

This means that the linear transformation of normalized values with preservation of dispositions (the constant C is the same for all attributes), and the simultaneous shift of the normalized values of each attribute by the value h_j (each attribute can define its own shift) does not change the RPI and the ranking of alternatives (for WSM, TOPSIS and analogues).

If the aggregation function F is nonlinear, for example, the Weighted Product Model (WPM), then the transformation (3.5) defines an infinite set of normalizations in the absence of a selection criterion, and as a consequence, the impossibility of a rational choice.

In accordance with the priority of operations in formula (3.5), it is necessary to first perform scaling, and then shift. In transformation (3.5), the scaling coefficient "C" must be the same for all attributes, and the shift h_j can be individual for each attribute j. This means that when C=Const, the domains of normalized values can be shifted independently and arbitrarily without damaging the result. For example, all values for the first attribute can be increased by 7, and all values for the second attribute can be decreased by 3 units. The relative rankings of the alternatives dQ_i will not change.

Thus, the transformation (3.5) can be applied for the purpose of comparing normalized attribute values for different normalization methods. In addition, in the extensive list of normalization methods, equivalent methods that determine equivalent rankings of alternatives are easily found. For example, in the notation of the paper [7], Norm16 and Norm18 are equivalent (for the justification, see the example in section 3.2).

In addition to rigorous proof, numerical experiments have shown that in aggregation methods such as WSM, RS, MABAC (Multi-Attributive Border approximation Area Comparison), ARAS (Additive Ratio Assessment methodology), GRA (Grey Relational Analysis), MARCOS (Measurement of Alternatives and Ranking according to Compromise Solution), RADAR-II (the RAnking based on Distance And Range), MAIRCA, RAWEC, and TOPSIS, the dQ indicator is invariant under the linear transformation (3.2). This means that the ranking based on the alternative rating in these methods does not change under linear transformations of the normalized values according to (3.2). Exceptions include the CODAS and COPRAS (Complex Proportional Assessment) methods, as well as nonlinear aggregation methods such as WPM, WASPAS, and others. The formalization of this property within the rank model (2.1) has the following form:

Suppose that within the chosen model (F, *Norm*, W), where *Norm* is a linear normalization method, the obtained rankings of alternatives are $Q^{(1)}$,

$$Q_i^{(1)} = F(x_{ij}, w_j),\ i = 1,...,m, \tag{3.7}$$

If we perform a linear transformation of the normalized values $u_{ij}=Cx_{ij}+h_j$, the rankings of the alternatives will change. We denote them as $Q^{(2)}$:

$$Q_i^{(2)} = F(Cx_{ij} + h_j, w_j),\ i = 1,...,m, \tag{3.8}$$

$$Q_i^{(1)} \neq Q_i^{(2)}. \tag{3.9}$$

However, if the model uses the aggregation methods:

F={WSM, TOPSIS, RS, MABAC, ARAS, GRA, MARCOS, RADAR-II, MAIRCA, RAWEC}, (3.10)

and linear transformations of normalized values, then

$$dQ_i^{(1)} = dQ_i^{(2)},\ \forall i. \tag{3.11}$$

According to Definition 2 of the previous section, the linear transformation produces the same rank, or the ranks of the alternatives are invariant with respect to the linear transformation of the form (3.5).

Note 1: In linear multivariate normalization methods of the form (2.11), the stretch-compression coefficients are different for each criterion, unlike the transformation (3.5). Therefore, the rankings of the alternatives may change.

Note 2: In linear multivariate normalization methods of the form (2.11) with bias ($a_j^*\neq 0$), the bias does not affect the ranking if the model uses the aggregation methods (3.10) specified above.

3.3. The same Essence of the Value Measurement Methods (WSM) and the Goal Level Model (TOPSIS) within the framework of linear normalization methods

Let us consider two models WSM and TOPSIS(L_1), which differ only in the aggregation method. For the TOPSIS model, according to formula (2.9), we use the L_1 distance metric (Manhattan or TaxiCab distance):

$$d(v_{ij}, v_j^*) = \sum_{j=1}^{n} \left| v_{ij} - v_j^* \right| . \tag{3.12}$$

We denote such a model as TOPSIS(L_1). The authors of [31] have previously shown the equivalence of these two models without specifying the choice of the normalization method.

In this study, it is specified that the WSM and TOPSIS(L_1) models are equivalent for all linear normalization methods with the same weight of criteria for both models. However, when using WSM, the order of applying the inversion and normalization is important for the Vec and Sum methods. $\mathrm{d}Q_i^{(1)} = \mathrm{d}Q_i^{(2)}$ with sequential application of ReS(*Norm*(a)). If we first perform the inversion and then normalization, then the results $\mathrm{d}Q_i^{(1)} \neq \mathrm{d}Q_i^{(2)}$. This remark was previously described in Section 3.1.

Despite the significant difference in the aggregation formulas of the WSM and TOPSIS(L_1) methods, these methods are identical in RPI values: WSM(x, w) ~TOPSIS(x, w, L_1), i.e. $\mathrm{d}Q^{\mathrm{WSM}} = \mathrm{d}Q^{\mathrm{TOPSIS}}$. In particular, the equivalence of WSM and TOPSIS(L_1) means that both approaches have the same character and shows that Value Measurement models and Reference Level models are interrelated.

4. NORMALIZATION TECHNIQUES FOR MULTIVARIATE DATA

In this section, multivariate normalization methods are grouped into five groups based on how they align the domains of normalized values of different features. It is assumed that domain alignment reduces the hidden priority of the contribution of individual features to the integral indicator that determines the ranking of alternatives. It is also assumed that the rankings of alternatives for MCDM models using different normalization methods from the same group are consistent.

In this study, each group includes either known methods or below present a linear transformation technique that transforms normalized values to a given type. Figure 6 shows a classification scheme that aligns domains by 1, 2, or 3 features.

The first group "1) Up.Norm" includes methods that produce identical upper domain boundaries.

The second group "2) IZ.Norm" includes methods that produce identical upper and lower domain boundaries.

The third group "3) Mean.Norm" includes methods that produce identical averages.

The fourth group "4) Dist.Norm" represents the normalization of the deviation matrix (ΔD). The methods in this group generate identical upper (or lower) bounds and identical mean values.

The fifth group "5) MS.Norm" includes methods that produce identical means and identical ranges.

In many situations, there are no priorities when comparing domains of different attributes, although with different normalization methods, the rankings of alternatives can differ significantly, even to changes the rank of alternatives. If domains bias occurs when applying a method, of normalization then this method is not recommended. Therefore, a domain bias criterion is necessary, but not sufficient, for selecting a normalization method.

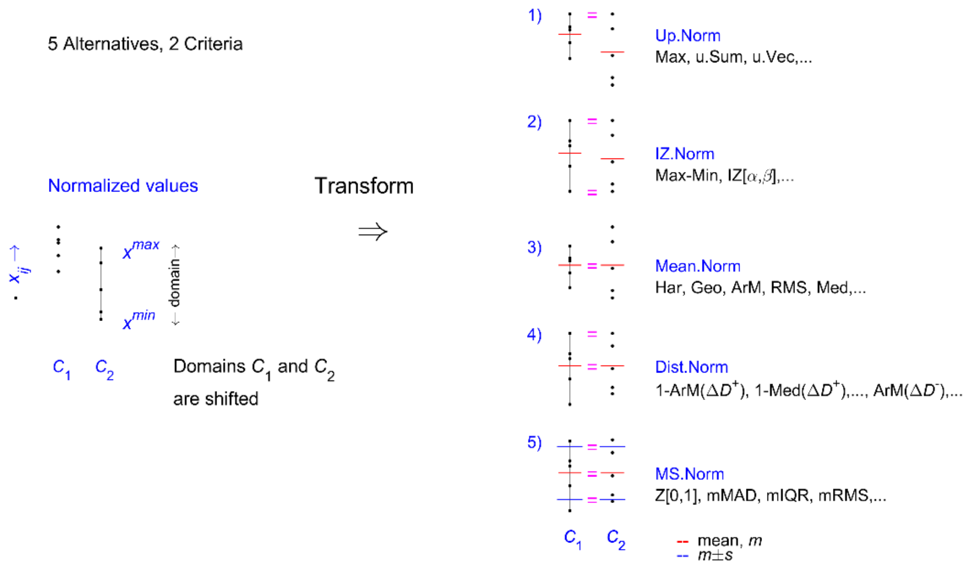

Figure 6: Classification scheme of normalization methods based on domain alignment of normalized values

4.1. Linear transformation of normalized values to 1 (Up.Norm)

One of the best normalization methods for interpreting normalized values is the Max method, which defines normalized values as fractions of the best value. Therefore, it is advisable to make the largest (best) normalized value for each attribute equal to 1. Let us transform the normalized values of the interval (0,1) of such methods as Sum, Vec, etc., into the interval of values (0, 1]. For this, we use the linear transformation (3.5). Considering that the rating of alternatives will not change when using the linear transformation (3.5), this will allow us to compare the results of different normalization methods with the Max method.

Let us denote such a transformation as normalization with displacement to the upper value and use the abbreviation u.Norm, for example, u.Sum, u.Vec.

For scaling, we apply a preliminary shift of the domain of each attribute to a fixed point 0, and then perform scaling by multiplying by the scale factor C:

$$v_{ij} = (x_{ij} - min_i(x_{ij}))\cdot C\ . \tag{4.1}$$

This allows us to control the domain span. We define the scale factor C so as to obtain the same value of the largest attribute span as with using the Max method:

$$x_{ij}^0=\mathrm{Max}(a_{ij}), \tag{4.2}$$

$$C = \frac{\max_j(rng_i(x_{ij}^0))}{\max_j(rng_i(x_{ij}))}\ . \tag{4.3}$$

Next, we shift each domain to 1.

$$u_{ij} = v_{ij} + 1 - max_i(v_{ij})\ . \tag{4.4}$$

Example: For a decision-making problem with matrix D_0 by Eq. (2.12) for the WSM(*Norm*(D_0), w) model with the ReS inversion and weight coefficients w=(⅙ ⅙ $^1/_{12}$ ⅓ ¼). In three cases, the Max, u.Sum, and u.Vec methods are used as *Norm*().

Two of the Sum and Vec methods listed in Table 1 have a range of values in the interval (0; 1). For the Sum method, the range of values is about $1/m$, and for the Vec method, it is about $1/\sqrt{m}$. A graphical illustration of the normalized values after such a transformation is shown below in Figure 7.

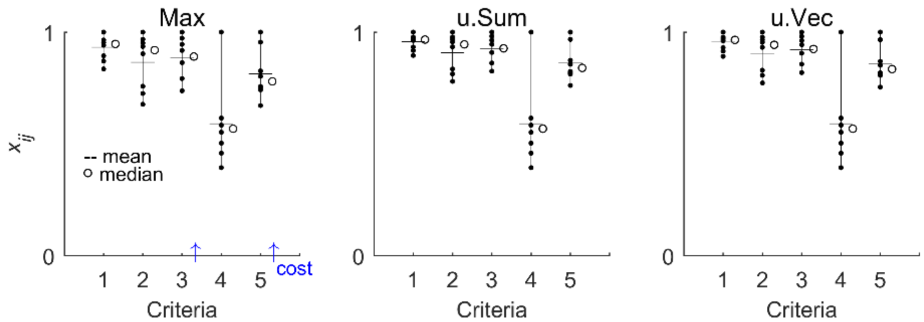

Figure 7: Linear transformation to 1 for all attributes. Decision matrix D_0

The scaling coefficient C according to formula (3.5) is defined by the 4th attribute, i.e. the range of normalized values for the 4th attribute for all methods Max, u.Sum and u.Vec coincides. This is not the case for other attributes.

The visual effect compared to Fig. 1 is impressive. It is easy to see that the areas of normalized values for methods Max, u.Sum and u.Vec almost coincide, while in Fig. 1 this is not determined. The corresponding ranks of alternatives obtained by the WSM method coincide: R_i={2 6 8 7 4 5 3 1}. But still, these are three different normalization methods Max, Sum and Vec, having three different lists of ratings regardless of the transformation and having different RPI for the Max, Sum and Vec methods and the same RPI for the pairs of methods Sum/ u.Sum, Vec/ u.Vec:

$dQ_i^{(Max)}$={43.0517 19.8687 5.5544 6.7775 9.5646 9.4885 5.6946}

$dQ_i^{(Sum)}$={41.8398 21.6570 3.9157 7.4670 9.6606 8.6735 6.7865}=$dQ_i^{(u.Sum)}$

$dQ_i^{(Vec)}$={41.9799 21.5105 4.0518 7.3640 9.7214 8.7075 6.6650}=$dQ_i^{(u.Vec)}$

The obtained result means that instead of the normalization methods Sum and Vec, it is better to use the equivalent methods u.Sum and u.Vec.

4.2. Normalization leading to the same range of values of attributes (IZ.Norm)

It is logical to aggregate different features when their ranges coincide. Let the worst (smallest) value be equal to *I* for all criteria, and the best (largest) value be equal to *Z*. Intermediate values are interpreted as a share of the range (*Z-I*). In Max-Min normalization, the domain [*I*, *Z*] is [0, 1]. But in some aggregation methods (WPM, COPRAS, WASPAS, etc.), it is necessary to exclude the zero value. Then we apply the IZ transformation [19]:

$$\mathrm{IZ}:\ [0,\ 1] \rightarrow [I, Z]. \tag{4.5}$$

$$x_{ij} = \frac{a_{ij} - a_j^{\min}}{a_j^{\max} - a_j^{\min}} \cdot (Z - I) + I = Max\text{-}Min(a) \cdot (Z - I) + I, \quad [I, Z] \subset [0,\ 1],\ \forall i, j\ . \tag{4.6}$$

Useful properties of transformation (4.6) [17]:
1. Max-Min(*Norm*(*a*))=Max-Min(*a*) for all linear normalization methods.
2. IZ is a linear transformation $u_{ij}=c \cdot x_{ij}+h_j$, that preserves ranking (and proportionality of the performance indicator of alternatives) for cases where the feature aggregation function is linear (WSM, etc.) or homogeneous (TOPSIS, etc.).

There are no problems with choosing the upper bound of *Z*. This value is equal to 1, which allows comparing the results of different tasks. For the lower bound, it is recommended to use the median value (robust characteristic) of the smallest normalized feature values obtained with Max-normalization [17], since Max-normalization has the simplest and most understandable interpretation as a share of the best value:

$$I = md\left(\mathrm{Max}\right) = median_j[min_i(\mathrm{Max}(a_{ij}))]\ . \tag{4.7}$$

An illustration of multivariate normalization with equal values of the upper and lower domain boundaries for all attributes is show in Figure 8.

The values of IZ[*md*(Max),1] in Figure 8 are consistent with the Max normalization when determining the lower boundary of *I*=*md*(Max) using formula (4.7). All three presented normalizations preserve the ranking for cases where the feature aggregation function is linear (WSM, etc.) or homogeneous (TOPSIS, etc.). In the aggregation methods WPM, COPRAS, WASPAS, etc., the IZ transformation is simply necessary. This is also relevant for nonlinear normalization (see section 5).

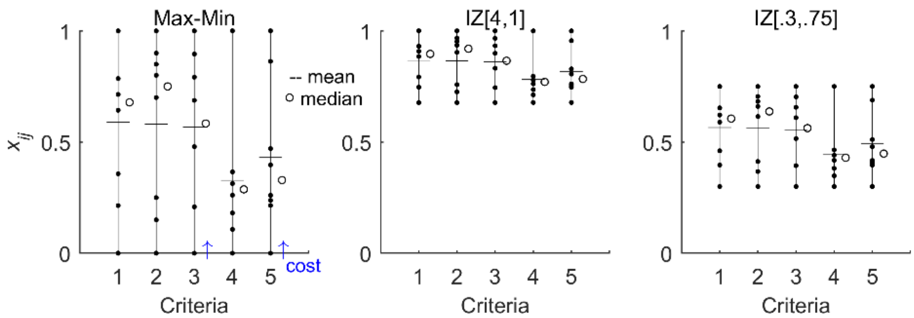

Figure 8: Normalizations with equal values of the upper and lower boundaries of the domain of all attributes. Decision matrix D_0

When Max-Min or IZ normalizing, the mean and median attribute values are shift relative to each other, and the data density (standard deviations) is different (Figure 8). If the shift is significant, this may affect the subsequent ranking of alternatives only for nonlinear aggregation methods.

4.3. Normalization technique of the multivariate data that results in equal "mean" values (Mean.Norm)

4.3.1. Generalization of Sum, Vec normalization as a proportion of the mean: Har, Geo, RMS, CoH, Med normalization

Let us represent the Sum normalization as follows:

$x_{ij} = \mathrm{Sum}(a_{ij}) = a_{ij}/(m \cdot k_j/m) = (1/m) \cdot a_{ij} / \overline{a}_j$,

Where, $\overline{a}_j$ is the arithmetic mean of the jth attribute over all alternatives and m the numbers of alternatives.

Note that for a linear or homogeneous aggregation function, scaling all attributes (dividing by m) does not affect the rating and rank results (property P.6). Therefore, the normalized values by the Sum method are interpreted as fractions of the arithmetic mean. Since the scale factor k_j for the Sum normalization method as a sum of natural values has no interpretation, it would be more correct to define this method as Arithmetic Mean (ArM) normalization.

Similarly, for the Vec normalization method, the characteristic scale does not have a vector meaning, as follows from the abbreviation of the method. The attribute values are defined on a one-dimensional scale, while the vector metric is multidimensional. As with the Sum method, multiplication and division by $\sqrt{m}$ allows the normalized value to be interpreted as a proportion of the mean square value. Therefore, it is more correct to define this method as Root Mean Square (RMS) normalization.

$x_{ij} = \mathrm{Vec}(a_{ij}) = a_{ij}/(\sqrt{m} \cdot k_j/\sqrt{m}) = (1/\sqrt{m}) \cdot a_{ij}/s_j$,

Where, s_j is the mean square value of the jth attribute across all alternatives.

By analogy, other normalizations can be defined, with the same interpretation of values as proportions of the mean. In particular, one can use the harmonic mean (HM), geometric mean (GM), median mean (Med), and contra-harmonic mean (CM) for which the following inequalities are satisfied:

$$min \le HM \le GM \le AM \le RMS \le CM \le max,\ min \le Med \le max, \qquad (4.8)$$

Where, $\mathrm{CM}=(\mathrm{RMS})^2/\mathrm{AM}$.

The normalization formulas have the same form. In all cases, only the formula for k_j changes. We use the contextual abbreviation to denote the methods of normalization based on means:

$$\mathbf{Har}: x_{ij} = \mathrm{Har}(a_{ij}) = \frac{1}{m} \cdot a_{ij} / k_j, \ \text{ Where, } k_j = m / \sum_{i=1}^{m} \frac{1}{a_{ij}},$$

$$\mathbf{Geo}: x_{ij} = \mathrm{Geo}(a_{ij}) = \frac{1}{m} \cdot a_{ij} / k_j, \ \text{ Where, } k_j = \left(\prod_{i=1}^{m} a_{ij} \right)^{1/m},$$

$$\mathbf{ArM}\ (\mathbf{Sum}): x_{ij} = \mathrm{ArM}(a_{ij}) = \frac{1}{m} \cdot a_{ij} / k_j, \ \text{ Where, } k_j = \frac{1}{m} \cdot \sum_{i=1}^{m} a_{ij},$$

$$\textbf{RMS}\ \left(\sqrt{m}\cdot\textbf{Vec}\right): x_{ij} = \text{RMS}(a_{ij}) = \frac{1}{m}\cdot a_{ij}/k_j,\ \ \text{Where, } k_j = \sqrt{\frac{1}{m}\cdot\sum_{i=1}^{m} a_{ij}^2}\ ,$$

$$\textbf{CoH}: x_{ij} = \text{CoH}(a_{ij}) = \frac{1}{m}\cdot a_{ij}/k_j,\ \ \text{Where, } k_j = \sum_{i=1}^{m} a_{ij}^2 / \sum_{i=1}^{m} a_{ij}\ .$$

$$\textbf{Med}: x_{ij} = \text{Med}(a_{ij}) = \frac{1}{m}\cdot a_{ij}/k_j,\ \ \text{Where, } k_j = median_i(a_{ij})\ .$$

The coefficient $1/m$ in all formulas scales the data so that the normalized values $x_{ij}\in(0,1)$. Note that for the Vec method, the normalization coefficient is $1/\sqrt{m}$ and the normalized values for the Vec and RMS methods differ by a factor of $\sqrt{m}$, i.e. Vec and RMS produce the same ranks. All normalization variants, within the proportions of the average, are approximately the same. The correspondence is stronger, the less the averages of each attribute differ from each other. The numerical results of normalization, illustrating a slight shift in various domains, are present in Table 3.

Table 3: The largest and smallest values x_{ij} for different normalizations of the "share of the mean"

		C_1	C_2	C_3	C_4	C_5	Range for C_2
Har	$\max_i x_{ij}$	0.1346	0.1474	0.1483	0.2262	0.1480	0.0475
	$\min_i x_{ij}$	0.1124	0.0998	0.1094	0.0891	0.0994	
Geo	$\max_i x_{ij}$	0.1343	0.1459	0.1476	0.2193	0.1467	0.0471
	$\min_i x_{ij}$	0.1122	0.0988	0.1089	0.0864	0.0986	
ArM (Sum)	$\max_i x_{ij}$	0.1341	0.1445	0.1469	0.2117	0.1456	0.0486
	$\min_i x_{ij}$	0.1120	0.0979	0.1083	0.0834	0.0978	
RMS ($\sqrt{m}\cdot$Vec)	$\max_i x_{ij}$	0.1338	0.1432	0.1462	0.2033	0.1445	0.0462
	$\min_i x_{ij}$	0.1118	0.0970	0.1078	0.0801	0.0971	
CoH	$\max_i x_{ij}$	0.1336	0.1419	0.1454	0.1953	0.1434	0.0458
	$\min_i x_{ij}$	0.1116	0.0962	0.1073	0.0770	0.0963	
Med	$\max_i x_{ij}$	0.1320	0.1360	0.1476	0.2200	0.1401	0.0439
	$\min_i x_{ij}$	0.1102	0.0921	0.1089	0.0867	0.0941	

In the presented example (WSM method) the ratings of the alternatives also differ slightly. The relative ratings of the alternatives for the list sorted in descending order of ranks (Table 4) are almost identical for different normalization methods.

Table 4: Relative performance indicator of alternatives (dQ) for the list sorted in descending order of ranks for different normalization methods. WSM method

WSM		*Norm*						
rank	Alternatives, #	Har	Geo	ArM (Sum)	RMS ($\sqrt{m}\cdot$Vec)	CoH	Max	*)
1	A_2	dQ_i, %						
2	A_6	41.59	41.71	41.84	41.98	42.12	43.05	↗
3	A_8	21.91	21.79	21.66	21.51	21.36	19.87	↘
4	A_7	3.68	3.79	3.92	4.05	4.19	5.55	↗
5	A_4	7.65	7.56	7.47	7.36	7.26	6.78	↘
6	A_5	9.54	9.60	9.66	9.72	9.78	9.56	↘
7	A_3	8.63	8.65	8.67	8.71	8.74	9.49	↘
8	A_1	6.99	6.90	6.79	6.66	6.54	5.69	↗

*) increases ↗, decreases ↘

Thus, all normalization variants, within the framework of shares from the average, approximately coincide. This correspondence is stronger, the less the averages of each attribute differ from each other. The effectiveness of one of the variants cannot be determined, and there are no weighty arguments in favor of one of the methods. The difference in ratings is insignificant, but a difference in ranks is possible.

In numerical experiments, it was found that the value of dQ in the WSM method maintains strict ordering in a series of normalizations, with the interpretation of values as shares from the average. If the normalizations are ordered in the same sequence as the sequence of averages: $min \leq HM \leq GM \leq AM \leq RMS \leq CM \leq max$, then the value of d$Q_i$ either monotonically increases or monotonically decreases. This property is indicated in Table 4 by arrows. The hypothesis requires verification. Based on this statement, it is possible to determine the influence of the normalization method on the rating of alternatives. Given that the maximum value closes the chain of inequalities for various means, the Max normalization closes the line of normalizations as a proportion of the mean. There is no reason to refuse to use normalization in the mean, since both principles of multivariate normalization are fulfilled. In content, the Sum and Vec methods have nothing in common with their abbreviation. This is a version of normalization based on the arithmetic mean and the root mean square. In the following illustrations, we will use a double abbreviation, for example, ArM(Sum).

The linear transformation (4.1)-(4.4) of the normalized values to the value 1 for all the presented methods will not change the relative rating dQ. Such a transformation can be used for comparison with the Max method, since in this case the largest values are equal to 1 and the contribution of the normalized values to the rating is determined only by the disposition of the values (coefficient k_j). The graphical results of the transformation of the domains of normalized values to the upper level are presented in Figure 9.

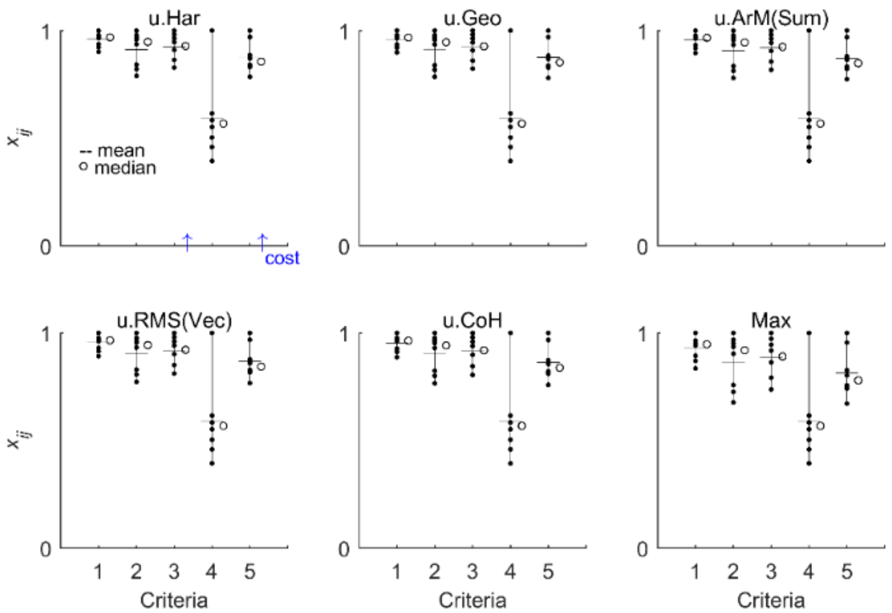

Figure 9: Transformation of domains of normalized values obtained within the proportions of the mean to 1. Comparison with Max normalization. Decision matrix D_0

Visual analysis of the domains in Figure 9 shows that the entire range of mean-based normalization methods is approximately the same. The numerical values, according to Table 3, have differences in the fourth or third decimal place. Therefore, it is not surprising that in most studies of multi-criteria choice, the rankings also coincide when using Max and Sum normalization.

4.3.2. Some interesting properties of "mean" normalization

Table 5 presents a number of statistical properties for mean-normalized data, discovered by the author in the process of numerical analysis. These properties may have been described in the literature earlier. However, the author was unable to find the relevant references.

Table 5: Statistical properties of data normalized using the "mean" technique

Normalization	Statistics	Equality of means for all attributes	Equality of statistics
x_{ij}=Har(a_{ij})	harmonic mean	hm_j=harmmean$_i$ (x_{ij})=h_0, $\forall j$	
x_{ij}=Geo(a_{ij})	geometric mean	gm_j=geomean$_i$ (x_{ij})=g_0, $\forall j$	
x_{ij}=Med(a_{ij})	median	md_j=median$_i$ (x_{ij})=md_0, $\forall j$	h_0=g_0=md_0=
x_{ij}=ArM(a_{ij}) (Sum)	arithmetic mean	am_j=mean$_i$ (x_{ij})=m_0, $\forall j$	m_0=r_0=c_0
x_{ij}=RMS(a_{ij})($\sqrt{m}\cdot$Vec)	root mean square	rm_j=rms$_i$ (x_{ij})=r_0, $\forall j$	=1/m *)
x_{ij}=CoH(a_{ij})	contra-harmonic mean	ch_j =chm$_i$ (x_{ij})=c_0, $\forall j$	

*) where m is the scale factor of the "mean" normalization technique, or the numbers of alternatives.

Property P.7: The "means" of all criteria calculated for normalized matrices are equal, if the method of normalization having the same name as the name of the statistic. For example, h_j=h_0, $\forall j$, if use the Har normalization.

Property P.8: The various "mean" statistics computed for normalized matrices are the same, for all cases, where the normalization method has the same name as the name of the statistic.

The property of equality of statistics when normalized by the mean follows from the obvious relationship:

$$\text{mean}_i(x_{ij})=\text{mean}_i(1/m\cdot a_{ij}/k_j)=1/(m\cdot k_j)\cdot\text{mean}_i(a_{ij})=1/(m\cdot k_j)\cdot k_j=1/m,$$

where any of the means or the median from Table 5 can be used as the "mean".

The equality property of the P.8 statistic is unexpected and appears to be a fundamental property relating linear transformations and different means.

Note: the property is true if no data inversion is required or if the ReS inversion is applied before normalization (to the original matrix D). If the data inversion is performed after normalization, the property is satisfied in two groups: separately for the benefit criteria and separately for the cost criteria.

A graphical illustration of these properties is shown in Figure 10. This illustration demonstrates that for all six normalization by "mean", the corresponding means (where the normalization method has the same name as the name of the statistic) are equal, both for all criteria and among themselves. Visual analysis of the domains in Figure 10 shows that the entire range of mean-based normalization methods is approximately the same.

Alignment of the average characteristics of different samples (columns of matrix D) for MCDM problems is levels out the priority of the normalized features of different criteria. Accordingly, when using inversion for individual criteria based on the nonlinear transformation $1/x$, together with the methods of normalization by the mean ArM(Sum) or

RMS(Vec), the property of equality of means is not fulfilled, which means that the goal of the methods: to equalize all means for all attributes will not be achieved. This is another argument against the use of nonlinear data inversion.

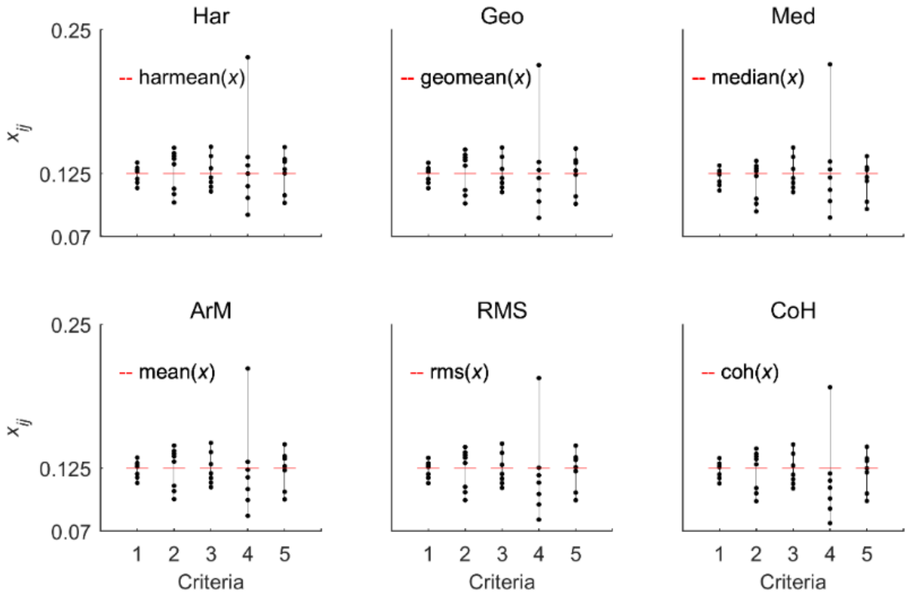

Figure 10: Normalization by "mean" (Mean.Norm). Decision matrix D_0

4.4. A technique for normalizing multidimensional data based on removal attribute values from the ideal (Dist.Norm)

4.4.1 dSum normalizations and basic properties

dSum normalization (displacement Sum) is also referred to in the literature [7] as the Enhanced Accuracy method. This term, in essence, has nothing to do with accuracy and was first used in the study [38]. dSum normalization has the following form:

$$x_{ij} = 1 - (a_j^{\max} - a_{ij}) / k_j \tag{4.9}$$

$$k_j = \sum_{i=1}^{m} (a_j^{\max} - a_{ij}) . \tag{4.10}$$

If we follow the form by Eq. (2.11) for a linear transformation, then the dSum method has a displacement of the form:

$$a_j^* = a_j^{\max} - k_j . \tag{4.11}$$

The smaller the deviation of a feature from the maximum value, the more preferable the alternative for this feature.

dSum normalization has two remarkable properties. First, the largest normalized values for all criteria coincide and equal 1:

$$\max_i(x_{ij})=1, \forall j. \tag{4.12}$$

Secondly, the average normalized values for all criteria are the same (ignoring the inversion):

$$m_j=\text{mean}_i(x_{ij})=(m-1)/m,\ \forall j, \tag{4.13}$$

and are determined only by the number m of alternatives (see also Table 5 and Fig. 1). Due to the second property, as with the Sum method, the name and abbreviation dSum are appropriate. Thus, dSum normalization aligns both the "upper" boundaries of all criteria domains, both in Up.Norm normalization and in the mean values, as in the Mean.Norm methods.

4.4.2 Normalization technique for the matrix of deviations from the ideal

A closer look at formulas (4.9)-(4.11) reveals that these formulas reflect the normalization of ArM for the matrix of deviations (or distances) b_{ij} from the best or ideal attribute value $a_j^{\max}$:

$$\Delta D^+ = b_{ij} = a_j^{\max} - a_{ij} \tag{4.14}$$

$$y_{ij} = \text{ArM}(b_{ij})\ . \tag{4.15}$$

For the STB objective, the best alternative is determined by the smallest sum of the deviations of the y_{ij} values. Alternatively, the y_{ij} values must first be inverted (see Section 2.9), for example, in the form x_{ij}=1-y_{ij}, and the LTB objective must be addressed, as implemented in formula (4.9).

Thus, dSum normalization is based on a preliminary transformation of the decision matrix. In fact, the normalized values define not attributes, but the distance of the decision matrix from the ideal attribute values. This is consistent with Goal or Reference Level Models of rank-based MCDM methods, such as TOPSIS, CODAS, and others. In this case, the researcher operates with new attributes of alternatives that define distances from the best value. However, it is necessary to first convert the deviations of the natural values to dimensionless form using one of the linear normalization methods. Accordingly, in the abbreviation dSum, the first letter "d" can be interpreted as "distance".

Following these comments, a generalization of dSum normalization is possible.

$$x_{ij} = 1 - Norm(b_{ij})\,, \tag{4.16}$$

Where, *Norm*() is one of the linear normalization methods.

Following Table 1, we apply other possible normalization options:

1) *Norm*=Max. It is easy to show that in this case 1-Max(ΔD^+) =Max-Min(D):

$$1 - x_{ij} = 1 - (a_j^{\max} - a_{ij}) / k_j\,,\ k_j = \max_i (a_{ij} - a_j^{\min}) = a_j^{\max} - a_j^{\min}\,,$$

from which it follows

$$1 - x_{ij} = 1 - (a_j^{\max} - a_{ij}) / k_j = \frac{a_{ij} - a_j^{\min}}{a_j^{\max} - a_j^{\min}}\,,$$

2) *Norm*=ArM. This is already a known normalization dSum (4.9)-(4.11), which it is appropriate to designate as 1-ArM(ΔD^+).

3) *Norm*=RMS. Then we get the new normalization 1-RMS(ΔD^+):

$$x_{ij} = 1 - \text{RMS}(b_{ij}) \tag{4.17}$$

$$\max_i(x_{ij})=1,\ \forall j. \tag{4.18}$$

4) *Norm*=Med (раздел 4.4). Then we get the new normalization 1-Med(ΔD^{+}):

$$x_{ij} = 1 - \mathrm{Med}(b_{ij}) \,. \tag{4.19}$$

Like 1-ArM(ΔD^{+}), 1-Med(ΔD^{+}) normalization has two remarkable properties. First, the largest normalized values for all criteria coincide and are equal to 1:

$$\max_i(x_{ij})=1,\ \forall j. \tag{4.20}$$

The median values of the normalized values for all criteria are the same (ignoring the inversion):

$$m_j = median_i(x_{ij}) = (m-1)/m,\ \forall j. \tag{4.21}$$

5) *Norm*=Max-Min. It is easy to show that in this case 1-Max-Min(ΔD^{+})=Max-Min(D).
6) *Norm*=Z[0,1]. It is easy to show that in this case 1-Z[0,1](ΔD^{+})= Z[0,1](D).

The two other normalizations considered, Har and Geo (see Section 4.3), have no solution, since b_{ij} contain zero values. This means that scaling is impossible, since k_j=Inf (for Har), or k_j=0 (for Geo).

So, we now have two more acceptable normalizations: 1-Med(ΔD^{+}), 1-RMS(ΔD^{+}). Both options belong to the 4) group of methods, since the normalization is performed on a distance matrix. Given that we are operate with a distance matrix, we will define this group as "Dist.Norm."

Normalization of 1-ArM(ΔD^{+}), 1-Med(ΔD^{+}), 1-RMS(ΔD^{+}) aligns the "upper" boundaries of all attributes. Furthermore, the means for dSum are equal, and the medians for 1-Med(ΔD^{+}) are equal. For the 1-RMS(ΔD^{+}) method, the equality of the root mean square values is only approximate. These properties allow for consistent normalization of the attributes of all criteria.

4.4.3 Normalization technique for the matrix of deviations from the anti-ideal

Similar to the previous version, we will form a matrix of deviations (or distances) b_{ij} from the smallest or anti-ideal value of the attribute $a_j^{\min}$:

$$\Delta D^{-} = b_{ij} = a_{ij} - a_j^{\min} \,. \tag{4.22}$$

For the LTB objective, the selection of the best alternative is determined by the largest sum of deviations of the y_{ij} values. In this case, the researcher operates with new attributes of the alternatives, which determine the distances from the anti-ideal negative ideal solution (NIS). This is consistent with Goal or Reference Level Models of rank-based MCDM methods, such as TOPSIS, CODAS, and others. However, it is necessary to first convert the deviations of the natural values to dimensionless form using one of the linear normalization methods.

$$x_{ij} = Norm(b_{ij}) \,, \tag{4.23}$$

Where, *Norm*() is one of the linear normalization methods.
Following Table 1, we apply other possible normalization options:

1) *Norm*=Max. It is easy to show that in this case Max(ΔD^{-}) =Max-Min(D).

2) *Norm*=ArM. Then we get the new normalization ArM(ΔD^{-})

$$x_{ij} = \frac{1}{m} \cdot (a_{ij} - a_j^{\min}) / k_j,\ k_j = \frac{1}{m} \sum_{i=1}^{m} (a_{ij} - a_j^{\min}) \,, \tag{4.24}$$

m_j=mean$_i$(x_{ij})=1/m, ∀j. (4.25)

3) *Norm*=RMS. Then we get the new normalization RMS(ΔD^-):

$$x_{ij} = 1 - \mathrm{RMS}(b_{ij}),\tag{4.26}$$

m_j=rms$_i$(x_{ij})=1/m, ∀j. (4.27)

4) *Norm*=Med (section 4.4). Then we get the new normalization Med(ΔD^-):

$$x_{ij} = 1 - \mathrm{Med}(b_{ij}),\tag{4.28}$$

m_j=median$_i$(x_{ij})=1/m, ∀j. (4.29)

5) *Norm*=Max-Min. It is easy to show that in this case Max-Min(ΔD^-) =Max-Min(D).
6) *Norm*=Z[0,1]. It is easy to show that in this case Z[0,1](ΔD^-) = Z[0,1](D).

The two other normalizations considered, Har and Geo (see Section 4.3), have no solution, since b_{ij} contain zero values. This means that scaling is impossible, since k_j=Inf (for Har), or k_j=0 (for Geo).

Thus, we now have three more perfectly acceptable normalizations: ArM(ΔD^-), Med(ΔD^-), and RMS(ΔD^-). All three options belong to the 4) group methods ("Dist.Norm"), since normalization is performed on the distance matrix. In all cases, min$_i$(x_{ij})=0. The mean values for ArM(ΔD^-) (4.25), the root mean square value for RMS(ΔD^-) (4.27) and the median values for Med(ΔD^-) (4.29) are same for all attributes. These properties ensure consistent normalization of attributes across all criteria.

An illustration of multidimensional normalization of the deviation matrices ΔD^+ and ΔD^- obtained from the matrix D_0 according to (2.12) is shown in Figure 11.

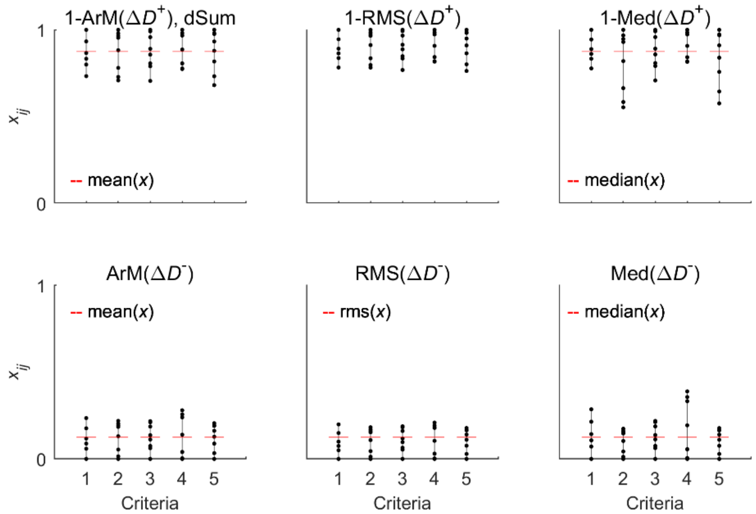

Figure 11: Normalization of the deviation matrices ΔD^+ and ΔD^- obtained from the matrix D_0

Note that for all six presented methods of the "Dist.Norm" class, the values of the dispositions of the alternatives' performance indicators are different. This means that each normalization is unique. Moreover, the implementation of solutions for the D_0 matrix, assuming that all criteria are benefit criteria, with weighting coefficients $w = (1\ 3\ 2\ 5\ 3)$, showed that the alternatives of the first three ranks change for different normalizations.

Interpretation of the normalization of the deviation matrix (ΔD^+ and ΔD^-):

1. Deviation matrix normalization automatically transforms the Value Measurement Methods into a Goal Level Modelsl, not in terms of distances, but in terms of value increments across all attributes. Since finite increments, to some extent, determine the velocity, and distances are the basis of value measurement models, the aggregation results using these two normalization approaches may differ. However, one thing is certain: deviation normalization complements value normalization.
2. Deviation matrix normalization automatically transforms data in the Goal Level Modelsl into second-order finite increments, which, to some extent, determine accelerations (growth rates). This extends the class of aggregation models to the form Growth Rate Models: the best alternative is the one with the largest increments across all attributes.

4.5. Multivariate normalization technique in the "Z-score" scale (MS.Norm)

This section presents a range of methods analogous to Z-score normalization.

Z-score normalization is attractive because the standardized values of all attributes have the same measurement scale defined by the standard deviation of each feature, the same means, and variances equal to 1. Also important is the equal density in the domain of values of all features: the range of values (-1, 1) contains about 68% of all observations of each feature:

$$u_{ij} = \frac{a_{ij} - a_j^*}{s_j},\ a_j^* = mean(a_{ij}),\ s_j = \sqrt{\frac{1}{m} \cdot \sum_{i=1}^{m} (a_{ij} - a_j^*)^2}\ . \tag{4.30}$$

However, MCDM has difficulty processing and interpreting negative values. The problem is solved using a sequence of linear transformations that map the values of u_{ij} to a given interval $[\alpha, \beta]$ while preserving the proportions in the data both within each attribute and between objects:

$$u_{ij} = Norm\left(a_{ij}\right) \in [u^{min}, u^{max}] \to x_{ij} \in [\alpha, \beta]\,, \tag{4.31}$$

Where, $u^{max} = \max_j \max_i (u_{ij})$, $u^{min} = \min_j \min_i (u_{ij})$, $[\alpha, \beta] \subseteq [0, 1]$.

The scaling technique is as follows:

$$x_{ij} = (u_{ij} - u^{min}) \cdot (\beta - \alpha) / \left(u^{max} - u^{min}\right) + \alpha\ . \tag{4.32}$$

As noted above, such a linear transformation preserves the information content of the decision matrix.

As a result, the range of values of all attributes is transformed into the segment $[\alpha, \beta]$ (in particular, [0, 1]) with preservation of means and variances. This means that all features are still measured in the same scale, tied to natural values and have the same density and disposition of values. Formula (4.32) is similar to formula (4.6) for the IZ transformation, with the difference that in (4.32) the maximum and minimum values are defined for all attributes and do not depend on j, unlike formula (4.6). The author designated the transformation of Z-scores to [0, 1] as the MS method (Mean & Standard deviation) [17] or Z[0, 1]. Note that after the MS transformation, the ranking and PRI of alternatives does not change (as well as for Z-scores) for cases where the feature aggregation function is linear (WSM, etc.) or homogeneous (TOPSIS, etc.).

If the zero value creates a problem or under other requirements of a specific multi-criteria choice problem, after the MS transformation, you can perform the IZ transformation and transform the normalized values into the interval $[I, Z] \subset [0, 1]$. The lower boundary matching is similar to the IZ method:

$$I = md\left(\text{Max}\right) = median_j[min_i(\text{Max}(a_{ij}))]\,. \tag{4.33}$$

Z-score normalization is not the only one of its kind. You can use similar and no less significant approaches (Table 6) if you use other parameters of displacement a_j^* and compression k_j for linear transformation and subsequent transformations (4.34)-(4.37):

Table 6: Analogues of Z-score transformation with equal "variance"

MAD[0,1] Mean Absolute Deviation	mdAD[0,1] median Absolute Deviation	mdIQR[0,1] median InterQuartile Range	mdRMS[0,1] Root Mean Square around a median
a_j^*=mean$_i(a_{ij})$ $k_j=\sum_i \mid a_{ij} - a_j^* \mid$	a_j^*=median$_i(a_{ij})$ $k_j=\sum_i \mid a_{ij} - a_j^* \mid$	a_j^*=median$_i(a_{ij})$ k_j=iqr (a_{ij})	a_j^*=median$_i(a_{ij})$ $k_j=\sqrt{\frac{1}{m}\cdot\sum_{i=1}^{m}(a_{ij}-a_j^*)^2}$
(4.34)	(4.35)	(4.36)	(4.37)

For the analogs presented in Table 5, the normalized values of each attribute have the same measurement scale, determined by the value k_j corresponding to formulas (4.34)-(4.37). For example, for the MAD[0,1] method, we successively apply formulas (4.34), (2.11), and (4.31)-(4.33). For all these methods, it is important to coordinate the data dispositions in a fixed scale for all attributes. The values of x_j^* are the same for all attributes, the unit of measurement of the k_j scale calculated for x_{ij} is also the same for all attributes j. An illustration of the normalization of Z-score analogs is shown in Figure 12.

When using the IZ transformation (property P.6), the ranking of alternatives does not change and the proportionality of the performance indicator of alternatives is preserved for cases where the feature aggregation function is linear (WSM, etc.) or homogeneous (TOPSIS, etc.).

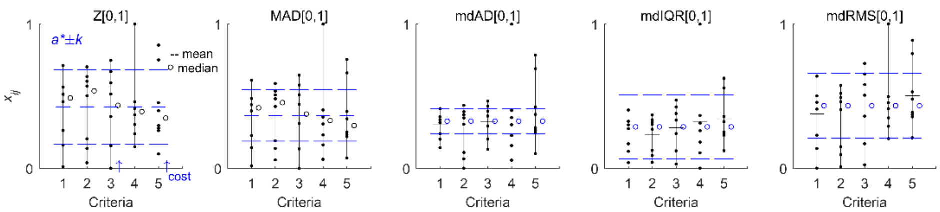

Figure 12: Analogues of Z-score transformation with equal "variance". Decision matrix D_0

Note: the peculiarity of the inversion for the Z[0,1] normalization method is that when using the ReS transformation for cost attributes, the data are inverted symmetrically with respect to the middle of the domain, and not with respect to the mean value (or median for the case of Z-score analogs). Therefore, to maintain equality of means, it is recommended to perform the inversion before normalization:

$x_{ij^*}=Z_{[0,1]}(ReS(a_{ij^*})),\ \forall\, j^* \in C^-$

The inversion option after normalization is also possible. However, the mean value (or median value) of the cost attribute for all alternatives will be shifted under the ReS transformation, unless the symmetry point of the data domain and the mean values coincide. In this case, it is necessary to perform an additional data shift for the cost attributes so that the mean values (or medians) of all attributes are equal. For Z-score normalization, the standardized values have the same mean m_1=mean($Z_{[0,1]}(a)$). Then the data shift for the cost attributes is determined by the transformation:

$$x_{ij^*} = ReS\left(Z_{[0,1]}\left(a_{ij^*}\right)\right) + m_1 - m_{2j^*}, \forall j^* \in C^-, \tag{4.38}$$

$$m_{2j^*} = \text{mean}\left(ReS\left(Z_{[0,1]}\left(a_{ij^*}\right)\right)\right).$$

With such a shift, a situation is possible that the normalized values will be outside the region [0, 1]. If this is the case, then additional data compression will be required in accordance with the scaling technique *Norm*[α, β] according to formula (4.32).

4.6. Non-linear normalization: nLinear(Max-Min) and nLinear(Z-score) techniques. Asymmetry in data

Any non-linear normalization (*n*Linear) distorts the information contained in the decision matrix. Strong arguments are needed for its use. Some studies even suggest taking logarithms?!

One of the arguments could be as follows [17]: to strengthen the contribution of the upper values and weaken the contribution of the lower values of attributes to the performance indicator of alternatives. Two simple techniques are suggested:

1) perform the Max-Min(a) transformation then apply the *S*-shaped function with the center of symmetry at point 0.5 — *n*Linear(Max-Min),

2) perform the Z-score(a) transformation and then apply the *S*-shaped function with the center of symmetry at point 0 — *n*Linear(Z-score),

The choice of the *S*-shaped function can be different, as well as the points of symmetry (mean, median). If necessary, you can perform an IZ transformation of the resulting normalized values into the interval $[I, Z] \subset [0, 1]$. An illustration of nonlinear normalization using techniques 1) and 2) with different choices of the *S*-shaped function is shown in Figure 13.

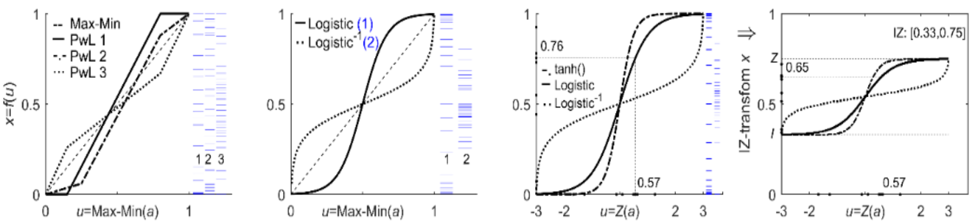

Figure 13: Non-linear normalization technique with different choice of S-shaped function: *n*Linear(Max-Min) – *a*), *b*) and *n*Linear(Z-score) – *c*), IZ transform – *d*)

The choice of the function and its parameters ensures the necessary amplification of the influence. If we use the inverse function for the S-shaped one, the result is inverted: the contribution of the upper values will be weakened, and the lower ones will be strengthened (dotted lines in Figure 13). There are several possible options for the S-shaped function:

piecewise linear function, spline function defined by a piecewise polynomial of degree 2, logistic function, hyperbolic tangent: Tanh(), Gauss error function: Ef(), cumulative distribution function: NormCDF(), etc.

One of the problems of data processing is asymmetry. It is initially unclear whether the asymmetry of the data is an internal property of the objects under study, or is due, for example, to the representativeness of the sample. If the researcher has determined that asymmetry is not a property of the object, then the correction of the asymmetry of the data is possible only with the use of a nonlinear transformation. Linear transformation does not change the asymmetry in the data (property 4, Table 2).

In order to implement this, it is necessary to shift the center of symmetry for the 1) *n*Linear(Max-Min) methods relative to the point 0.5 or for the 2) *n*Linear(Z-score) methods relative to the point 0 [17].

4.7. Target normalization: coordinated technique with LTB (STB) criteria normalization

The nominal value of an attribute is some intermediate value between the largest and smallest. Typically, this value represents the optimal characteristic or performance, or may be stated by the customer. The following generalization of linear normalization methods is proposed for the case of target nominal criteria and LTB goals [17]:

$$x_{ij} = \begin{cases} \dfrac{a_{ij} - a_j^*}{k_j}, & a_{ij} < a_j^t \\ \dfrac{2a_j^t - a_{ij} - a_j^*}{k_j}, & a_{ij} \ge a_j^t \end{cases} . \tag{4.39}$$

Next, it is necessary to perform a data shift according to formulas (4.39)-(4.41) so that the target value x^t shifts to x^*, corresponding to the largest of the values of a_{it}. Such a shift ensures that all normalized values $\overline{x}_{ij} \in [0, 1]$.

$$x_t^* = \max_{1 \le i \le m} \frac{a_{it} - a_t^*}{k_t}, \; for\ j = t \text{ (target criteria)}, \tag{4.40}$$

$$\overline{x}_{ij} = x_{ij} + x_j^* - x_j^t . \tag{4.41}$$

The graphical diagram is presented in Figure 14 for the case of applying the Z[0,1] normalization. The advantage of formulas (4.39)-(4.41) over their analogues is that the compression and displacement coefficients are consistent with linear normalization methods of other attributes (not TTB).

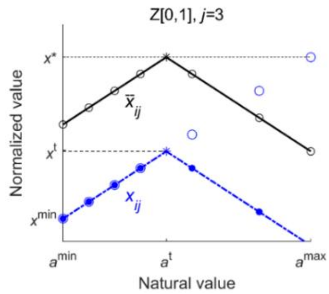

Figure 14: Data transformation scheme for TTB criteria 3 using formulas (4.39)-(4.41)

In particular, the same rate of data change (increment) is preserved as for typical linear normalization methods Max, Sum, …, Z[0, 1]. This is essential for tasks in which there is a mixture of benefit, cost and target criteria.

5. PRACTICAL RECOMMENDATIONS FOR CHOOSING A NORMALIZATION METHOD

The title of this section reflects the fact that there are no criteria for choosing a normalization method, and the title of this article indicates that the choice is based on two main principles (section 2.7) and on the interpretation of normalization scales. Various modifications of normalization are based on the properties of linear transformations (section 3). A practical recommendation for choosing a normalization method for solving MCDM problems is as follows:

Include in the solution of the problem several basic linear methods of multivariate normalization that provide a variety of basic normalization scales.

The author's recommendation is as follows:

1) Max, 2) Max-Min, 3) ArM, 4) 1-ArM(ΔD^{+}), 5) Z[0, 1], followed by transformation of normalized values into the domain [α, 1].

Linear transformation of normalized values into the domain [α, 1] will not change the ranking in the case of linear or homogeneous aggregation functions, and in other variants such transformation minimizes the shift of the domains of normalized values relative to each other.

Justification of choice: all methods satisfy the basic principles of normalization of multivariate data, have a rational interpretation of normalized values The harmonization consists of leveling the domains of the values of individual attributes for subsequent aggregation of private features.

The Max method aligns the upper limit of the normalized values of all attributes. The Max-Min method defines a scale of equal ranges for all attributes — the [0, 1] scale. The ArM method is a representative of the group of normalization methods based on the "typical" value and ensures the equality of the arithmetic mean of all attributes. The dSum method simultaneously aligns the upper limit of the normalized values and the average (arithmetic) values of all attributes. The Z[0,1] method define scale based on equal typical values and equal range.

Note 1: Before normalization, it is necessary to analyze the data for asymmetry and outliers, since linear normalization does not affect these parameters.

Note 2: Since the "typical values" (mean, median, etc.) are shifted when the data is inverted, the inversion of the cost attributes should be performed before normalization, i.e. for natural values, following formula (2.14). This applies to methods based on equal typical values and equal range.

Let us give an example of visualization of the normalization of the decision matrix D_0, by Eq. (2.12) using the recommended methods. To visualize the normalization scales, we perform a linear transformation of the normalized values into the interval $[\alpha, 1] \subseteq [0, 1]$, where, $\alpha = \min_i \min_j (\mathrm{Max}(a_{ij}))$ the smallest normalized value for the Max method. In this way, we will ensure the same range of normalized values for all normalization methods and proportionality of the alternatives efficiency indicator (dQ) for cases where the feature aggregation function is linear (WSM, etc.) or homogeneous (TOPSIS, etc.).

Normalizations:
1) Max method, α=$\min_i \min_j$ (Max(a_{ij})),
2) Max-Min method or IZ[α, 1],
3) ArM (or Sum) method,
4) 1-ArM(ΔD^+) (or dSum) method,
5) Z[0,1] method.

Next, for all normalization options, we perform the transformation into [α, 1] using Eq. (4.15).

Figure 15 shows the domains of normalized values of various attributes of the recommended methods after linear transformation in [α, 1].

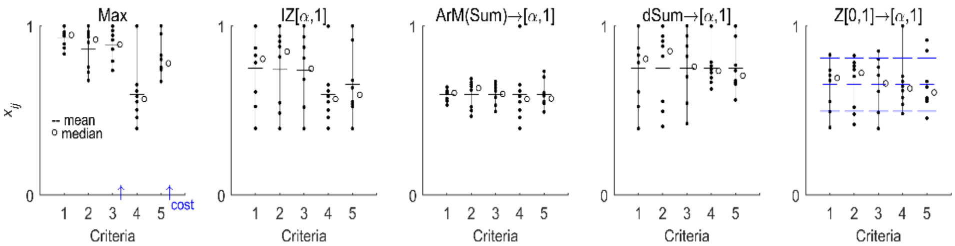

Figure 15: Domains of normalized values after linear transformation to [α, 1]

Figure 15 demonstrates possible shifts in the domains of normalized values of various attributes due to the choice of method. These deviations cause changes in the ratings. The set of basic linear methods of multidimensional normalization selected by the authors provides a variety of basic normalization scales.

Let us give an example of solving the MCDM problem with the decision matrix D_0, where the 3rd and 5th criteria are cost criteria, the weights of the criteria are specified as: w=(⅙ ⅙ $^1/_{12}$ ⅓ ¼). We will perform the inversion of the cost criteria using the ReS transformation (2.14). As aggregation functions, we use the WSM and TOPSIS(L_2) methods as two basic MCDM methods.

Using 2 aggregation methods and 5 normalization methods, we have 10 results for analysis, presented in Table 7.

Table 7: Ranks and ratings of alternatives for the decision matrix D_0. WSM and TOPSIS methods

rank	Max			ArM			1-ArM(ΔD^+)			Max-Min			Z[0,1]		
	transform *Norm* to [α,1], α=$\min_j(\min_i(\text{Max}(a_{ij})))$														
	A_i	Q_i	dQ,%	A_i	Q_i	dQ,%	A_i	Q_i	dQ,%	A_i	Q_i	dQ,%	A_i	Q_i	dQ,%
	WSM														
I	2	0.94	-	2	0.16	-	2	0.96	-	2	0.82	-	2	0.71	-
II	7	0.78	68.0	7	0.13	67.6	6	0.89	55.7	6	0.49	63.0	6	0.45	65.9
III	6	0.78	0.3	6	0.13	1.5	7	0.88	3.0	7	0.48	2.8	7	0.44	2.0
	TOPSIS(L_2)														
I	2	0.83	-	2	0.87	-	2	0.71	-	2	0.75	-	2	0.76	-
II	7	0.39	71.0	8	0.40	66.9	6	0.51	55.0	6	0.46	66.8	4	0.45	65.8
III	8	0.39	0.5	7	0.38	3.6	4	0.49	5.9	4	0.45	1.0	6	0.43	3.7

The ranking results in Table 7 for this problem weakly depend on the choice of the normalization method. Alternative A_2 has rank 1 in all cases. The superiority (by dQ) in relation to the rating of the alternatives of rank 2 is from 50 to 71%. Alternatives of ranks

2 and 3 for different models are alternatives A_4, A_6, A_7, A_8. Such competition is due to the low RPI value (from 0.3 to 3.5%). Thus, the proposed normalization methods reflect the full picture of the rating competition and provide complete information to the decision maker.

The situation with rank reversal occurs in most problems. But in some variants rank reversal is significant.This reduces the reliability of the formalized solution, and in some cases, the formal conclusions are contradictory.

6. THE IMPACT OF NORMALIZATION ON RANKING. THE RANK REVERSAL PHENOMENON

Let us set the task of generating a decision matrix that is sensitive to the choice of the normalization method, all other parameters of the decision-making model being the same.

For a specific problem defined by the decision-making matrix D, the choice of one of the linear normalization methods always changes the ratings of the alternatives. But the ranking can be either the same or different for different methods. The same ranking means the same ordering of the rating of the alternatives Q_i.

The technique for generating such a decision matrix D is based on the generation of random values (uniform law). It is necessary to generate m random values (m alternatives) for each attribute from the range of values determined by setting the range of values. For each such decision matrix, ranking is performed for different normalization procedures, all other parameters of the MCDM model being fixed, according to formula (2.1). The iterative search procedure is completed when all alternatives of 1-rank are different for the selected set of normalization methods. Additionally, we require that the dispositions, according to formula (3.1), between the ratings Q_i of the alternatives of the first and second ranks constitute no less than γ% (3-5%) of the range Q_m-Q_1. This ensures a significant difference in the ratings.

Table 8 shows two decision matrices D_1 and D_2 with the same range of attribute variation as matrix D_0 by Eq. (2.12). Decision matrices D_1 and D_2 are generated in such a way that for them the ranking of alternatives using the WSM and TOPSIS methods, respectively for D_1 and D_2, is different for different 5 normalization methods: Max, u.Sum, dSum, Max-Min, Z[0,1]. These matrices define a different choice problem than matrix D_0. Such decision matrices can hypothetically define the same decision-making problem that is defined by setting matrix D_0, but with a different set of alternatives.

Table 8: Decision matrix D_1 and D_2 which is highly sensitive to the normalization methods

	D_1 (for WSM)					D_2 (for TOPSIS)				
	C_1^+	C_2^-	C_3^+	C_4^+	C_5^-	C_1^+	C_2^-	C_3^+	C_4^+	C_5^-
A_1	78	5504	163	1085	535	71	4761	167	1421	520
A_2	75	5738	143	1815	582	85	4707	176	1600	649
A_3	74	5499	164	2203	568	79	4810	149	1850	465
A_4	73	6356	144	2584	506	85	6329	180	1245	472
A_5	80	5544	150	1662	471	76	6459	150	2517	645
A_6	81	4828	167	1256	661	72	4884	153	1252	577
A_7	84	5944	168	1182	531	78	4579	164	1147	666
A_8	75	6121	176	1363	450	84	6280	182	2088	612

In the numerical experiment, we purposefully generated two such matrices D_1 and D_2 so that the alternatives with rank 1 were different for 5 different linear normalization methods. In this example, equal weights of criteria were assumed. However, such a result also occurs in the presence of priorities in the criteria. The presented matrices D_1 and D_2 determine the high sensitivity of the problem to the choice of the normalization method. For the WSM aggregation method, the frequency of such a result is low. Only in 32 cases out of 2,500,000 generated matrices are the results different for 5 normalization methods, in 4124 cases the results differ for 4 normalization methods and about 5% for 3 normalization methods. For the TOPSIS aggregation method, in the computational experiments, in 189 cases out of 2,500,000 generated matrices, the 1st rank alternatives are different for 5 normalization methods, in 14840 cases the results differ for 4 normalization methods and about 10% of cases for 3 normalization methods. The ranks and ratings of alternatives for the decision matrices D_1, D_2 are present in Table 9.

Table 9: Ranks and ratings of alternatives for the decision matrix D_1, D_2. WSM and TOPSIS methods

rank	Max			u.ArM			1-ArM(ΔD^+),			Max-Min			Z[0,1]		
	A_i	Q_i	dQ,%	A_i	Q_i	dQ,%	A_i	Q_i	dQ,%	A_i	Q_i	dQ,%	A_i	Q_i	dQ,%
	D_1, WSM														
I	**3**	0.88	-	**4**	0.91	-	**6**	0.90	-	**7**	0.54	-	**5**	0.57	-
II	4	0.87	5.6	3	0.90	9.2	7	0.89	7.2	5	0.53	3.4	7	0.56	4.0
III	5	0.86	17.2	5	0.88	26.4	5	0.89	3.9	6	0.51	7.7	6	0.54	7.5
	D_2, TOPSIS(L_2)														
I	**3**	0.60	-	**5**	0.62	-	**4**	0.63	-	**2**	0.59	-	**8**	0.58	-
II	5	0.55	17.7	3	0.58	10.8	2	0.58	15.6	8	0.56	10.5	2	0.57	5.1
III	8	0.54	3.9	8	0.57	4.1	8	0.58	2.0	3	0.56	0.5	3	0.55	5.3

The numbers of alternatives that have a rank of 1 when using one of the 5 normalization methods are highlighted in yellow. Only alternative A_1 does not have rank 1 (and does not even make it into the top three). The example demonstrates that a formal choice is impossible!

High sensitivity of the problem to the choice of the normalization method creates not only uncertainty, but also the impossibility of multi-criteria choice. Therefore, sensitivity analysis is a necessary component when solving MCDM using rank models. For problems with high sensitivity of the result to the normalization method, it is necessary to use Outranking MCDM methods [40], which do not use normalization.

7. CONCLUSIONS

In the absence of criteria for the effectiveness of multidimensional data normalization methods, it is important keep it informative of the original data as much as possible and to coordinate the normalized values (domains) of different features in order to prevent hidden priority of the contribution of individual features to the integrated indicator that determines the ranking of alternatives. The priority of attributes should be regulated only by weight coefficients.

Inattention to the details of multidimensional data normalization can be traced in many studies. Automatically, all the results of such studies become unreliable.

This study presents a wide range of normalization options based on linear transformations. Linear transformation of normalized values does not change the ranking in the case of linear or homogeneous aggregation functions, and in other options such a transformation minimizes the shift of the domains of normalized values relative to each other. The entire pool of solutions is aimed at preventing the hidden priority of the contribution of individual features to the integral indicator determining the ranking of alternatives.

This article presents five different techniques:

1) aligning the upper bounds of the normalized values of all attributes. This is the Max method or any linear method with the Up.Norm transform applied;

2) aligning the upper and lower bounds of the normalized values of all attributes. This is the Max-Min method or any linear method with the IZ.Norm transform applied;

3) aligning the "typical values" of the normalized values of all attributes. These are methods of the Mean.Norm class:

ArM (or Sum) — typical values of the arithmetic mean of the values of each attribute.

Har — typical values of the harmonic mean of the values of each attribute,

Geo — typical values of the geometric mean of the values of each attribute,

RMS (or Vec) — typical values of the root mean square of the values of each attribute,

CoH — typical values of the contra-harmonic value of the values of each attribute,

Med — typical values of the median of the values of each attribute.

4) aligning the upper (lower) boundary and the "average" of the normalized values of all attributes. These are methods of the Dist.Norm class: ArM(ΔD^-), Med(ΔD^-), RMS(ΔD^-), 1-ArM(ΔD^+), 1-Med(ΔD^+), 1-RMS(ΔD^+).

5) aligning the "means" and "range." These are methods of the MS.Norm class with values transformed to [0,1]: Z[0,1] — Z-score, MAD[0,1] — Mean Absolute Deviation, mdAD[0,1] — median Absolute Deviation, mdIQR[0,1] — median InterQuartile Range, mdRMS[0,1] — Root Mean Square around a median.

Although citing a large number of different linear and nonlinear normalization options, the author is unable to single out a preferred effective method. One approach to overcoming uncertainty in choosing a normalization method is to solve the problem using a set of basic normalization methods. This is the practical recommendation for choosing a normalization method presented in Section 5, which recommends using all five consistent normalization approaches when solving MCDM problems. Then, if conflicting results arise, a thorough analysis of the causes and justifications will be required.

Identifying a group of basic normalization methods and linking them to the structure of the source data is an important continuation of multivariate normalization research.

Based on the results of the study, the author proposed a set of basic linear multivariate normalization methods that provide a variety of normalization scales: Max(D), ArM(D), 1-ArM(ΔD^+), Max-Min(D), and Z[0,1](D) methods, followed by transformation of the normalized values into a fixed subdomain [α, 1]. Such a transformation of normalized values will not change the ranking in the case of linear or homogeneous aggregation functions, while in other cases, such a transformation minimizes the shift in the domains of normalized values relative to each other. These cases require further study.

The high sensitivity of the problem to the choice of normalization method creates not only uncertainty but also the impossibility of multicriteria choice. Therefore, sensitivity analysis is a necessary component when solving MCDM problems using rank models. For problems with high sensitivity to the normalization method, it is necessary to use outranking MCDM methods that do not use normalization.

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