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Research Article

A NOVEL ENTROPY BASED VIKOR METHOD WITH MINKOWSKI TYPE DISTANCE MEASURE ON GTHF-NUMBERS AND ITS APPLICATIONS

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Abstract: Various set theories have been developed to model the uncertainties frequently encountered in real-world scenarios. This study proposes an entropy-based VIKOR method based on Generalized Trapezoidal Fuzzy Numbers (GTHFNs) due to their high representation capacity, with the aim of addressing uncertainty more effectively. Entropy is used to express the mathematical values of the fuzziness of GTHFNs. To ensure flexibility, the proposed method has been formulated using a Minkowski-type distance measure. This decision-making framework not only provides a way to solve the MCDM problem but also incorporates an important mathematical idea as a different solution approach. The applicability of the proposed algorithm is demonstrated through a numerical case study. Comparative results show that the method provides a more precise and effective distinction among alternatives, proving its validity in environments characterized by high uncertainty and incomplete information.

Keywords: Generalized hesitant trapezoidal fuzzy numbers, Entropy measure, Minkowski type distance measure, VIKOR method.

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1. INTRODUCTION

In contemporary discourse, numerous terms employed in daily communication are inherently complex and frequently associated with uncertainty. To effectively address decision-making problems characterized by such uncertainties, a variety of mathematical frameworks have been proposed, including fuzzy set theory [1], intuitionistic fuzzy set theory [2], and more recently, hesitant fuzzy set theory [3]. Among these, hesitant fuzzy sets, introduced by Torra [3], represent a significant advancement by allowing the membership degree of an element to be described through a set of possible values rather than a single value, thereby capturing hesitation in expert opinions more accurately. Hesitant fuzzy sets have been extensively studied by many researchers and extended to many sets. For instance, probabilistic hesitant fuzzy sets [4, 5], dual hesitant fuzzy sets [6], complex hesitant fuzzy sets [7], intuitionistic hesitant fuzzy sets [8], pythogorean hesitant fuzzy sets [9]. Some similarity and distance measures are also defined on these sets [10, 11, 12, 13]. Additionally, the TOPSIS method has also been adapted to function under hesitant fuzzy set environments. At the same time, hesitant fuzzy set theory has been studied on decision making methods in [14] for ANP, [15] for AHP, [16] for TOPSIS [17] for EDAS, [18] for VIKOR.

Despite these advancements, traditional hesitant fuzzy sets still fall short in representing all nuances of original data. To address this limitation, Deli and Karaaslan [19] introduced the concept of generalized trapezoidal hesitant fuzzy numbers over the real number set $\mathbb{R}$. Subsequently, Deli [20] extended the TOPSIS method to accommodate these generalized structures. Furthermore, several studies have proposed alternative approaches to effectively manage uncertain and qualitative information [21, 22, 23].

The Minkowski distance measure is a flexible and powerful measurement method that unifies various distance calculation techniques under a single general formula. This method can be customized through the use of a parameter p, which allows it to encompass widely used metrics such as the Euclidean distance, depending on the value of p. Thanks to this adaptability, the most appropriate distance measure can be selected according to the structure of the data and the context of the analysis, thereby optimizing model performance. Minkowski distance measure offers not only theoretical flexibility but also a broad range of practical applications. In particular, it plays an important role in multi-criteria decision-making (MCDM) methods [24, 25, 26], where it is used to analyze the similarity or dissimilarity between alternatives. This contributes to increased model flexibility and enables decision-makers to conduct analyses based on varying levels of risk perception and sensitivity. As a result, decision-making processes become more dynamic, adaptable, and context-specific.

The concept of fuzzy entropy was first introduced by Shannon [27] as a measure for quantifying the degree of fuzziness in fuzzy sets. This foundational idea was further extended by Lin [28], who applied it to information-theoretic divergence measures. Since then, a wide range of research has focused on applying entropy-based approaches in fuzzy environments, as demonstrated in studies such as [29, 30].

In the domain of multi-criteria decision-making (MCDM), particularly for problems involving conflicting objectives, the VIKOR method has proven to be an efficient tool for identifying compromise solutions [31, 32]. The VIKOR method aims to determine the most appropriate solution among conflicting criteria by identifying the alternative closest to the ideal solution based on the decision maker's preferences. Considering both benefit and cost criteria, it has been widely adopted in multi-criteria decision-making (MCDM) problems. Due to its effectiveness, the VIKOR method has found extensive applications across various disciplines [33, 34, 35, 36, 37]. Building upon this foundation, Opricovic [38] proposed a fuzzy extension of the VIKOR method.

Further research has integrated both entropy measures and VIKOR-based strategies for decision-making in uncertain environments, as shown in the works of various scholars [39, 40, 41, 42]. Entropy is used to quantify the fuzziness inherent in generalized trapezoidal hesitant fuzzy numbers (GHTF-numbers), making it an essential tool for measuring uncertainty in such sets. The main contributions of this paper are as follows: Firstly, we define a Minkowski-type distance measure with GTHF-numbers. Then we define a entropy on Minkowski-type distance measure with GTHF-numbers and define a entropy based VIKOR method on Minkowski-type distance measure with GTHF-numbers. After that we have developed a decision-making method based on this approach and we have presented an application that illustrates the advantages and simplicity of this method. Finally compared our proposed method with existing ones to highlight its advantages.

Accordingly, the remainder of the paper is organized as follows. In the Preliminary section, we outlined key definitions and notions relevant to the remainder of the paper. The section titled "Entropy Based VIKOR Method on Minkowski-Type Distance Measures with GTHF-Numbers" begins with the axiomatic definition of the Minkowski-type distance for GTHF-numbers, followed by the introduction of the entropy measure based on this distance measure. In the last part of the section, we present a VIKOR method based on the entropy measure for GTHF-numbers. A numerical example is illustrated in the "Illustrative Case Study" section to validate the proposed approach. In the section titled "Comparison and Analysis Discussion", we discuss the performance of the proposed method in comparison with other established methods. The proposed VIKOR method is better suited than existing techniques for dealing with uncertain and imprecise data, offering decision-makers multiple options among potentially infinite alternatives. Furthermore, the outcomes are more informative, as the distinctions between alternatives are clearer. In "Conclusion" section, we summarized the main features and findings of the proposed method and suggested some directions for future research.

2. PRELIMINARY

In this section we will introduce some basic notations of generalized trapezoidal hesitant fuzzy numbers [19] are required for the next sections.

Definition 1. *[19] Let* U *be a universe of discourse,* $\xi_i \in [0,1]$ $(i \in I = \{1,2,\dots,n\}$ *or* $\{1,2,\dots,m\}$ *or* $\dots)$ *and* $a_1,a_2,a_3,a_4 \in \mathbb{R}$ *such that* $a_1 \le a_2 \le a_3 \le a_4$. *Then, a generalized hesitant trapezoidal fuzzy number (GTHF-number)* $\mathrm{A}_\Theta = \langle (a_1,a_2,a_3,a_4); \{\xi_i : \xi_i \in \xi(x), \xi(x)$ *is a set of some values in [0,1]*$\}\rangle$ *is described by membership functions given as*

$$T^i(x) = \begin{cases} (x-a_1)\xi_i/(a_2-a_1) & a_1 \le x < a_2 \\ \xi_i & a_2 \le x \le a_3 \\ (a_4-x)\xi_i/(a_4-a_3) & a_3 < x \le a_4 \\ 0 & otherwise \end{cases}$$

The set of all GTHF-numbers on $\mathbb{R}$ is denoted by Θ.

Definition 2. *[19] Let $\mathrm{A}^1_\Theta = \langle (a_1,a_2,a_3,a_4); \xi^1 = \xi^1(x)\rangle$ and $\mathrm{A}^2_\Theta = \langle (b_1,b_2,b_3,b_4); \xi^2 = \xi^2(x)\rangle \in \Theta$. Then,*

(i) $\mathrm{A}^1_\Theta \oplus \mathrm{A}^2_\Theta = \langle (a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4); \cup_{\xi^1_1 \in \xi^1, \xi^2_1 \in \xi^2}\{\xi^1_1 + \xi^2_1 - \xi^1_1 \cdot \xi^2_1\}\rangle$;

(ii) $$\mathrm{A}^1_\Theta \odot \mathrm{A}^2_\Theta = \begin{cases} \langle (a_1b_1, a_2b_2, a_3b_3, a_4b_4); \cup_{\xi^1_1 \in \xi^1, \xi^2_1 \in \xi^2}\{\xi^1_1 . \xi^2_1\}\rangle & (a_4>0, b_4>0) \\ \langle (a_1b_4, a_2b_3, a_3b_2, a_4b_1); \cup_{\xi^1_1 \in \xi^1, \xi^2_1 \in \xi^2}\{\xi^1_1 . \xi^2_1\}\rangle & (a_4<0, b_4>0) \\ \langle (a_4b_4, a_3b_3, a_2b_2, a_1b_1); \cup_{\xi^1_1 \in \xi^1, \xi^2_1 \in \xi^2}\{\xi^1_1 . \xi^2_1\}\rangle & (a_4<0, b_4<0) \end{cases}$$

(iii) $\lambda \mathrm{A}_\Theta = \langle (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4); \cup_{\xi \in \xi(x)}\{1-(1-\xi)^\lambda\}\rangle (\lambda \ge 0)$

(iv) $(\mathrm{A}_\Theta)^\lambda = \langle (a_1^\lambda, a_2^\lambda, a_3^\lambda, a_4^\lambda); \cup_{\xi \in \xi(x)}\{\xi^\lambda\}\rangle (\lambda \ge 0)$.

Definition 3. *[43] Let $\mathrm{A}^1_\Theta = \langle (a_1,a_2,a_3,a_4); \xi^1 = \xi^1(x)\rangle$, $\mathrm{A}^2_\Theta = \langle (b_1,b_2,b_3,b_4); \xi^2 = \xi^2(x)\rangle \in \Theta$ and $\lambda \neq 0$. The normalized Hamming distance between A^1_Θ and A^2_Θ is given by:*

$$\begin{aligned}\delta(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta) \quad &= \tfrac{1}{16\cdot\kappa\cdot\ell} \cdot \textstyle\sum_{\xi^1_i \in \xi^1, \xi^2_i \in \xi^2}(|(1+\xi^1_i)a_1 - (1+\xi^2_i)b_1| \\ &+|(1+\xi^1_i)a_2 - (1+\xi^2_i)b_2| \\ &+|(1+\xi^1_i)a_3 - (1+\xi^2_i)b_3| \\ &+|(1+\xi^1_i)a_4 - (1+\xi^2_i)b_4|)\end{aligned}$$

here $\kappa = \max\{|a_1|,|a_2|,|a_3|,|a_4|,|b_1|,|b_2|,|b_3|,|b_4|\}$ and $\ell = \xi^1 \cdot \xi^2$.

Definition 4. *[19] Consider a collection of GTHF-numbers $\{\mathrm{A}^j_\Theta\}_{j \in I_n} \subset \Theta$, and define a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \in [0,1]$ for all $j \in I_n$, and $\sum_{j=1}^n \omega_j = 1$.*

(i) For $H^G_\omega : \Theta^n \to \Theta$, if

$$\begin{aligned} H^G_\omega(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta, \ldots, \mathrm{A}^n_\Theta) = &\bigotimes_{j=1}^n (\mathrm{A}^j_\Theta)^{\omega_j} \\ &= \langle (\prod_{j=1}^n {a_1}_j^{\omega_j}, \prod_{j=1}^n {a_2}_j^{\omega_j}, \prod_{j=1}^n {a_3}_j^{\omega_j}, \prod_{j=1}^n {a_4}_j^{\omega_j}); \\ &\cup_{\xi^1_1 \in \xi^1, \xi^2_1 \in \xi^2, \ldots, \xi^n_1 \in \xi^n}\{\prod_{j=1}^n (\xi^j_1)^{\omega_j}\}\rangle \end{aligned}$$

then H^G_ω is defined as the GTHF-number weighted geometric operator.

(ii) For $H^A_\omega : \Theta^n \to \Theta$, *if*

$$\begin{aligned} H^A_\omega(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta, \ldots, \mathrm{A}^n_\Theta) =& \bigoplus_{j=1}^{n} \omega_j . \mathrm{A}^j_\Theta \\ =& \langle (\sum_{j=1}^{n} \omega_j \cdot a_{1j}, \sum_{j=1}^{n} \omega_j \cdot a_{2j}, \sum_{j=1}^{n} \omega_j \cdot a_{3j}, \sum_{j=1}^{n} \omega_j \cdot a_{4j}); \\ & \cup_{\xi_1^1 \in \xi^1, \xi_1^2 \in \xi^2, \ldots, \xi_1^n \in \xi^n} \{(1 - \prod_{j=1}^{n} (1 - \xi_1^j)^{\omega_j})\} \rangle \end{aligned}$$

then H^A_ω *is defined as the GTHF-number weighted arithmetic operator.*

(iii) For $H^A_{\omega,\eta} : \Theta^n \to \Theta$, *if*

$$\begin{aligned} H^G_{\omega,\eta}(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta, \ldots, \mathrm{A}^n_\Theta) =& \frac{1}{\eta} \cdot \bigotimes_{j=1}^{n} \mathrm{A}^{j\ \omega_j}_\Theta \\ =& \langle (\frac{1}{\eta} \cdot \prod_{j=1}^{n} a_{1j}^{\omega_j}, \frac{1}{\eta} \cdot \prod_{j=1}^{n} a_{2j}^{\omega_j}, \frac{1}{\eta} \cdot \prod_{j=1}^{n} a_{3j}^{\omega_j}, \frac{1}{\eta} \cdot \prod_{j=1}^{n} a_{4j}^{\omega_j}); \\ & \cup_{\xi_1^1 \in \xi^1, \xi_1^2 \in \xi^2, \ldots, \xi_1^n \in \xi^n} \{1 - (1 - \prod_{j=1}^{n} (1 - (1 - \xi_1^j)^\eta)^{\omega_j})^{\frac{1}{\eta}} \} \rangle \end{aligned}$$

then $H^G_{\omega,\eta}$ *is defined as the generalized GTHF-number weighted geometric operator, where* $\eta > 0$.

(iv) For $H^A_{\omega,\eta} : \Theta^n \to \Theta$, *if*

$$\begin{aligned} H^A_{\omega,\eta}(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta, \ldots, \mathrm{A}^n_\Theta) =& (\bigoplus_{j=1}^{n} \omega_j \cdot \mathrm{A}^{j\ \eta}_\Theta)^{\frac{1}{\eta}} \\ =& \langle (\sum_{j=1}^{n} \omega_j \cdot a_{1j}, \sum_{j=1}^{n} \omega_j \cdot a_{2j}, \sum_{j=1}^{n} \omega_j \cdot a_{3j}, \sum_{j=1}^{n} \omega_j \cdot a_{4j}); \\ & \cup_{\xi_1^1 \in \xi^1, \xi_1^2 \in \xi^2, \ldots, \xi_1^n \in \xi^n} \{(1 - \prod_{j=1}^{n} (1 - \xi_1^{j\lambda})^{\omega_j})^{\frac{1}{\eta}} \} \rangle \end{aligned}$$

then $H^A_{\omega,\eta}$ *is defined as the generalized GTHF-number weighted arithmetic operator, where* $\eta > 0$.

(v) The maximum of the GTHF-numbers A^j_Θ *is denoted by* H^+_Θ *and is defined as follows:*

$$\begin{aligned} H^+_\Theta =& \langle \max_{i=1,2,\ldots,n} \{a_{1i}\}, \max_{i=1,2,\ldots,n} \{a_{2i}\}, \max_{i=1,2,\ldots,n} \{a_{3i}\}, \max_{i=1,2,\ldots,n} \{a_{4i}\}); \\ & \bigcup_{\xi_1^1 \in \xi^1, \xi_1^2 \in \xi^2, \ldots, \xi_1^n \in \xi^n} \max\{\xi_1^1, \xi_1^2, \ldots, \xi_1^n\} \rangle \end{aligned}$$

(vi) The minimum of the GTHF-numbers A_Θ^j *is denoted by* H_Θ^- *and is defined as follows:*

$$H_\Theta^- = \langle \min_{i=1,2,\dots,n}\{a_{1i}\}, \min_{i=1,2,\dots,n}\{a_{2i}\}, \min_{i=1,2,\dots,n}\{a_{3i}\}, \min_{i=1,2,\dots,n}\{a_{4i}\}); \bigcup_{\xi_1^1\in\xi^1,\xi_1^2\in\xi^2,\dots,\xi_1^n\in\xi^n} \min\{\xi_1^1,\xi_1^2,\dots,\xi_1^n\}\rangle$$

3. ENTROPY BASED VIKOR METHOD ON MINKOWSKI-TYPE DISTANCE MEASURES WITH GTHF-NUMBERS

3.1. Minkowski-Type Distance Measures on GTHF-Numbers

Definition 5. *Let* $\mathrm{A}_\Theta^1 = \langle (a_1,a_2,a_3,a_4);\{\xi^1\}\rangle$, $\mathrm{A}_\Theta^2 = \langle (b_1,b_2,b_3,b_4);\{\xi^2\}\rangle \in \Theta$ *and* $\varphi \neq 0$. *Then, the Minkowski-type distance between* A_Θ^1 *and* A_Θ^2 *is defined as follows:*

$$\begin{aligned}\delta_\varphi(\mathrm{A}_\Theta^1,\mathrm{A}_\Theta^2) = \frac{1}{16\cdot\kappa\cdot\ell}\cdot \sum_{\xi_i^1\in\xi^1,\xi_i^2\in\xi^2} &[(|(1+\xi_i^1)a_1-(1+\xi_i^2)b_1|)^\varphi \\ &+(|(1+\xi_i^1)a_2-(1+\xi_i^2)b_2|)^\varphi \\ &+(|(1+\xi_i^1)a_3-(1+\xi_i^2)b_3|)^\varphi \\ &+(|(1+\xi_i^1)a_4-(1+\xi_i^2)b_4|)^\varphi]^{\frac{1}{\varphi}}\end{aligned} \tag{1}$$

In Equation 1, given $\varphi = 1$, *the Hamming distance measure [43] is given as follows:*

$$\begin{aligned}\delta_1(\mathrm{A}_\Theta^1,\mathrm{A}_\Theta^2) = \frac{1}{16\cdot\kappa\cdot\ell}\cdot \sum_{\xi_i^1\in\xi^1,\xi_i^2\in\xi^2} &[(|(1+\xi_i^1)a_1-(1+\xi_i^2)b_1|) \\ &+(|(1+\xi_i^1)a_2-(1+\xi_i^2)b_2|) \\ &+(|(1+\xi_i^1)a_3-(1+\xi_i^2)b_3|) \\ &+(|(1+\xi_i^1)a_4-(1+\xi_i^2)b_4|)]\end{aligned}$$

In Equation 1, given $\varphi = 2$, *the Euclidean distance measure is given as follows:*

$$\begin{aligned}\delta_2(\mathrm{A}_\Theta^1,\mathrm{A}_\Theta^2) = \frac{1}{16\cdot\kappa\cdot\ell}\cdot \sum_{\xi_i^1\in\xi^1,\xi_i^2\in\xi^2} &[(|(1+\xi_i^1)a_1-(1+\xi_i^2)b_1|)^2 \\ &+(|(1+\xi_i^1)a_2-(1+\xi_i^2)b_2|)^2 \\ &+(|(1+\xi_i^1)a_3-(1+\xi_i^2)b_3|)^2 \\ &+(|(1+\xi_i^1)a_4-(1+\xi_i^2)b_4|)^2]^{\frac{1}{2}}\end{aligned}$$

In Equation 1, given $\varphi = 27$, *the distance measure is given as follows;*

$$\begin{aligned}\delta_{27}(\mathrm{A}_\Theta^1,\mathrm{A}_\Theta^2) = \frac{1}{16\cdot\kappa\cdot\ell}\cdot \sum_{\xi_i^1\in\xi^1,\xi_i^2\in\xi^2} &[(|(1+\xi_i^1)a_1-(1+\xi_i^2)b_1|)^{27} \\ &+(|(1+\xi_i^1)a_2-(1+\xi_i^2)b_2|)^{27} \\ &+(|(1+\xi_i^1)a_3-(1+\xi_i^2)b_3|)^{27} \\ &+(|(1+\xi_i^1)a_4-(1+\xi_i^2)b_4|)^{27}]^{\frac{1}{27}}\end{aligned}$$

here $\varphi > 0$, $\kappa = \max\{|a_1|, |a_2|, |a_3|, |a_4|, |b_1|, |b_2|, |b_3|, |b_4|\}$ and $\ell = \xi^1 \cdot \xi^2$.

Equation (1) defines a generalized Minkowski-type distance metric for every positive value of φ. Increasing the value of the φ parameter significantly reduces the computational cost of the method, and the use of higher-order absolute differences proves effective in minimizing uncertainty. This effect has been clearly observed in example cases such as $\varphi = 27$. In practical applications, higher φ values (e.g., $\varphi = 27$ or $\varphi = 70$) are often preferred over lower ones in order to ensure numerical stability.

Example 6. *Assume*

$$\mathrm{A}^1_\Theta = \langle (0.3, 0.35, 0.41, 0.52); \{0.19, 0.41\}\rangle \text{ and } \mathrm{A}^2_\Theta = \langle (0.15, 0.28, 0.34, 0.42); \{0.13, 0.39\}\rangle$$

are normalized GTHF-numbers. Then;

$$\begin{aligned}
\delta_{27}(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta) &= \tfrac{1}{16\cdot\kappa\cdot\ell} \cdot \textstyle\sum_{\xi^1_i \in \xi^1, \xi^2_i \in \xi^2} [(|(1+\xi^1_i)a_1 - (1+\xi^2_i)b_1|)^{27} \\
&+ (|(1+\xi^1_i)a_2 - (1+\xi^2_i)b_2|)^{27} \\
&+ (|(1+\xi^1_i)a_3 - (1+\xi^2_i)b_3|)^{27} \\
&+ (|(1+\xi^1_i)a_4 - (1+\xi^2_i)b_4|)^{27}]^{\frac{1}{27}} \\
&= \tfrac{1}{16.(0.52).4} \cdot ((|(1+0.19)(0.3) - (1+0.13)(0.15)|)^{27} \\
&+ (|(1+0.19)(0.35) - (1+0.13)(0.28)|)^{27} \\
&+ (|(1+0.19)(0.41) - (1+0.13)(0.34)|)^{27} \\
&+ (|(1+0.19)(0.52) - (1+0.13)(0.42)|)^{27} \\
&+ (|(1+0.39)(0.3) - (1+0.41)(0.15)|)^{27} \\
&+ (|(1+0.39)(0.35) - (1+0.41)(0.28)|)^{27} \\
&+ (|(1+0.39)(0.41) - (1+0.41)(0.34)|)^{27} \\
&+ (|(1+0.39)(0.52) - (1+0.41)(0.42)|)^{27})^{\frac{1}{27}} \\
&= 0.051091
\end{aligned}$$

Theorem 7. *Let* $\mathrm{A}^1_\Theta = \langle (a_1, a_2, a_3, a_4); \xi^1(x)\rangle$, $\mathrm{A}^2_\Theta = \langle (b_1, b_2, b_3, b_4); \xi^2(x)\rangle$, $\mathrm{A}^3_\Theta = \langle (c_1, c_2, c_3, c_4); \xi^3(x)\rangle \in \Theta$ *and* $\varphi \neq 0$. *Then, the distance measure* $\delta_\varphi(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta)$ *satisfies the following properties:*

(i) $0 \leq \delta_\varphi(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta) \leq 1$

(ii) $\delta_\varphi(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta) = \delta_\varphi(A^2_\Theta, \mathrm{A}^1_\Theta)$

(iii) $\mathrm{A}^1_\Theta = \mathrm{A}^2_\Theta \Rightarrow \delta_\varphi(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta) = 0$

(iv) $\delta_\varphi(\mathrm{A}^1_\Theta, \mathrm{A}^3_\Theta) + \delta_\varphi(\mathrm{A}^3_\Theta, \mathrm{A}^2_\Theta) \geq \delta_\varphi(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta)$

Proof. Let $\alpha = \max\{|a_1|, |b_1|\}$, $\beta = \max\{|a_2|, |b_2|\}$, $\gamma = \max\{|a_3|, |b_3|\}$, $\theta = \max\{|a_4|, |b_4|\}$, $\kappa = \max\{|a_1|, |a_2|, |a_3|, |a_4|, |b_1|, |b_2|, |b_3|, |b_4|\}$ and $\ell = \xi^1 \cdot \xi^2$.

(i) It can be seen from Definition 5 that $0 < \delta_\varphi(\mathrm{A}^1_\Theta, \mathrm{A}^2_\Theta) < 1$ for any GTHF-number.

(ii)

$$\begin{aligned}\delta_\varphi(\mathrm{A}^1_\Theta,\mathrm{A}^2_\Theta) = & \tfrac{1}{16\cdot\kappa\cdot\ell}\cdot\textstyle\sum_{\xi^1_1\in\xi^1,\xi^2_1\in\xi^2}((|(1+\xi^1_1)a_1-(1+\xi^2_1)b_1|)^\varphi \\ & +(|(1+\xi^1_1)a_2-(1+\xi^2_1)b_2|)^\varphi+ \\ & (|(1+\xi^1_1)a_3-(1+\xi^2_1)b_3|)^\varphi \\ & +(|(1+\xi^1_1)a_4-(1+\xi^2_1)b_4|)^\varphi)^{\frac{1}{\varphi}} \\ = & \tfrac{1}{16\cdot\kappa\cdot\ell}\cdot\textstyle\sum_{\xi^1_1\in\xi^1,\xi^2_1\in\xi^2}((|(1+\xi^2_1)b_1-(1+\xi^1_1)a_1|)^\varphi \\ & +(|(1+\xi^2_1)b_2-(1+\xi^1_1)a_2|)^\varphi+ \\ & (|(1+\xi^2_1)b_3-(1+\xi^1_1)a_3|)^\varphi \\ & +(|(1+\xi^2_1)b_4-(1+\xi^1_1)a_4|)^\varphi)^{\frac{1}{\varphi}} \\ = & \delta_\varphi(\mathrm{A}^2_\Theta,\mathrm{A}^1_\Theta).\end{aligned}$$

(iii) Given that A^1_Θ and A^2_Θ are identical, we have $a_1=b_1$, $a_2=b_2$, $a_3=b_3$, $a_4=b_4$, and $\xi=\xi^1=\xi^2$. Therefore, the degree of distance between them is calculated as:

$$\begin{aligned}\delta_\varphi(\mathrm{A}^1_\Theta,\mathrm{A}^2_\Theta) = & \tfrac{1}{16\cdot\kappa\cdot\ell}\cdot\textstyle\sum_{\xi^1_1\in\xi^1,\xi^2_1\in\xi^2}((|(1+\xi^1_1)a_1-(1+\xi^2_1)b_1|)^\varphi \\ & +(|(1+\xi^1_1)a_2-(1+\xi^2_1)b_2|)^\varphi+ \\ & (|(1+\xi^1_1)a_3-(1+\xi^2_1)b_3|)^\varphi \\ & +(|(1+\xi^1_1)a_4-(1+\xi^2_1)b_4|)^\varphi)^{\frac{1}{\varphi}} \\ = & \tfrac{1}{16\cdot\kappa\cdot\ell}\cdot\textstyle\sum_{\xi^1_1\in\xi^1,\xi^2_1\in\xi^2}((|(1+\xi)a_1-(1+\xi)a_1|)^\varphi \\ & +(|(1+\xi)a_2-(1+\xi)a_2|)^\varphi+ \\ & (|(1+\xi)a_3-(1+\xi)a_3|)^\varphi+(|(1+\xi)a_4-(1+\xi)a_4|)^\varphi)^{\frac{1}{\varphi}} \\ = & 0\end{aligned}$$

(iv)

$$\begin{aligned}\delta_\varphi(\mathrm{A}^1_\Theta,\mathrm{A}^2_\Theta) = & \tfrac{1}{16\cdot\kappa\cdot\ell}\cdot\textstyle\sum_{\xi^1_1\in\xi^1,\xi^2_1\in\xi^2}((|(1+\xi^1_1)a_1-(1+\xi^2_1)b_1|)^\varphi+ \\ & (|(1+\xi^1_1)a_2-(1+\xi^2_1)b_2|)^\varphi+ \\ & (|(1+\xi^1_1)a_3-(1+\xi^2_1)b_3|)^\varphi \\ & +(|(1+\xi^1_1)a_4-(1+\xi^2_1)b_4|)^\varphi)^{\frac{1}{\varphi}} \\ = & \tfrac{1}{16\cdot\kappa\cdot\ell}\cdot\textstyle\sum_{\xi^1_1\in\xi^1,\xi^2_1\in\xi^2,\xi^3_1\in\xi^3}((|(1+\xi^1_1)a_1+(1+\xi^3_1)c_1-(1+\xi^3_1)c_1-(1+\xi^2_1)b_1|)^\varphi+ \\ & (|(1+\xi^1_1)a_2+(1+\xi^3_1)c_2-(1+\xi^3_1)c_2-(1+\xi^2_1)b_2|)^\varphi+ \\ & (|(1+\xi^1_1)a_3+(1+\xi^3_1)c_3-(1+\xi^3_1)c_3-(1+\xi^2_1)b_3|)^\varphi+ \\ & (|(1+\xi^1_1)a_4+(1+\xi^3_1)c_4-(1+\xi^3_1)c_4-(1+\xi^2_1)b_4|)^\varphi)^{\frac{1}{\varphi}} \\ \leq & \tfrac{1}{16\cdot\kappa\cdot\ell}\cdot\textstyle\sum_{\xi^1_1\in\xi^1,\xi^3_1\in\xi^3}((|(1+\xi^1_1)a_1-(1+\xi^3_1)c_1|)^\varphi+(|(1+\xi^1_1)a_2-(1+\xi^3_1)c_2|)^\varphi+ \\ & (|(1+\xi^1_1)a_3-(1+\xi^3_1)c_3|)^\varphi+(|(1+\xi^1_1)a_4-(1+\xi^3_1)c_4|)^\varphi)^{\frac{1}{\varphi}}+ \\ & \tfrac{1}{16\cdot\kappa\cdot\ell}\cdot\textstyle\sum_{\xi^3_1\in\xi^3,\xi^2_1\in\xi^2}((|(1+\xi^3_1)c_1-(1+\xi^2_1)b_1|)^\varphi+ \\ & (|(1+\xi^3_1)c_2-(1+\xi^2_1)b_2|)^\varphi \\ & +(|(1+\xi^3_1)c_3-(1+\xi^2_1)b_3|)^\varphi+(|(1+\xi^3_1)c_4-(1+\xi^2_1)b_4|)^\varphi)^{\frac{1}{\varphi}} \\ = & \delta_\varphi(\mathrm{A}^1_\Theta,\mathrm{A}^3_\Theta)+\delta_\varphi(\mathrm{A}^3_\Theta,\mathrm{A}^2_\Theta)\end{aligned}$$

□

3.2. Entropy on Minkowski-Type Distance Measures with GTHF-Numbers

Definition 8. *A function $\mathscr{E}:\Theta\to[0,1]$ is called an entropy on GTHF-numbers if it holds the following properties:*

$(\mathscr{E}1)$: $A_\Theta = \langle (a_1, a_2, a_3, a_4); 1\rangle$ *or* $A_\Theta = \langle (a_1, a_2, a_3, a_4); 0\rangle \Rightarrow \mathscr{E}(A_\Theta) = 0$,

$(\mathscr{E}2)$: $\mathscr{E}(A_\Theta) = 1 \Leftrightarrow \delta_\varphi(A_\Theta, A_\Theta^+) = \delta_\varphi(A_\Theta, A_\Theta^-)$ *for all* $A_\Theta \in \Theta$,

$(\mathscr{E}3)$: $\mathscr{E}(A_\Theta) = \mathscr{E}(A_\Theta^c)$, *for all* $A_\Theta \in \Theta$,

$(\mathscr{E}4)$: *For all* $A_\Theta^1, A_\Theta^2 \in \Theta, if$

$$|\frac{\delta_\varphi(A_\Theta^1, A_\Theta^-)}{\delta_\varphi(A_\Theta^1, A_\Theta^-) + \delta_\varphi(A_\Theta^1, (A_\Theta^+))} - \frac{1}{2}| \geq |\frac{\delta_\varphi(A_\Theta^2, A_\Theta^-)}{\delta_\varphi}(A_\Theta^2, A_\Theta^-) + \delta_\varphi(A_\Theta^2, (A_\Theta^+)) - \frac{1}{2}|$$

then $\mathscr{E}(A_\Theta^1) \leq \mathscr{E}(A_\Theta^2)$,

where $A_\Theta^+ = \langle (a_1, a_2, a_3, a_4); \{1\}\rangle, A_\Theta^- = \langle (a_1, a_2, a_3, a_4); \{0\}\rangle$ *and* $A_\Theta^c = \langle (a_1, a_2, a_3, a_4); \{1 - \xi\}\rangle$.

Theorem 9. *The entropy of GTHF-numbers is defined for each* $A_\Theta \in \Theta$ *as*

$$\mathscr{E}(A_\Theta) = 1 - 2\left|\frac{\delta_\varphi(A_\Theta, A_\Theta^-)}{\delta_\varphi(A_\Theta, A_\Theta^-) + \delta_\varphi(A_\Theta, A_\Theta^+)} - \frac{1}{2}\right|,$$

where δ_φ *is a Minkowski-type distance measure for GTHF-numbers.*

Proof. It can be readily verified that $\mathscr{E}(A_\Theta)$ satisfies all four conditions outlined in Definition 8. Therefore, the detailed proof is omitted for conciseness. □

3.3. VIKOR Method Based on the Entropy Measure for GTHF-numbers

In this section, we propose a VIKOR method based on GTHF-numbers by extending the original VIKOR framework [31, 32, 38] through the incorporation of entropy-based weights.

Definition 10. *Let* $\mathfrak{U} = (u_1, u_2, \ldots, u_m)$ *denote a set of alternatives and* $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_n\}$ *be the set of criterions,* $\{c_{j1}, c_{j2}, ..., c_{jk_j}\}$ *be the set of sub-criteria for* $\mathfrak{C}_j (j \in \{1, 2, \ldots, n\})$ *such that* k_j *is a number sub-criteria of criterion* $\mathfrak{C}_j$ *and* $A_{\Theta}{}_{ir}^j$ *be a GTHF-number for all* $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$ *and* $(r \in \{1, 2, \ldots, k_j\})$. *In here,* $A_{\Theta}{}_{ir}^j r \in \{1, 2, \ldots, k_j\}$ *denotes evaluation of the alternative* u_i *with respect to the sub-criteria* $c_{jr} (r \in \{1, 2, \ldots, k_j\})$ *made by expert or decision maker. Then,*

$$[A_{\Theta}{}^j{}_{ir}]_{m \times k_j} = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{array} \begin{array}{c} \begin{array}{cccc} c_{j1} & c_{j2} & \cdots & c_{jk_j} \end{array} \\ \begin{pmatrix} A_{\Theta}{}_{11}^j & A_{\Theta}{}_{12}^j & \cdots & A_{\Theta}{}_{1k_j}^j \\ A_{\Theta}{}_{21}^j & A_{\Theta}{}_{22}^j & \cdots & A_{\Theta}{}_{2k_j}^j \\ \vdots & \vdots & \vdots & \vdots \\ A_{\Theta}{}_{m1}^j & A_{\Theta}{}_{m2}^j & \cdots & A_{\Theta}{}_{mk_j}^j \end{pmatrix} \end{array}$$

is defined as the GTHF-number multi-criteria decision sub-matrix based on the set of sub-criteria $\{c_{j1}, c_{j2}, ..., c_{jk_j}\}$ *of the decision maker.*

Moreover, table of $[A_\Theta{}^j{}_{ir}]_{m\times k_j}$ is as follows;

$\mathfrak{C}_j$	c_{j1}	c_{j2}	...	c_{jm}
u_1	$A_{\Theta}{}^j_{11}$	$A_{\Theta}{}^j_{12}$	$\cdots$	$A_{\Theta}{}^j_{1k_j}$
u_2	$A_{\Theta}{}^j_{21}$	$A_{\Theta}{}^j_{22}$	$\cdots$	$A_{\Theta}{}^j_{2k_j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	$A_{\Theta}{}^j_{m1}$	$A_{\Theta}{}^j_{m2}$	$\cdots$	$A_{\Theta}{}^j_{mk_j}$

Algorithm:

Step 1. Construct the sub-matrix $[A_\Theta{}^j ir] m \times k_j$ of GTHF-numbers corresponding to the sub-criteria set $c_{j1}, c_{j2}, \ldots, c_{jk_j}$ for use in the decision-making process.

Step 2. Calculate the overall decision values matrix

$$A_{\Theta ij} = \langle (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}); \xi^{ij}(x)\rangle = \quad H^G_w(A_\Theta{}^j{}_{i1}, A_\Theta{}^j{}_{i2}, \ldots, A_\Theta{}^j{}_{ik_j}) = \bigotimes_{r=1}^{k_j} A_\Theta{}^j{}_{ir}$$

For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, $A_{\Theta ij}$ indicates how alternative u_i is evaluated according to criterion $\mathfrak{C}_j$.

Step 3. Construct the entropy matrix $[\mathscr{E}_{ij}]_{m\times n}$ derived from the overall decision matrix. $[A_{\Theta ij}]_{m\times n}$ by using

$$\mathscr{E}_{ij} = 1 - 2\left|\frac{\delta_\varphi(A_{\Theta ij}, A_{\Theta}{}^-_{ij})}{\delta_\varphi(A_{\Theta ij}, A_{\Theta}{}^-_{ij}) + \delta_\varphi(A_{\Theta ij}, A_{\Theta}{}^+_{ij})} - \frac{1}{2}\right|$$

where $A_{\Theta}{}^+_{ij} = \langle (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}); \{1\}\rangle$ and $A_{\Theta}{}^-_{ij} = \langle (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}); \{0\}\rangle$.

Step 4. Calculate the weights of the criteria ω_j where $j \in \{1, 2, \ldots, n\}$ as;

$$\omega_j = \frac{1 - \sum_{i=1}^n \bar{\mathscr{E}}_{ij}}{n - \sum_{i=1}^m \sum_{j=1}^n \bar{\mathscr{E}}_{ij}}; \quad j \in \{1, 2, \ldots, n\}.$$

where $\bar{\mathscr{E}}_{ij} = \frac{\mathscr{E}_{ij}}{\max\{\mathscr{E}_{i1}, \mathscr{E}_{i2}, \ldots, \mathscr{E}_{in}\}}$ for $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$.

Step 5. Calculate the $\mathscr{P}_i$ corresponding to the maximum values and the $\mathscr{R}_i$ corresponding to the opponent values, respectively, as:

$$\mathscr{P}_i = \sum_{j=1}^n \omega_j \frac{\delta_\varphi(H^+_\Theta, A_{\Theta ij})}{\delta_\varphi(H^+_\Theta, H^-_\Theta)}; \quad i \in \{1, 2, \ldots, m\}$$

and

$$\mathscr{R}_i = max_j\{\frac{\omega_j \delta_\varphi(H_\Theta^+, \mathrm{A}_{\Theta ij})}{\delta_\varphi(H_\Theta^+, H_\Theta^-)}\}; \quad i \in \{1,2,\ldots,m\}$$

where

$$H_\Theta^+ = \langle max_{i,j}\{a_{1ij}\}, max_{i,j}\{a_{2ij}\}, max_{i,j}\{a_{3ij}\}, max_{i,j}\{a_{4ij}\}); max_{\xi_{kl}^{ij} \in \xi_{ij}(x)}\{\xi_{ij}^{ij}\}\rangle$$

and

$$H_\Theta^- = \langle min_{i,j}\{a_{1ij}\}, min_{i,j}\{a_{2ij}\}, min_{i,j}\{a_{3ij}\}, min_{i,j}\{a_{4ij}\}); min_{\xi_{kl}^{ij} \in \xi_{ij}(x)}\{\xi_{ij}^{ij}\}\rangle$$

for $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,n\}$.

Step 6. Determine the values of $\mathscr{Q}(u_i)$ for $i \in \{1,2,\ldots,m\}$ as follows,

$$\mathscr{Q}^v(u_i) = v(\frac{\mathscr{P}_i - \mathscr{P}^-}{\mathscr{P}^+ - \mathscr{P}^-}) + (1-v)(\frac{\mathscr{R}_i - \mathscr{R}^-}{\mathscr{R}^+ - \mathscr{R}^-}), (i \in \{1,2,\ldots,m\}, j \in \{1,2,\ldots,n\})$$

where $\mathscr{P}^+ = max\{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_m\}$, $\mathscr{P}^- = min\{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_m\}$, $\mathscr{R}^+ = max\{\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_m\}$, $\mathscr{R}^- = min\{\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_m\})$ $v \in [0,1]$ Here,

1. $\mathscr{Q}^v(u_i)$ is the minimal if $v < 0.5$
2. $\mathscr{Q}^v(u_i)$ is the maximum if $v > 0.5$
3. $\mathscr{Q}^v(u_i)$ is both minimal and maximum if $v = 0.5$.

Step 7. Create a table based on $\mathscr{Q}^v(u_i)$, $\mathscr{P}_i$, and $\mathscr{R}_i$ for each u_i where $i = 1,2,\ldots,m$.

Step 8. Sort the alternatives based on $\mathscr{Q}^v(u_i)$, $\mathscr{P}_i$, and $\mathscr{R}_i$ by using;

1. $\mathscr{Q}^v(u_k) < \mathscr{Q}^v(u_l) \Rightarrow u_k > u_l(k,l \in \{1,2,\ldots,m\})$,
2. $\mathscr{P}_k < \mathscr{P}_l \Rightarrow u_k > u_l(k,l \in \{1,2,\ldots,m\})$,
3. $\mathscr{R}_k < \mathscr{R}_l \Rightarrow u_k > u_l(k,l \in \{1,2,\ldots,m\})$

where both of the following conditions are satisfied for all $v \in [0,1]$.

1. $$\mathscr{Q}^v(u_k) - \mathscr{Q}^v(u_l) \geq \frac{1}{m-1}(k,l \in \{1,2,\ldots,m\})$$

   where m denotes the number of supplier alternatives, and u_l and u_k are the first and second best alternatives in the ranking list based on $\mathscr{Q}^v(u_i)$, $i \in \{1,2,\ldots,m\}$.

2. u_l must be the highest-ranked alternative by $\mathscr{P}_i$ and/or $\mathscr{R}_i$, with $i,l \in \{1,2,\ldots,m\}$.

The algorithm of the proposed method can be given as follows:

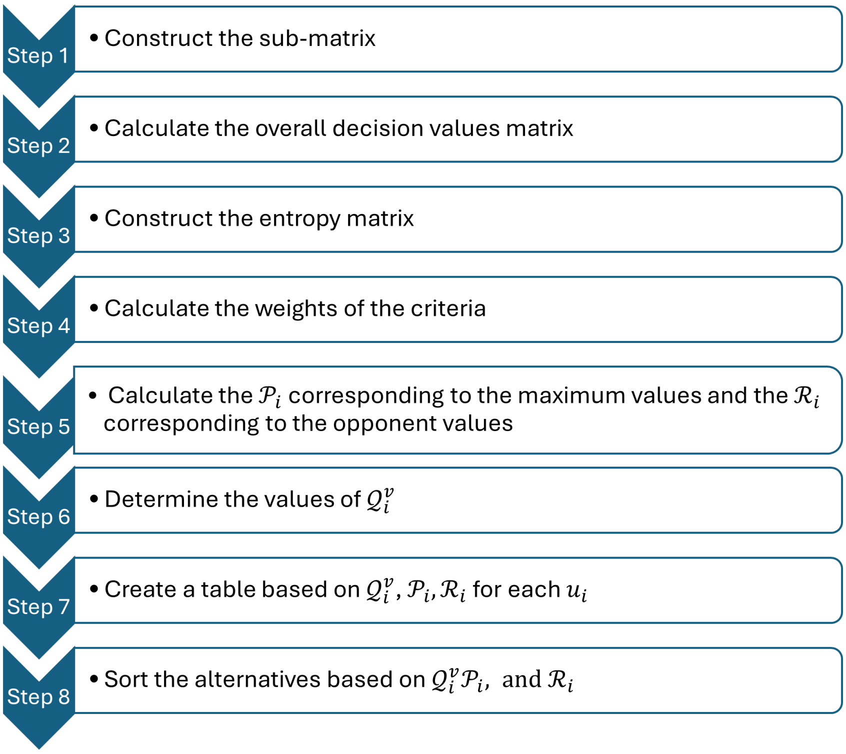

Figure 1: Flowchart of Proposed Method

3.4. Illustrative case study

The effective supply of maintenance and repair services in hospitals is of critical importance to help ensure that patients are appropriately managed and that medical equipment is kept in good working order. Many believe that effective maintenance and repair services are essential for safe and continuous treatment in hospitals. In light of today's technological advances, the need for smart robots to provide maintenance and repair services has emerged. The amount of hospital resources allocated is increasing in direct proportion to the scale of the need to be met. In this context, finding the most appropriate solution becomes even more important. Therefore, robots need to be able to work in the most efficient way in order to properly perform maintenance of hospital equipment. This method, which sorts the alternatives for maintenance and repair operations under several conflicting criteria and selects the most optimal solution, uses a multi-criteria ranking index based on finding the closest result to the ideal solution. Thus, the most suitable robots to be used in hospital maintenance and repair services will be selected. For this reason, we have given an example as follows:

Example 11. *(It's adapted from [43]). A technology company is considering the use of robots to speed up the maintenance and repair of medical equipment used in the health-care industry. The company selects u_1, u_2 and u_3 models from the existing robots on the market for testing. These robots will perform maintenance and repair operations in a hospital environment and the robot that provides the best service will be selected. During the selection process, a commission will be established to evaluate each robot according to various performance criteria. The commission will select the most suitable robot by considering the following technical characteristics of the robots: Evaluation Criteria:*

- $\mathfrak{C}_1$ (**Processing Speed**)= How quickly robots can complete maintenance and repair processes. This is critical to how quickly robots can repair medical devices and ensure continuity of service.
- $\mathfrak{C}_2$ (**Precision**)= How well robots can protect the vulnerability of the devices they work with. Especially in the sensitive parts of medical devices, the error rates of the robot are very important for the safety of the devices.
- $\mathfrak{C}_3$ (**Operating Time**)= The working time of battery-powered robots. It is important that the robots can operate for a long time without downtime and can be maintained in emergencies.
- $\mathfrak{C}_4$ (**Cost Effectiveness**)= The costs of purchasing and maintaining the robots. This is of great importance, especially in terms of how it can impact the company's budget in the long term. The cost is evaluated based on the robot's technical characteristics and its long-term use.
- $\mathfrak{C}_5$ (**Adaptability**)= How quickly robots can be adapted to different medical equipment and various hospital environment. The flexibility of the robot, its ability to adapt to different device models, will enhance efficiency in maintenance processes.

$$\begin{aligned}
\mathfrak{C}_1(Processing\ \ Speed) = &\ \{c_{11} = \textit{Clock Speed}, c_{12} = \textit{Core Count}, c_{13} = \textit{Processor},\\
&\ c_{14} = \textit{Frequency}\ \ \textit{Scaling}\}\\
\mathfrak{C}_2(Precision) = &\ \{c_{21} = \textit{Accuracy}, c_{22} = \textit{Resolution}, c_{23} = \textit{Margin}\ \ \textit{of}\ \ \textit{Error}\}\\
\mathfrak{C}_3(Operating\ \ Time) = &\ \{c_{31} = \textit{Battery}\ \ \textit{Capacity}, c_{32} = \textit{Workload}\}\\
\mathfrak{C}_4(Cost\ \ Effectiveness) = &\ \{c_{41} = \textit{Quality}, c_{42} = \textit{Time}\ \ \textit{Management}\}\\
\mathfrak{C}_5(Adaptability) = &\ \{c_{51} = \textit{Multi-Device}\ \ \textit{Integration},\\
&\ c_{52} = \textit{Adaptability to Use Cases},\\
&\ c_{53} = \textit{Energy Efficiency}, c_{54} = \textit{Regional Adaptability},\\
&\ c_{55} = \textit{Performance Improvement}\}
\end{aligned}$$

The diagrammatic representation of these criterias is shown in Figure 2.

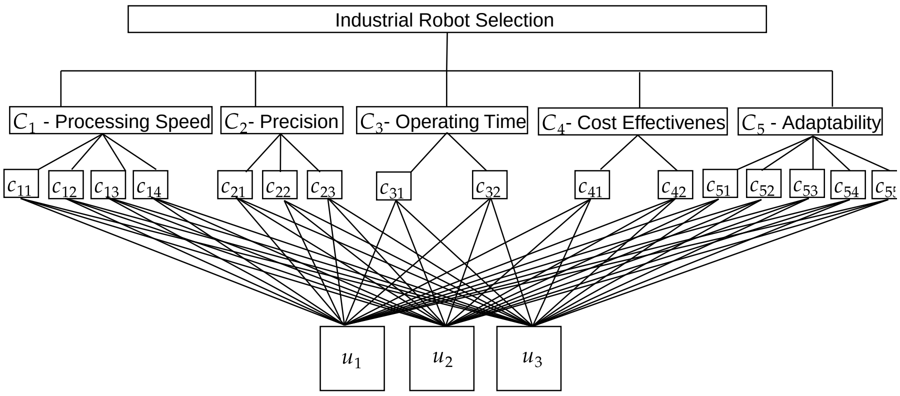

Figure 2: Robot Selection Diagram

According to Table 1, the alternatives will be evaluated.

Table 1: Linguistic terms

Linguistic terms	Linguistic values of GTHF-numbers
Extremely High	$\langle(1.0, 1.0, 1.0, 1.0); \{1.0\}\rangle$
Very Very High	$\langle(0.9, 0.9, 1.0, 1.0); \{0.8, 0.9, 1.0\}\rangle$
Very High	$\langle(0.5, 0.6, 0.7, 0.8); \{0.8, 0.9\}\rangle$
Good	$\langle(0.4, 0.5, 0.6, 0.7); \{0.6, 0.7\}\rangle$
Medium High	$\langle(0.4, 0.5, 0.5, 0.6); \{0.4, 0.7, 0.8\}\rangle$
Medium	$\langle(0.4, 0.5, 0.5, 0.6); \{0.5\}\rangle$
Medium Low	$\langle(0.2, 0.3, 0.4, 0.5); \{0.2, 0.3, 0.4\}\rangle$
Low	$\langle(0.1, 0.2, 0.3, 0.5); \{0.1, 0, 3\}\rangle$
Very Low	$\langle(0.1, 0.2, 0.3, 0.4); \{0.2\}\rangle$
Very Very Low	$\langle(0.1, 0.2, 0.2, 0.3); \{0.1\}\rangle$

The decision-making process is calculated as follows:

Step 1. For each $j \in \{1, 2, 3, 4, 5, 6\}$, the GTHF-number decision sub-matrices $[A_{\Theta}{}^{j}{}_{ir}]_{3\times k_j}$ were constructed based on the sub-criteria set $\{c_{j1}, c_{j2}, \ldots, c_{jk_j}\}$ of $\mathfrak{C}_j$, as detailed in Tables $2-6$.

Table 2: Table presenting the GTHF-number multi-criteria decision sub-matrices of $[A_{\Theta}{}^{1}{}_{ir}]_{3\times4}$

$\mathfrak{C}_1$	c_{11}	c_{12}
u_1	$\langle(0.9,0.9,1.0,1.0);\{0.8,0.9,1.0\}\rangle$	$\langle(1.0,1.0,1.0,1.0);\{1.0\}\rangle$
u_2	$\langle(0.4,0.5,0.5,0.6);\{0.5\}\rangle$	$\langle(0.2,0.3,0.4,0.5);\{0.2,0.3,0.4\}\rangle$
u_3	$\langle(0.9,0.9,1.0,1.0);\{0.8,0.9,1.0\}\rangle$	$\langle(0.1,0.2,0.3,0.4);\{0.2\}\rangle$
	c_{13}	c_{14}
u_1	$\langle(0.5,0.6,0.7,0.8);\{0.8,0.9\}\rangle$	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$
u_2	$\langle(0.4,0.5,0.6,0.7);\{0.6,0.7\}\rangle$	$\langle(0.4,0.5,0.5,0.6);\{0.5\}\rangle$
u_3	$\langle(0.1,0.2,0.3,0.5);\{0.1,0,3\}\rangle$	$\langle(0.1,0.2,0.3,0.4);\{0.2\}\rangle$

Table 3: Table presenting the GTHF-number multi-criteria decision sub-matrices of $[A_{\Theta}{}^{2}{}_{ir}]_{3\times3}$

$\mathfrak{C}_2$	c_{21}	c_{22}	c_{23}
u_1	$\langle(1.0,1.0,1.0,1.0);\{1.0\}\rangle$	$\langle(0.4,0.5,0.5,0.6);\{0.5\}\rangle$	$\langle(0.5,0.6,0.7,0.8);\{0.8,0.9\}\rangle$
u_2	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$	$\langle(0.4,0.5,0.6,0.7);\{0.6,0.7\}\rangle$	$\langle(0.1,0.2,0.3,0.4);\{0.2\}\rangle$
u_3	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$	$\langle(1.0,1.0,1.0,1.0);\{1.0\}\rangle$	$\langle(0.1,0.2,0.3,0.5);\{0.1,0,3\}\rangle$

Table 4: Table presenting the GTHF-number multi-criteria decision sub-matrices of $[A_{\Theta}{}^{3}{}_{ir}]_{3\times2}$

$\mathfrak{C}_3$	c_{31}	c_{32}
u_1	$\langle(1.0,1.0,1.0,1.0);\{1.0\}\rangle$	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$
u_2	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$	$\langle(0.4,0.5,0.5,0.6);\{0.5\}\rangle$
u_3	$\langle(0.1,0.2,0.3,0.4);\{0.2\}\rangle$	$\langle(1.0,1.0,1.0,1.0);\{1.0\}\rangle$

Table 5: Table presenting the GTHF-number multi-criteria decision sub-matrices of $[A_{\Theta}{}^{4}{}_{ir}]_{3\times2}$

$\mathfrak{C}_4$	c_{41}	c_{42}
u_1	$\langle(0.4,0.5,0.6,0.7);\{0.6,0.7\}\rangle$	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$
u_2	$\langle(0.5,0.6,0.7,0.8);\{0.8,0.9\}\rangle$	$\langle(0.4,0.5,0.5,0.6);\{0.5\}\rangle$
u_3	$\langle(1.0,1.0,1.0,1.0);\{1.0\}\rangle$	$\langle(0.5,0.6,0.7,0.8);\{0.8,0.9\}\rangle$

Table 6: Table presenting the GTHF-number multi-criteria decision sub-matrices of $[A_{\Theta}{}^{5}{}_{ir}]_{3\times3}$

$\mathfrak{C}_5$	c_{51}	c_{52}	c_{53}
u_1	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$	$\langle(0.5,0.6,0.7,0.8);\{0.8,0.9\}\rangle$
u_2	$\langle(0.1,0.2,0.3,0.4);\{0.2\}\rangle$	$\langle(1.0,1.0,1.0,1.0);\{1.0\}\rangle$	$\langle(0.1,0.2,0.3,0.5);\{0.1,0,3\}\rangle$
u_3	$\langle(0.4,0.5,0.6,0.7);\{0.6,0.7\}\rangle$	$\langle(0.4,0.5,0.5,0.6);\{0.5\}\rangle$	$\langle(0.1,0.2,0.2,0.3);\{0.1\}\rangle$

Step 2. The overall decision values matrix, derived from Tables 2 to 6, is presented in Table 7.

Table 7: Aggregated values of the GTHF-number multi-criteria decision sub-matrices $[{A_\Theta}^j{}_{ir}]_{3\times k_j}(j=\{1,2,\ldots,6\})$ computed with $[A_{\Theta ij}]_{3\times 6}=H^G_\omega({A_\Theta}^j{}_{i1},{A_\Theta}^j{}_{i2},\ldots,{A_\Theta}^j{}_{ik_j})$ as;

	$\mathfrak{C}_1$	$\mathfrak{C}_2$
u_1	$\langle(0.12,0.14,0.15,0.17);\{0.50,0.52,0.53,0.55\}\rangle$	$\langle(0.11,0.14,0.15,0.19);\{0.74,0.77\}\rangle$
u_2	$\langle(0.03,0.04,0.06,0.09);\{0.42,0.43,0.46,0.48,0.49,0.51\}\rangle$	$\langle(0.03,0.06,0.08,0.12);\{0.23,0.24\}\rangle$
u_3	$\langle(0.01,0.02,0.04,0.07);\{0.24,0.31,0.32,0.25,0.33\}\rangle$	$\langle(0.07,0.11,0.13,0.18);\{0.22,0.31\}\rangle$
	$\mathfrak{C}_3$	$\mathfrak{C}_4$
u_1	$\langle(0.05,0.10,0.10,0.15);\{0.32\}\rangle$	$\langle(0.03,0.07,0.08,0.13);\{0.24,0.26\}\rangle$
u_2	$\langle(0.06,0.11,0.11,0.16);\{0.22\}\rangle$	$\langle(0.14,0.19,0.21,0.27);\{0.63,0.67\}\rangle$
u_3	$\langle(0.16,0.22,0.27,0.32);\{0.45\}\rangle$	$\langle(0.25,0.30,0.35,0.40);\{0.89,0.95\}\rangle$
	$\mathfrak{C}_5$	
u_1	$\langle(0.01,0.03,0.03,0.06);\{0.20,0.21\}\rangle$	
u_2	$\langle(0.07,0.11,0.15,0.19);\{0.27,0.39\}\rangle$	
u_3	$\langle(0.05,0.08,0.08,0.12);\{0.31,0.33\}\rangle$	

Step 3. Based on the overall decision values matrix $[A_{\Theta ij}]_{3\times 6}$ given in Table 7, we derived the entropy matrix $[\mathscr{E}_{ij}]_{3\times 5}$ as follows;

$$[\mathscr{E}_{ij}]_{m\times n}=\begin{pmatrix}0.95 & 0.49 & 0.64 & 0.5 & 0.41\\ 0.93 & 0.47 & 0.44 & 0.7 & 0.66\\ 0.58 & 0.53 & 0.705 & 0.16 & 0.66\end{pmatrix}$$

Step 4. We computed the weights of the criterias $\omega_1,\omega_2,\omega_3,\omega_4,\omega_5$as;

$$\begin{aligned}\omega_1 &= \frac{1-(\mathscr{E}_{11}+\mathscr{E}_{21}+\mathscr{E}_{31})}{5-(\mathscr{E}_{11}+\mathscr{E}_{12}+...+\mathscr{E}_{15}+\mathscr{E}_{21}+\mathscr{E}_{22}+...+\mathscr{E}_{15}+\mathscr{E}_{31}+\mathscr{E}_{32}+...+\mathscr{E}_{35})}\\ &= \frac{1-(0.95+0.93+0.58)}{5-(0.95+0.49+0.64+0.5+0.41+0.93+0.47+0.44+0.7+0.66+0.58+0.53+0.705+0.16+0.66)}\\ &=0.3817\end{aligned}$$

similarly we have $\omega_2=0.1281,\omega_3=0.2053,\omega_4=0.0941$ and $\omega_5=0.1908$.

Step 5. We calculated the $\mathscr{P}_i(i=1,2,3)$ values representing the maximum and $\mathscr{R}_i(i=1,2,3)$ representing the opponent, respectively, as follows:

$$\begin{aligned}\mathscr{P}_1 &= \frac{\omega_1\delta_6(H^+_\Theta,A_{\Theta 11})+\omega_2\delta_6(H^+_\Theta,A_{\Theta 12})+\omega_3\delta_6(H^+_\Theta,A_{\Theta 13})+\omega_4\delta_6(H^+_\Theta,A_{\Theta 14})+\omega_5\delta_6(H^+_\Theta,A_{\Theta 15})}{\delta_6(H^+_\Theta,H^-_\Theta)}\\ &= \frac{(0.38167)(0.021767)+(0.1281)(0.038006)+(0.2053)(0.098695)+(0.0941)(0.053695)+(0.1908)(0.060456)}{(0.020429)}\\ &= 0.4898\end{aligned}$$

similarly we have $\mathscr{P}_2=0.4492$ and $\mathscr{P}_3=0.3041$.

$$\mathscr{R}_i=max\{\frac{\omega_j\delta_\varphi(H^+_\Theta,A_{\Theta ij})}{\delta_\varphi(H^+_\Theta,H^-_\Theta)}\}$$

$$\begin{aligned}\mathscr{R}_1 =max\{&\frac{\omega_1\delta_\varphi(H^+_\Theta,A_{\Theta 11})}{\delta_\varphi(H^+_\Theta,H^-_\Theta)},\frac{\omega_2\delta_\varphi(H^+_\Theta,A_{\Theta 12})}{\delta_\varphi(H^+_\Theta,H^-_\Theta)}\\ &\frac{\omega_3\delta_\varphi(H^+_\Theta,A_{\Theta 13})}{\delta_\varphi(H^+_\Theta,H^-_\Theta)},\frac{\omega_4\delta_\varphi(H^+_\Theta,A_{\Theta 14})}{\delta_\varphi(H^+_\Theta,H^-_\Theta)},\frac{\omega_5\delta_\varphi(H^+_\Theta,A_{\Theta 15})}{\delta_\varphi(H^+_\Theta,H^-_\Theta)}\}\end{aligned}$$

$$\mathscr{R}_1 = max\{0.0083, 0.0049, 0.0203, 0.0051, 0.011537\} = 0.0202$$

likewise we have $\mathscr{R}_2 = 0.0202$ and $\mathscr{R}_3 = 0.0108$. Let $v = 0.5$, compute the values $\mathscr{Q}_i$ $(i = 1,2,3)$: where

$$H^+_\Theta = \langle(0.25, 0.3, 0.35, 0.4); \{0.89, 0.95, 0.95, 0.0.95, 0.95, 0.95\}\rangle$$

and

$$H^-_\Theta = \langle(0.01, 0.02, 0.03, 0.06); \{0.20, 0.21, 0.21, 0.21, 0.21, 0.21\}\rangle.$$

Using the following equation, we determined the distance between the alternatives u_i and H^+_Θ;

$$\begin{aligned}
\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 11}) = \quad & \tfrac{1}{16\cdot\kappa\cdot\ell} \cdot \textstyle\sum_{\xi^1_i \in \xi^1, \xi^2_i \in \xi^2} [(|(1+\xi^1_i)a_1 - (1+\xi^2_i)b_1|)^6 \\
& +(|(1+\xi^1_i)a_2 - (1+\xi^2_i)b_2|)^6 \\
& +(|(1+\xi^1_i)a_3 - (1+\xi^2_i)b_3|)^6 \\
& +(|(1+\xi^1_i)a_4 - (1+\xi^2_i)b_2|)^4]^{\frac{1}{6}} \\
= \quad & \tfrac{1}{16.(0.4).16}.((|(1+0.5)(0.12) - (1+0.89)(0.25)|)^6 \\
& +(|(1+0.5)(0.14) - (1+0.89)(0.3)|)^6 + \\
& (|(1+0.5)(0.15) - (1+0.89)(0.35)|)^6 \\
& +(|(1+0.5)(0.17) - (1+0.89)(0.4)|)^6 + \\
& (|(1+0.52)(0.12) - (1+0.95)(0.25)|)^6 \\
& +(|(1+0.52)(0.14) - (1+0.95)(0.3)|)^6 + \\
& (|(1+0.52)(0.15) - (1+0.95)(0.35)|)^6 \\
& +(|(1+0.52)(0.17) - (1+0.95)(0.4)|)^6 + \\
& (|(1+0.53)(0.12) - (1+0.95)(0.25)|)^6 \\
& +(|(1+0.53)(0.14) - (1+0.95)(0.3)|)^6 + \\
& (|(1+0.53)(0.15) - (1+0.95)(0.35)|)^6 \\
& +(|(1+0.53)(0.17) - (1+0.95)(0.4)|)^6 + \\
& (|(1+0.55)(0.12) - (1+0.95)(0.25)|)^6 \\
& +(|(1+0.55)(0.14) - (1+0.95)(0.3)|)^6 + \\
& (|(1+0.55)(0.15) - (1+0.95)(0.35)|)^6 \\
& +(|(1+0.55)(0.17) - (1+0.95)(0.4)|)^6)^{\frac{1}{6}} \\
= \quad & 0.0218
\end{aligned}$$

similarly we have $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 12}) = 0.0380$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 13}) = 0.0987$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 14}) = 0.0537$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 15}) = 0.0605$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 21}) = 0.0189$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 22}) = 0.0545$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 23}) = 0.0986$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 24}) = 0.0295$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 25}) = 0.0456$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 31}) = 0.0240$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 32}) = 0.0077$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 33}) = 0.0525$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 34}) = 0$, $\delta_6(H^+_\Theta, \mathrm{A}_{\Theta 35}) = 0.0532$ and $\delta_6(H^+_\Theta, H^-_\Theta) = 0.0204$

Step 6. The index values of $\mathscr{Q}(u_i)$ corresponding to $u_i\,(\in \{1,2,3\})$ were determined as follows:,

$$\begin{aligned}
\mathscr{Q}_i = \quad & v(\tfrac{\mathscr{P}_i - \mathscr{P}^-}{\mathscr{P}^+ - \mathscr{P}^-}) + (1-v)(\tfrac{\mathscr{R}_i - \mathscr{R}^-}{\mathscr{R}^+ - \mathscr{R}^-}) \\
\mathscr{Q}_1 = \quad & v(\tfrac{\mathscr{P}_1 - \mathscr{P}^-}{\mathscr{P}^+ - \mathscr{P}^-}) + (1-v)(\tfrac{\mathscr{R}_1 - \mathscr{R}^-}{\mathscr{R}^+ - \mathscr{R}^-}) \\
= \quad & (0.5)\tfrac{(0.4898-0.3041)}{(0.4898-0.3041)} + (1-0.5)\tfrac{(0.0203-0.0108)}{(0.0203-0.0108)} \\
= \quad & 1
\end{aligned}$$

where

$$\begin{aligned}\mathscr{P}^{+} =& \quad max\{\mathscr{P}_1 = 0.4898, \mathscr{P}_2 = 0.4492, \mathscr{P}_3 = 0.3041\}\\ \mathscr{P}^{-} =& \quad min\{\mathscr{P}_1 = 0.4898, \mathscr{P}_2 = 0.4492, \mathscr{P}_3 = 3041\},\\ \mathscr{R}^{+} =& \quad max\{\mathscr{P}_1 = 0.0203, \mathscr{P}_2 = 0.0202, \mathscr{P}_3 = 0.0108\}\\ \mathscr{P}^{-} =& \quad min\{\mathscr{P}_1 = 0.0203, \mathscr{P}_2 = 0.0202, \mathscr{P}_3 = 0.0108\}\end{aligned}$$

Similarlty, we have
$\mathscr{Q}_2 = 0.8900$ and $\mathscr{Q}_3 = 0$.

Step 7. Table 8 was constructed based on the values of $\mathscr{Q}^{v}(u_i)$, $\mathscr{P}_i$ and $\mathscr{R}_i$ for $u_i (\in \{1,2,3\})$;

Table 8: List of $\mathscr{P}_i$, $\mathscr{R}_i$ and $\mathscr{Q}^{v}(u_i)$ values

	$\mathscr{P}_i$	$\mathscr{R}_i$	$\mathscr{Q}_i(v=0.1)$	$\mathscr{Q}_i(v=0.2)$	$\mathscr{Q}_i(v=0.5)$	$\mathscr{Q}_i(v=0.7)$	$\mathscr{Q}_i(v=1)$
u_1	0.4898	0.0203	1	1	1	1	1
u_2	0.4493	0.0202	0.9767	0.9550	0.8900	0.8466	0.7816
u_3	0.3041	0.0108	0	0	0	0	0

Step 8. As presented in Table 9, we ranked the alternatives using the index values $\mathscr{Q}^{v}(u_i)$, group utility $\mathscr{P}i$, and individual regret $\mathscr{R}i$.

Table 9: Ranking of Alternatives

	u_1	u_2	u_3	GTHF-numbers ranking
$\mathscr{P}_i$	0.4898	0.4492	0.3041	$u_3 > u_2 > u_1$
$\mathscr{R}_i$	0.0203	0.0202	0.0108	$u_3 > u_2 > u_1$
$\mathscr{Q}_i(v=0.1)$	1	0.9767	0	$u_3 > u_2 > u_1$
$\mathscr{Q}_i(v=0.2)$	1	0.9550	0	$u_3 > u_2 > u_1$
$\mathscr{Q}_i(v=0.5)$	1	0.8900	0	$u_3 > u_2 > u_1$
$\mathscr{Q}_i(v=0.7)$	1	0.8466	0	$u_3 > u_2 > u_1$
$\mathscr{Q}_i(v=1)$	1	0.7816	0	$u_3 > u_2 > u_1$

where the alternative u_2 have the smallest index value $\mathscr{Q}^{v}(u_2)(k \in \{1,2,\ldots,m\})$ is a compromise solution since the following two conditions are satisfied for any $v \in [0,1]$ as seen in Table 10;

Table 10: Evaluation of $\mathscr{Q}^{v}(u_i)$ values

	$v=0.1$	$v=0.2$	$v=0.5$	$v=0.7$	$v=1$
$\mathscr{Q}_{u_2}$	0.9767	0.9550	0.8900	0.8466	0.7816
$\mathscr{Q}_{u_3}$	0	0	0	0	0
$\mathscr{Q}_{u_2} - \mathscr{Q}_{u_3}$	0.9767	0.9550	0.8900	0.8466	0.7816
$\mathscr{Q}^{v}(u_2) - \mathscr{Q}^{v}(u_3) \geq \frac{1}{3-1}$	0.5	0.5	0.5	0.5	0.5
1-Status	Approved	Approved	Approved	Approved	Approved
2-Status	Approved	Approved	Approved	Approved	Approved

3.5. Comparison and Analysis Discussion

In the section, Example 11 is used to compare the proposed method with the existing approaches presented in [20] and [43]. As the weight vectors are calculated based on Minkowski-type distance and entropy measures, the proposed approach helps in minimizing computational complexity. Table-11 shows that entropy measures are applicable to decision-making problems, and the results obtained align with those in [20] and [43] using the TOPSIS method under Hamming distance measures. When the distance measure applied in [43] is replaced with a generalized Minkowski distance, the results no longer match. The proposed method not only provides a solution to MCDM problems but also introduces a significant mathematical concept that offers an alternative solution approach.

Compared to existing methods, the proposed VIKOR-based decision-making process is better suited for managing uncertainty and imprecision, and it facilitates the evaluation of numerous alternatives by decision-makers. Moreover, the differences between alternatives become more distinct, thereby enhancing the significance of the results. Consequently, combining the VIKOR method with the entropy measure and the Minkowski-type distance measure yields higher accuracy in solving MCDM problems.

Table 11: The results from the several different measures

	u_1	u_2	u_3	ranking
TOPSIS via Euclidean [20]	$0,4322$	0.7590	0.6387	$u_2 > u_3 > u_1$
TOPSIS via Hamming [20]	0.4951	0.7129	0.6444	$u_2 > u_3 > u_1$
VIKOR via Hamming [43] $(v = 0.5)$	1	0	0.7090	$u_2 > u_3 > u_1$
VIKOR via Hamming [43] $(v = 0.1)$	1	0	0.9001	$u_2 > u_3 > u_1$
VIKOR via Hamming [43] $(v = 0.2)$	1	0	0.8529	$u_2 > u_3 > u_1$
VIKOR via Hamming [43] $(v = 0.7)$	1	0	0.6131	$u_2 > u_3 > u_1$
VIKOR via Hamming [43] $(v = 1)$	1	0	0.4692	$u_2 > u_3 > u_1$
Developed VIKOR (6. degree) $(v = 0.5)$	1	0.8900	0	$u_3 > u_2 > u_1$
Developed VIKOR (6. degree) $(v = 0.1)$	1	0.9767	0	$u_3 > u_2 > u_1$
Developed VIKOR (6. degree) $(v = 0.2)$	1	0.9550	0	$u_3 > u_2 > u_1$
Developed VIKOR (6. degree) $(v = 0.7)$	1	0.8466	0	$u_2 > u_3 > u_1$
Developed VIKOR (6. degree) $(v = 1)$	1	0.7816	0	$u_3 > u_2 > u_1$

In order to highlight the advantages of the proposed method, a comparative analysis is conducted against several well-known MCDM approaches, including classical VIKOR, GTHF-based TOPSIS, AHP-VIKOR, and AHP-Entropy-TOPSIS. The comparison focuses on key aspects such as uncertainty handling, weighting mechanisms, computational complexity, and decision-maker influence.

Table 12: Comparison of the Different MCDM Methods

	Proposed Method	VIKOR [31]	TOPSIS Under GTHF [43]	AHP-VIKOR [44]
Uncertainty Modeling	Yes	No	Yes	No
Criterion Weighting	Yes	No	Yes	No
Compromise Solution	Yes	Yes	No	Yes
Advantage/Disadvantage Distinction ($\mathscr{P}_i$&$\mathscr{R}_i$)	Yes	Yes	No	Yes
Parametric Control (v)	Yes	Yes	No	Yes
Criterion Type Benefit/Cost	Yes	Yes	Yes	Yes
Suitability for Complex Hierarchies	Yes	Moderate	Moderate	Moderate
Decision-Maker Influence	Moderate	Low	Moderate	High
Interpretability	Moderate	High	Moderate	High
Computational Complexity	High	Low	Moderate	Moderate
Application Flexibility	Yes	Moderate	Moderate	Moderate
Multi-Decision-Maker Compatibilitiy	Yes	No	Moderate	Yes

4. SENSITIVITY ANALYSIS OF THE PROPOSED METHOD

In order to assess the robustness and stability of the proposed decision-making method, a sensitivity analysis was performed by varying the compromise parameter v in the VIKOR algorithm. The figure illustrates how the three alternatives (u_1, u_2, and u_3) behave across different values of $v \in \{0.1, 0.2, 0.5, 0.7, 1\}$, in terms of the performance indicators $\mathscr{P}_i$, $\mathscr{R}_i$, and the aggregated ranking score $\mathscr{Q}_i$.

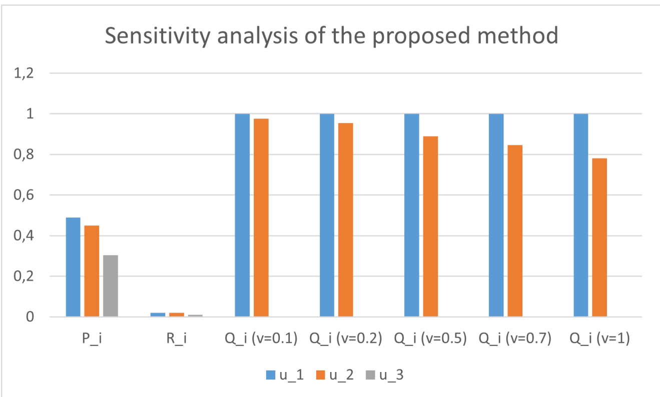

Figure 3: Sensitivity Analysis

As shown in the Figure 3, alternative u_3 consistently achieves the **lowest** $\mathscr{Q}_i$ values, indicating the best rank among all candidates regardless of the value of v. In contrast, u_1 maintains the **highest** $\mathscr{Q}_i$ scores, reflecting the weakest performance. Alternative u_2 remains in the intermediate position. This results in a stable and consistent ranking order for all levels of decision-maker preference:

Final ranking (based on $\mathscr{Q}_i$): $u_3 > u_2 > u_1$

These findings highlight the robustness of the proposed method, as the final rankings are not significantly affected by changes in the parameter v. Furthermore, they underscore the importance of using aggregated performance indicators to capture the overall behavior of alternatives, especially in decision environments involving trade-offs between group benefit and individual regret.

5. CONCLUSION

Generalized trapezoidal hesitant fuzzy numbers (GTHF-numbers) on $\mathbb{R}$ have shown strong potential in addressing numerous practical decision-making problems, as evidenced by [20, 19, 43]. GTHF-numbers are capable of capturing all relevant aspects of the decision-making process. The primary objective of this study is to develop a VIKOR-based decision-making method that incorporates an entropy measure specifically designed for GTHF-numbers. To achieve this, we first proposed an entropy measure tailored to GTHF-numbers and examined its fundamental properties. Subsequently, we introduced a VIKOR-based decision-making framework utilizing the proposed entropy measure, in which the entropy axioms for GTHF-numbers are both intuitive and computationally efficient. In the proposed method, the parametric flexibility and ease of application of the Minkowski distance measure are utilized to distinguish between alternatives more clearly and effectively. Moreover, comparative and sensitivity analyses confirm the stability of the top-ranked alternative across various scenarios, highlighting the model's practical relevance. Overall, this study systematically constructs decision matrices and assesses robotic technologies through advanced MCDM frameworks, offering a robust basis for well-informed and reliable decision-making. Future research will explore several promising directions to extend and refine this approach. Advancing ELECTRE, PROMETHEE, and AHP methods through the use of GTHF-numbers and the integration of both qualitative and quantitative information is vital for solving complex challenges such as decision-making under uncertainty, risk analysis, and personnel selection. This requires researchers to define novel concepts related to GTHF-numbers, such as centroid points, similarity measures, distance measures, and operations. Second, the proposed method can be further validated by comparing it with other methods such as ELECTRE, PROMETHEE, AHP, and TOPSIS, all applied to GTHF-numbers with similar functionalities. Third, we hope that our work will contribute to advancing research in this field.

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