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Research Article

A NOVEL PRIORITIZED AGGREGATION OPERATOR WITH TRAPEZOIDAL HESITANT INTUITIONISTIC FUZZY NUMBERS AND ITS APPLICATIONS TO SELECTION OF AN E-LEARNING PLATFORM

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Abstract: In the digital age of education, e-learning has become an integral part of the learning process. However, decision-making processes in this field involve elements of uncertainty and hesitation that need to be managed effectively. This study addresses Multi-Criteria Decision Making (MADM) problems using Trapezoidal Hesitant Fuzzy Numbers (THIFN), a powerful tool for representing uncertain and hesitant information. Two new operators are proposed for situations where feature prioritization is required: THIFN-Prioritized Weighted Average (THIFNPWA) and THIFN-Prioritized Weighted Geometric (THIFNPWG). The fundamental properties of the proposed operators, such as uniformity, boundedness, and monotonicity, are theoretically examined. The developed approach is applied to a real case study for selecting the most suitable e-learning platform. A comprehensive comparative analysis is conducted with existing methods in the literature to prove the validity of the methodology. The analysis results showed that the proposed method accurately preserved priority relationships between features and produced more consistent and discriminatory ranking results compared to existing approaches. Findings from the comparative analysis demonstrate that this method exhibits superior performance in processing prioritized and ambiguous data and offers a reliable information fusion model for hesitant decision-making environments.

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MSC: 03E72, 03E75.

1. INTRODUCTION

In contemporary society, marked by rapid development and increasing complexity, practical problems are frequently characterized by uncertainty, vagueness, and imprecision. To address these challenges, Multiple Attribute Decision Making (MADM) has emerged as a critical framework for evaluating and ranking alternatives based on conflicting criteria. In recent years, MADM methodologies have been significantly enhanced to handle complex environments, such as medical health selection [1], risk prioritization in safety models [2], and green supplier selection [3]. These modern approaches integrate advanced fuzzy structures with sophisticated decision algorithms to ensure robustness and reliability in the presence of incomplete or imprecise data. To model this uncertainty, fuzzy set theory, originally proposed by Zadeh [4], has been widely applied. While it provides a robust framework, numerous extensions have been introduced to capture different forms of imprecision more effectively. These include intuitionistic fuzzy sets [5], hyperfuzzy and superhyperfuzzy [6, 7], type-2 fuzzy sets [8], and extensions of q–fuzzy soft expert sets [9, 10]. Building on this work, Torra [11] introduced hesitant fuzzy sets (HFSs), allowing membership degrees to be represented by multiple possible values. Following this, Xu and Xia [12] proposed correlation measures, while Wei [13] introduced prioritized aggregation operators for HFSs, focusing on criteria with different importance levels. Despite these advancements, many methods fall short in capturing the full scope of original data. Consequently, Deli [14] proposed trapezoidal hesitant intuitionistic fuzzy numbers (THIFNs), and Ali et al. [15] introduced flexible Aczel-Alsina operators. These developments build upon the fundamental theorems of Atanassov [16] and the prioritized scoring methods of Yager [17]. In the domain of complex representations, Deli and Karaaslan [18] introduced generalized trapezoidal hesitant fuzzy (GTHF) numbers, later extended by Deli [19] through Bonferroni mean operators and advanced TOPSIS methods [20]. Recent trends include Archimedean Copulas with q-rung dual hesitant fuzzy sets [21] and similarity measures for linguistic term sets [22].

1.1. Literature review of related operators

As in classical set theory, aggregation operators are pivotal in synthesizing information from multiple attributes into a single representative value—an essential process in MADM. For instance, Sarfraz and Gul [1] proposed Hamacher operators for interval-valued complex T-spherical fuzzy sets, while Karamat and Sarfraz [23] utilized Aczél–Alsina t-norms for complex Pythagorean fuzzy information. Wan [24] utilized power average operators, and Aydemir et al. [25] developed Dombi prioritized operators for q-rung orthopair fuzzy sets. Further advancements include Einstein hybrid operators [26], linguistic prioritized averages [27], and intuitionistic trapezoidal fuzzy OWA operators [28].Prioritized aggregation operators have been extensively adapted to handle hierarchical criteria across diverse environments. Akram et al. [29] introduced prioritized operators for spherical fuzzy sets, while Garg and Rani [30] focused on complex intuitionistic fuzzy information. In the linguistic domain, Kumar and Chen [31] proposed the ALIFWA operator to refine group decision-making.

1.2. Research gaps and novelty of this study

Upon examining the above studies, it is evident that each method performs well under specific conditions where criteria share the same priority. However, these methods may produce undesirable decision-making outcomes when criteria have differing priority levels. Motivated by the concept of prioritized aggregation operators [32], we developed a novel approach to address MCDM problems involving THIFNs as proposed by Deli [14], specifically accommodating criteria with varying priority levels.

1.3. Main contributions of this study

The study's motivation and contributions are regarded as follows:

1. To address uncertainty and imprecision in MADM problems by utilizing THIFNs, which effectively represent hesitant and vague information.
2. To develop two novel prioritized aggregation operators:

   The THIFN-Prioritized Weighted Average (Δ_A) Operator.

   The THIFN-Prioritized Weighted Geometric (Δ_G) Operator. These operators are designed to incorporate attribute prioritization in the aggregation process under the THIF environment.,
3. To analyze the theoretical properties of the proposed operators, including idempotency, boundedness, and monotonicity, ensuring their mathematical validity and practical applicability,
4. To construct a MADM approach based on the proposed operators for solving decision problems involving prioritized attributes and uncertain data
5. To apply the proposed method to a real-world problem, specifically the selection of the most suitable e-learning platform, demonstrating the effectiveness and practicality of the approach,
6. To compare the proposed method with existing approaches, highlighting its advantages in handling prioritized, hesitant, and intuitionistic fuzzy information in complex decision-making scenarios.

1.4. Organization of the study

The remainder of this paper is organized as follows: Section 2 provides a brief overview of fundamental concepts and definitions related to THIFNs and other relevant preliminaries necessary for understanding the proposed method. Section 3 introduces the proposed THIFN-prioritized aggregation operators, including the Δ_A and Δ_G operators.

This section also presents and proves key properties of the operators, such as idempotency, boundedness, and monotonicity. Section 4 outlines the proposed decision-making approach based on the developed aggregation operators for solving MADM problems under the THIF environment. The demonstrates the applicability of the proposed method through a numerical case study, focusing on the selection of the most suitable e-learning platform. A comparison with existing methods is also provided to validate the effectiveness of the approach. Section 5 discusses the findings, limitations, and potential applications of the proposed method in various real-world domains. The concludes the study and outlines future research directions, including possible extensions to other fuzzy set theories and decision-making frameworks.

2. PRELIMINARIES

This section gives the preliminary concepts necessary for discussing the main results.

Definition 1. *[5] Let X be a universe of discourse. An intuitionistic fuzzy set (IFS) A in X is defined as follows:*

$$A = \{\langle x; \mu_A(x), \nu_A(x)\rangle : x \in X\} \tag{1}$$

where $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *represent the membership degree and non-membership degree of the element* $x \in X$ *with respect to the set A, respectively. These functions satisfy the condition:*

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \quad \text{for all } x \in X.$$

Definition 2. *[33] Let* $\tilde{a}$ *is a trapezoidal intuitionistic fuzzy number, its membership function and non-membership function as given, respectively.*

$$\mu_{\tilde{a}}(x) = \begin{cases} (x-a)\eta_{\tilde{a}}/(b-a) & a \le x \le b \\ \eta_{\tilde{a}} & b \le x \le c \\ (d-x)\eta_{\tilde{a}}/(d-c) & c \le x \le d \\ 0 & otherwise \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} (b-x)+\psi_{\tilde{a}}(x-a)/(b-a) & a \le x \le b \\ \psi_{\tilde{a}} & b \le x \le c \\ (x-c)\psi_{\tilde{a}}(d-x)/(d-c) & c \le x \le d \\ 1 & otherwise \end{cases}$$

where $0 \le \mu_A(x) \le 1$*;* $0 \le \nu_A(x) \le 1$*;* $0 \le \mu_A(x) + \nu_A(x) \le 1$*;* $a,b,c,d \in R$ *such that* $a \le b \le c \le d$*. Then, a trapezoidal intuitionistic fuzzy number (TIF-number)* $\tilde{a} = \langle([a,b,c,d];\mu_{\tilde{a}}),([a,b,c,d];\nu_{\tilde{a}})\rangle$ *is called trapezoidal intuitionistic fuzzy number. For convenience, let* $\tilde{a} = \langle[a,b,c,d];\mu_{\tilde{a}},\nu_{\tilde{a}}\rangle$*.*

Definition 3. *[34] Let* $\tilde{a} = \langle(a,b,c,d);\eta_{\tilde{a}}\rangle$ *be a TF-number with it's membership function* $\mu_{\tilde{a}}(x)$*. Centroid point of* $\tilde{a}$*, denoted by* $\tilde{a^*}$*, computed as;*

$$\tilde{a^*} = \frac{\int x.\mu_{\tilde{a}}(x)dx}{\int \mu_{\tilde{a}}(x)dx} = \frac{\eta_{\tilde{a}}(d^2-2c^2+2b^2-a^2+dc-ab)+3(c^2-b^2)}{3\eta_{\tilde{a}}(d-c+b-a)+6(c-b)}$$

Definition 4. *[35] Let X be a universe. Then, a hesitant fuzzy set(HFS), is denoted by H, is defined as:*

$$H = \{\langle x, \xi(x)\rangle\rangle : x \in X\} \tag{2}$$

where $\xi(x)$ is set of some values in $[0,1]$ and $\xi = \xi(x)$ is called a hesitant fuzzy element(HFE).

Definition 5. *[17] Let $G = \{G_j : j \in \{1,2,\ldots,n\}\}$ be a set of attributes, where a prioritization exists among them, expressed by the linear ordering $G_1 \succ G_2 \succ G_3 \succ \ldots \succ G_n$. This means that attribute G_j has a higher priority than attribute G_k if $j < k$.*

For any alternative x, the value $G_j(x) \in [0,1]$ represents its performance under attribute G_j. The prioritized average operator, denoted as $PA(G_1(x), G_2(x), \ldots, G_n(x))$, is defined by:

$$PA(G_1(x), G_2(x), \ldots, G_n(x)) = \sum_{j=1}^{n} w_j G_j(x)$$

where the weights w_j are given by:

$$w_j = \frac{T_j}{\sum_{i=1}^{n} T_i}, \quad with \quad T_j = \prod_{k=1}^{j-1} G_k(x), \quad and \quad T_1 = 1.$$

Definition 6. *[13] Let $\xi^j(j \in \{1,2,...,n\})$ be a collection of HFEs. Then,*

1. *the hesitant fuzzy prioritized weighted average operator of collection $\xi^j(j \in \{1,2,...,n\})$ is defined as:*

$$\begin{aligned} HFPWAO(\xi^1, \xi^2, ..., \xi^n) = & \quad \bigoplus_{j=1}^{n} \left(\frac{T_j}{\sum_{i=1}^{n} T_i} \xi^j \right) \\ = & \quad \bigcup_{h_1^1 \in \xi^1, h_1^2 \in \xi^2, ..., h_1^n \in \xi^n} \{1 - \prod_{j=1}^{n} (1 - h_1^j)^{\frac{T_j}{\sum_{i=1}^{n} T_i}}\} \end{aligned} \tag{3}$$

where $T_i = \prod_{k=1}^{i-1} \sum_{h_1^k \in \xi^k} h_1^k, T_j = \prod_{k=1}^{j-1} \sum_{h_1^k \in \xi^k} h_1^k (i,j = 2,...,n)$, $T_1 = 1$.

2. *the hesitant fuzzy prioritized weighted geometric operator of collection $\xi^j(j \in \{1,2,...,n\})$ is defined as:*

$$\begin{aligned} HFPWGO(\xi^1, \xi^2, ..., \xi^n) = & \quad \bigotimes_{j=1}^{n} (\xi^j)^{\frac{T_j}{\sum_{i=1}^{n} T_i}} \\ = & \quad \bigcup_{h_1^1 \in \xi^1, h_1^2 \in \xi^2, ..., h_1^n \in \xi^n} \{\prod_{j=1}^{n} (\xi^n)^{\frac{T_j}{\sum_{j=1}^{n} T_j}}\} \end{aligned} \tag{4}$$

where $T_i = \prod_{k=1}^{i-1} \sum_{h_1^k \in \xi^k} h_1^k, T_j = \prod_{k=1}^{j-1} \sum_{h_1^k \in \xi^k} h_1^k (i,j = 2,...,n)$, $T_1 = 1$.

Theorem 7. *[13] Let ξ^j and $\acute{\xi}^j(j \in \{1,2,...,n\})$ be two collection of HFEs.*

1. *If all* $\xi^j (j = 1,2,...,n)$ *are equal, i.e.* $\xi^j = \xi$ *for all* $j \in \{1,2,...,n\}$ *then,*

$$HFPWGO(\xi^1,\xi^2,...,\xi^n) = HFPWAO(\xi^1,\xi^2,...,\xi^n) = \xi \tag{5}$$

2. *Let* $\xi^- = \min_{j\in\{1,2,...,n\}}\{\xi^j\}$, $\xi^+ = \max_{j\in\{1,2,...,n\}}\{\xi^j\}$ *then*

$$\xi^- \leq HFPWAO(\xi^1,\xi^2,...,\xi^n), HFPWGO(\xi^1,\xi^2,...,\xi^n) \leq \xi^+ \tag{6}$$

3. *If* $\xi_i \leq \acute{\xi}_j$ *for all* $j \in \{1,2,...,n\}$, *then*

$$HFPWA(\xi^1,\xi^2,...,\xi^n) \leq HFPWA(\acute{\xi}^1,\acute{\xi}^2,...,\acute{\xi}^n) \tag{7}$$

$$HFPWA(\xi^1,\xi^2,...,\xi^n) \leq HFPWA(\acute{\xi}^1,\acute{\xi}^2,...,\acute{\xi}^n) \tag{8}$$

Definition 8. *[14] Let* $\mathbb{R}$ *be a set of real numbers such that* $a \leq b \leq c \leq d$ *with* $i \in \{1,2,3,...,n\} \vee \in \{1,2,3,...,m\} \vee ...$. *Then, a THIF-number, is denoted by* $\hbar$, *is defined as;*

$$\hbar = \langle (a,b,c,d); \{(\alpha_i,\beta_i) \in [0,1]\times[0,1]\}\rangle$$

is a special hesitant intuitionistic fuzzy set on the real number set $\mathbb{R}$, *whose membership functions and non-membership functions as given, respectively.*

$$\mu^i(x) = \begin{cases} (x-a)\alpha_i/(b-a) & a \leq x < b \\ \alpha_i & b \leq x \leq c \\ (d-x)\alpha_i/(d-c) & c < x \leq d \\ 0 & otherwise, \end{cases}$$

and

$$\nu^i(x) = \begin{cases} (b-x)+\beta_i(x-a)/(b-a) & a \leq x < b \\ \beta_i & b \leq x \leq c \\ (x-c)\beta_i(d-x)/(d-c) & c < x \leq d \\ 1 & otherwise, \end{cases}$$

For convenience, we will denote the THIF-number with

$$\hbar = \langle (a,b,c,d); (\alpha,\beta)\rangle.$$

In the paper, for focusing on THIF- numbers, note that the set of all THIF-number on $\mathbb{R}^+$ will be denoted by ϕ.

Definition 9. *[14] Let* $\hbar = \langle (a,b,c,d);(\alpha,\beta)\rangle, \hbar^1 = \langle (a_1,b_1,c_1,d_1);(\alpha^1,\beta^1)\rangle$, $\hbar^2 = \langle (a_2,b_2,c_2,d_2);(\alpha^2,\beta^2)\rangle \in \phi$ *and* $\gamma \neq 0$ *be any real number. Then,*

1. $\hbar_1 \oplus \hbar_2 = \langle (a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2); \cup_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2)}\{(\alpha_1^1+\alpha_1^2-\alpha_1^1.\alpha_1^2, \beta_1^1.\beta_1^2)\}\rangle$;

2. $$\hbar^1 \odot \hbar^2 = \begin{cases} \langle (a_1a_2,b_1b_2,c_1c_2,d_1d_2); \cup_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2)}\{(\alpha_1^1.\alpha_1^2, \beta_1^1+\beta_1^2-\beta_1^1.\beta_1^2)\}\rangle (d_1>0,d_2>0); \\ \langle (a_1d_2,b_1c_2,c_1b_2,d_1a_2); \cup_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2)}\{(\alpha_1^1.\alpha_1^2, \beta_1^1+\beta_1^2-\beta_1^1.\beta_1^2)\}\rangle (d_1<0,d_2>0); \\ \langle (d_1d_2,c_1c_2,b_1b_2,a_1a_2); \cup_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2)}\{(\alpha_1^1.\alpha_1^2, \beta_1^1+\beta_1^2-\beta_1^1.\beta_1^2)\}\rangle (d_1<0,d_2<0). \end{cases}$$

3. $\gamma\hbar = \begin{cases} \langle(\gamma a, \gamma b, \gamma c, \gamma d); \cup_{(\alpha_1,\beta_1)\in(\alpha,\beta),}\{1-(1-\alpha_1)^\gamma, \beta_1^\gamma\}\rangle(\gamma > 0); \\ \langle(\gamma d, \gamma c, \gamma b, \gamma a); \cup_{(\alpha_1,\beta_1)\in(\alpha,\beta)}\{1-(1-\alpha_1)^\gamma, \beta_1^\gamma\}\rangle(\gamma < 0). \end{cases}$

4. $(\hbar)^\gamma = \langle(a^\gamma, b^\gamma, c^\gamma, d^\gamma); \cup_{(\alpha_1,\beta_1)\in(\alpha,\beta)}\{\alpha_1^\gamma, 1-(1-\beta_1)^\gamma\}\rangle(\gamma \geq 0)$.

Definition 10. *[14] Let* $\hbar_j = \langle(a_j, b_j, c_j, d_j); (\alpha^j, \beta^j)\rangle$, $j \in I_n$ *be a collection of THIF-numbers. Then,*

1. *the THIF-numbers weighted average operator of collection* $\hbar_j (j \in \{1,2,...,n\})$ *is defined as:*

$$\begin{aligned} H_w^A(\hbar^1, \hbar^2, ..., \hbar^n) &= \bigoplus_{j=1}^n w_j \hbar^j \\ &= \langle(\sum_{j=1}^n w_j.a_j, \sum_{j=1}^n w_j.b_j, \sum_{j=1}^n w_j.c_j, \sum_{j=1}^n w_j.d_j); \\ &\quad \cup_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)}\{1-\prod_{j=1}^n(1-\alpha_1^j)^{w_j}, \prod_{j=1}^n(\beta_1^j)^{w_j}\}\rangle \end{aligned}$$

where $w = (w_1, w_2, ..., w_n)^T$ *is the weight vector of* $\hbar^j$, $j \in I_n$ *such that* $w_j \in [0,1]$ *and* $\sum_{j=1}^n w_j = 1$.

2. *the THIF-numbers weighted geometric operator of collection* $\hbar_j (j \in \{1,2,...,n\})$ *is defined as:*

$$\begin{aligned} H_w^G(\hbar_1, \hbar_2, ..., \hbar_n) &= \bigotimes_{j=1}^n \hbar_j^{w_j} \\ &= \langle(\prod_{j=1}^n a_j^{w_j}, \prod_{j=1}^n b_j^{w_j}, \prod_{j=1}^n c_j^{w_j}, \prod_{j=1}^n d_j^{w_j}); \\ &\quad \cup_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)}\{\prod_{j=1}^n(\alpha_1^j)^{w_j}, 1-\prod_{j=1}^n(1-\beta_1^j)^{w_j}\}\rangle \end{aligned}$$

3. AVERAGING/GEOMETRIC PRIORITIZED AGGREGATION OPERATORS

In this section, we investigate the prioritized aggregation averaging and geometric operators within the framework of THIFNs. We conducted a detailed examination of their basic properties such that as idempotency, boundedness and monotonicity.

3.1. THIFN-prioritized average operator

In this section, we introduce a weighted averaging prioritized aggregation operator, referred to as the Δ_A operator, designed for collections of THIF-numbers.

Definition 11. *Let* $\kappa_j = \langle(a_j, b_j, c_j, d_j); (\alpha^j, \beta^j)\rangle$, $j \in I_n$, *be a collection of THIFNs defined on* $[0,1] \subseteq \mathbb{R}$. *Let* $G = \{G_j : j \in \{1,2,\ldots,n\}\}$ *be a set of attributes with a prioritization as given in Definition 5. Then, the THIFN-prioritized average operator is denoted by* $\Delta_A(\kappa_1, \kappa_2, ..., \kappa_n)$, *is defined by*

$$\Delta_A(\kappa_1, \kappa_2, ..., \kappa_n) = \bigoplus_{j=1}^n \left(\frac{T_j}{\sum_{i=1}^n T_i} \kappa_j\right) \tag{9}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (j = 2, ..., n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (i = 2, ..., n).$$

Theorem 12. *Let $\kappa_j = \langle (a_j, b_j, c_j, d_j); (\alpha^j, \beta^j) \rangle$, $j = 1, 2, \ldots, n$, be a collection of THIF-numbers defined on $[0,1] \subseteq \mathbb{R}$. Let $G = \{G_j : j \in \{1, 2, \ldots, n\}\}$ be a set of attributes with a prioritization as given in Definition 5. Then, their aggregated value using the operator $\Delta_A(\kappa_1, \kappa_2, \ldots, \kappa_n)$ is also a THIF-number, and*

$$\begin{aligned}\Delta_A(\kappa_1, \kappa_2, ..., \kappa_n) = &\ \bigoplus_{j=1}^{n} \left(\frac{T_j \kappa_j}{\sum_{i=1}^{n} T_i} \right) \\ = &\ \langle (\textstyle\sum_{j=1}^{n} \frac{T_j}{\sum_{i=1}^{n} T_i} a_j, \sum_{j=1}^{n} \frac{T_j}{\sum_{i=1}^{n} T_i} b_j, \sum_{j=1}^{n} \frac{T_j}{\sum_{i=1}^{n} T_i} c_j, \sum_{j=1}^{n} \frac{T_j}{\sum_{i=1}^{n} T_i} d_j); \\ &\ \bigcup_{(\alpha_1^1, \beta_1^1) \in (\alpha^1, \beta^1), (\alpha_1^2, \beta_1^2) \in (\alpha^2, \beta^2), \ldots, (\alpha_1^n, \beta_1^n) \in (\alpha^n, \beta^n)} \{1 - \textstyle\prod_{j=1}^{n} (1 - \alpha_1^j)^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, \\ &\ \textstyle\prod_{j=1}^{n} (\beta_1^j)^{\frac{T_j}{\sum_{i=1}^{n} T_i}} \} \rangle. \end{aligned} \tag{10}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\alpha_1^1, \beta_1^1) \in (\alpha^1, \beta^1), (\alpha_1^2, \beta_1^2) \in (\alpha^2, \beta^2), \ldots, (\alpha_1^n, \beta_1^n) \in (\alpha^n, \beta^n)} \frac{(\alpha_1^k - \beta_1^k)(d_k^2 - 2c_k^2 + 2b_k^2 - a_k^2 + d_k c_k - a_k b_k) + 3(c_k^2 - b_k^2)}{3(\alpha_1^k - \beta_1^k)(d_k - c_k + b_k - a_k) + 6(c_k - b_k)} \, (j = 2, ..., n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\alpha_1^1, \beta_1^1) \in (\alpha^1, \beta^1), (\alpha_1^2, \beta_1^2) \in (\alpha^2, \beta^2), \ldots, (\alpha_1^n, \beta_1^n) \in (\alpha^n, \beta^n)} \frac{(\alpha_1^k - \beta_1^k)(d_k^2 - 2c_k^2 + 2b_k^2 - a_k^2 + d_k c_k - a_k b_k) + 3(c_k^2 - b_k^2)}{3(\alpha_1^k - \beta_1^k)(d_k - c_k + b_k - a_k) + 6(c_k - b_k)} \, (i = 2, ..., n).$$

In order to define Δ_A as a valid aggregation operator, it must satisfy the essential aggregation properties of idempotency, boundedness, and monotonicity. These properties are formally verified through Theorems 3.3, 3.5, and 3.7, respectively.

Theorem 13. ***(Idempotency)*** *Let $\kappa_j = \langle (a_j, b_j, c_j, d_j); (\alpha^j, \beta^j) \rangle$, $j = 1, 2, \ldots, n$, be a collection of THIF-numbers on $[0,1] \subseteq \mathbb{R}$, and let $G = \{G_j : j \in \{1, 2, \ldots, n\}\}$ be a set of attributes with a prioritization as given in Definition 5.*

If $\kappa_j = \kappa = \langle (a, b, c, d); (\alpha^j, \beta^j) \rangle$ for all $j \in \{1, 2, \ldots, n\}$, then

$$\Delta_A(\kappa_1, \kappa_2, ..., \kappa_n) = \kappa \tag{11}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\alpha_1^1, \beta_1^1) \in (\alpha^1, \beta^1), (\alpha_1^2, \beta_1^2) \in (\alpha^2, \beta^2), \ldots, (\alpha_1^n, \beta_1^n) \in (\alpha^n, \beta^n)} \frac{(\alpha_1^k - \beta_1^k)(d_k^2 - 2c_k^2 + 2b_k^2 - a_k^2 + d_k c_k - a_k b_k) + 3(c_k^2 - b_k^2)}{3(\alpha_1^k - \beta_1^k)(d_k - c_k + b_k - a_k) + 6(c_k - b_k)} \, (j = 2, ..., n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\alpha_1^1, \beta_1^1) \in (\alpha^1, \beta^1), (\alpha_1^2, \beta_1^2) \in (\alpha^2, \beta^2), \ldots, (\alpha_1^n, \beta_1^n) \in (\alpha^n, \beta^n)} \frac{(\alpha_1^k - \beta_1^k)(d_k^2 - 2c_k^2 + 2b_k^2 - a_k^2 + d_k c_k - a_k b_k) + 3(c_k^2 - b_k^2)}{3(\alpha_1^k - \beta_1^k)(d_k - c_k + b_k - a_k) + 6(c_k - b_k)} \, (i = 2, ..., n).$$

Proof. For $\kappa_j = \kappa$ for all $j \in \{1,2,...,n\}$, and by definition of Δ_A operator, we have

$$\begin{aligned}
\Delta_A(\kappa_1,\kappa_2,...,\kappa_n) =& \quad \bigoplus_{j=1}^{n}\left(\frac{T_j\kappa_j}{\sum_{i=1}^{n}T_i}\right)\\
=& \quad \langle(\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}a_j,\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}b_j,\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}c_j,\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}d_j);\\
& \quad \bigcup_{h_1^1\in\xi^1,h_1^2\in\xi^2,...,h_1^n\in\xi^n}\{1-\prod_{j=1}^{n}(1-h_1^j)^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\}\rangle\\
=& \quad \langle(\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}a,\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}b,\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}c,\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}d);\\
& \quad \bigcup_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)}\{1-\prod_{j=1}^{n}(1-\alpha_1^j)^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\\
& \quad \prod_{j=1}^{n}(\beta_1^j)^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\}\rangle(\sum_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}=1)\\
=& \quad \langle(a,b,c,d);(\alpha^j,\beta^j)\rangle(j=1,2,...,n)\}\rangle\\
=& \quad \kappa
\end{aligned} \tag{12}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1}\sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)}(j=2,...,n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1}\sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)}(i=2,...,n).$$

With this, the proof is finished. □

Theorem 14. ***(Boundedness)*** *Let $\kappa_j = \langle(a_j,b_j,c_j,d_j);(\alpha^j,\beta^j)\rangle$, $j = 1,2,\ldots,n$, be a collection of THIF-numbers defined on $[0,1] \subseteq \mathbb{R}$, and let $G = \{G_j : j \in \{1,2,\ldots,n\}\}$ be a set of attributes with a prioritization as given in Definition 5. Let*

$$\kappa^- = \langle(\min_{j\in\{1,2,...,n\}}\{a_j\},\min_{j\in\{1,2,...,n\}}\{b_j\},\min_{j\in\{1,2,...,n\}}\{c_j\},\min_{j\in\{1,2,...,n\}}\{d_j\});\min_{j\in\{1,2,...,n\}}\{\min_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)}\{(\alpha_1^j,\beta_1^j)\}\}\rangle$$

and

$$\kappa^+ = \langle(\max_{j\in\{1,2,...,n\}}\{a_j\},\max_{j\in\{1,2,...,n\}}\{b_j\},\max_{j\in\{1,2,...,n\}}\{c_j\},\max_{j\in\{1,2,...,n\}}\{d_j\});\max_{j\in\{1,2,...,n\}}\{\max_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)}\{(\alpha_1^j,\beta_1^j)\}\}\rangle$$

If

"$a_j \le a_i, b_j \le b_i, c_j \le c_i, d_j \le d_i$ and $\alpha_1^j \le \alpha_1^iand\beta_1^j \ge \beta_1^i$ for all $\alpha_1^j \in \alpha^j, \alpha_1^i \in \alpha^i$ and $\beta_1^j \in \beta^j, \beta_1^i \in \beta^i$" $\Rightarrow$ "$\kappa_j = \langle(a_j,b_j,c_j,d_j);(\alpha^j,\beta^j)\rangle \le \kappa_i = \langle(a_i,b_i,c_i,d_i);(\alpha^i,\beta^i)\rangle$"

then

$$\kappa^- \le \Delta_A(\kappa_1,\kappa_2,...,\kappa_n) \le \kappa^+. \tag{13}$$

Proof. We have $\frac{\sum_{j=1}^{n} T_j}{\sum_{i=1}^{n} T_i} = 1$, since

$$\min_{j\in\{1,2,...,n\}}\{a_j\} \le \{a_j\} \le \max_{j\in\{1,2,...,n\}}\{a_j\}$$

$$\min_{j\in\{1,2,...,n\}}\{b_j\} \le \{b_j\} \le \max_{j\in\{1,2,...,n\}}\{b_j\}$$

$$\min_{j\in\{1,2,...,n\}}\{c_j\} \le \{c_j\} \le \max_{j\in\{1,2,...,n\}}\{c_j\}$$

$$\min_{j\in\{1,2,...,n\}}\{d_j\} \le \{d_j\} \le \max_{j\in\{1,2,...,n\}}\{d_j\}$$

$$\{\min_{(\alpha_1^1)\in(\alpha^1),(\alpha_1^2)\in(\alpha^2),...,(\alpha_1^n)\in(\alpha^n)}\{(\alpha_1^j)\}\} \le \alpha^j \le \{\max_{(\alpha_1^1)\in(\alpha^1),(\alpha_1^2)\in(\alpha^2,),...,(\alpha_1^n)\in(\alpha^n)}\{(\alpha_1^j)\}\}.$$

and

$$\{\max_{(\beta_1^1)\in(\beta^1),(\beta_1^2)\in(\beta^2),...,(\beta_1^n)\in(\beta^n)}\{(\beta_1^j)\}\} \ge \beta^j \ge \{\min_{(\beta_1^1)\in(\beta^1),(\beta_1^2)\in(\beta^2),...,(\beta_1^n)\in(\beta^n)}\{(\beta_1^j)\}\}.$$

Let Δ_A operator. Then by the Theorem 3.2, it yields that

$$\kappa^- \le \Delta_A(\kappa_1,\kappa_2,...,\kappa_n) \le \kappa^+.$$

□

Theorem 15. ***(Monotonicity)*** *Let $\kappa_j = \langle(a_j,b_j,c_j,d_j);(\alpha^j,\beta^j)\rangle$ and $\acute{\kappa}_j = \langle(\acute{a}_j,\acute{b}_j,\acute{c}_j,\acute{d}_j);(\acute{\alpha}^j,\acute{\beta}^j)\rangle$, for $j = 1,2,\dots,n$, be two collections of THIF-numbers defined on $[0,1]\subseteq\mathbb{R}$. Let $G = \{G_j : j\in\{1,2,\dots,n\}\}$ be a set of attributes with prioritization as in Definition 5.*

If $\kappa_j \le \acute{\kappa}_j$ for all $j\in\{1,2,\dots,n\}$, based on Equation 19, then

$$\Delta_A(\kappa_1,\kappa_2,...,\kappa_n) \le \Delta_A(\acute{\kappa}_1,\acute{\kappa}_2,...,\acute{\kappa}_n). \tag{14}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1}\sum_{(\acute{\alpha}_1^1,\acute{\beta}_1^1)\in(\acute{\alpha}^1,\acute{\beta}^1),(\acute{\alpha}_1^2,\acute{\beta}_1^2)\in(\acute{\alpha}^2,\acute{\beta}^2),...,(\acute{\alpha}_1^n,\acute{\beta}_1^n)\in(\acute{\alpha}^n,\acute{\beta}^n)} \frac{(\acute{\alpha}_1^k-\acute{\beta}_1^k)(\acute{d}_k^2-2\acute{c}_k^2+2\acute{b}_k^2-\acute{a}_k^2+\acute{d}_k\acute{c}_k-\acute{a}_k\acute{b}_k)+3(\acute{c}_k^2-\acute{b}_k^2)}{3(\acute{\alpha}_1^k-\acute{\beta}_1^k)(\acute{d}_k-\acute{c}_k+\acute{b}_k-\acute{a}_k)+6(\acute{c}_k-\acute{b}_k)} (j=2,...,n)$$

$$T_1 = 1, T_i = \prod_{k=1}^{i-1}\sum_{(\acute{\alpha}_1^1,\acute{\beta}_1^1)\in(\acute{\alpha}^1,\acute{\beta}^1),(\acute{\alpha}_1^2,\acute{\beta}_1^2)\in(\acute{\alpha}^2,\acute{\beta}^2),...,(\acute{\alpha}_1^n,\acute{\beta}_1^n)\in(\acute{\alpha}^n,\acute{\beta}^n)} \frac{(\acute{\alpha}_1^k-\acute{\beta}_1^k)(\acute{d}_k^2-2\acute{c}_k^2+2\acute{b}_k^2-\acute{a}_k^2+\acute{d}_k\acute{c}_k-\acute{a}_k\acute{b}_k)+3(\acute{c}_k^2-\acute{b}_k^2)}{3(\acute{\alpha}_1^k-\acute{\beta}_1^k)(\acute{d}_k-\acute{c}_k+\acute{b}_k-\acute{a}_k)+6(\acute{c}_k-\acute{b}_k)} (i=2,...,n).$$

$$T_1 = 1, T_j = \prod_{k=1}^{j-1}\sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (j=2,...,n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1}\sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (i=2,...,n).$$

3.2. THIFN-prioritized geometric operator

In this section, we introduce a weighted averaging prioritized geometric operator, referred to as the Δ_G operator, designed for collections of THIF-numbers.

Definition 16. *Let $\kappa_j = \langle (a_j, b_j, c_j, d_j); (\alpha^j, \beta^j) \rangle$, $j = 1, 2, \ldots, n$, be a collection of THIF-numbers defined on $[0,1] \subseteq \mathbb{R}$. Let $G = \{G_j : j \in \{1, 2, \ldots, n\}\}$ be a set of attributes with a prioritization as described in Definition 5. Then, the THIFN-prioritized geometric operator, denoted by $\Delta_G(\kappa_1, \kappa_2, \ldots, \kappa_n)$, is defined as*

$$\Delta_G(\kappa_1, \kappa_2, ..., \kappa_n) = \kappa_1^{\frac{T_1}{\sum_{i=1}^n T_i}} \otimes \kappa_2^{\frac{T_2}{\sum_{i=1}^n T_i}} \otimes ... \otimes \kappa_n^{\frac{T_n}{\sum_{i=1}^n T_i}} \tag{15}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (j = 2, ..., n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (i = 2, ..., n).$$

Theorem 17. *Let $\kappa_j = \langle (a_j, b_j, c_j, d_j); (\alpha^j, \beta^j) \rangle$, $j = 1, 2, \ldots, n$, be a collection of THIF-numbers defined on $[0,1] \subseteq \mathbb{R}$. Let $G = \{G_j : j \in \{1, 2, \ldots, n\}\}$ be a set of attributes with a prioritization as specified in Definition 5. Then, the aggregated value obtained by applying the $\Delta_G(\kappa_1, \kappa_2, \ldots, \kappa_n)$ operator is also a THIF-number, and*

$$\begin{aligned}
\Delta_G(\kappa_1, \kappa_2, ..., \kappa_n) &= \kappa_1^{\frac{T_1}{\sum_{i=1}^n T_i}} \bigotimes \kappa_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \bigotimes ... \bigotimes \kappa_n^{\frac{T_n}{\sum_{j=1}^n T_j}} \\
&= \langle (a_j^{\frac{T_j}{\sum_{j=1}^n T_j}}, b_j^{\frac{T_j}{\sum_{j=1}^n T_j}}, c_j^{\frac{T_j}{\sum_{j=1}^n T_j}}, d_j^{\frac{T_j}{\sum_{j=1}^n T_j}}); \\
& \bigcup_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \{\prod_{j=1}^n (\alpha_1^j)^{\frac{T_j}{\sum_{i=1}^n T_i}}, \\
& 1 - \prod_{j=1}^n (1-\beta_1^j)^{\frac{T_j}{\sum_{i=1}^n T_i}}\} \rangle.
\end{aligned} \tag{16}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (j = 2, ..., n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (i = 2, ..., n).$$

It is straightforward to verify that the Δ_G operator possesses the following properties.

Theorem 18. ***(Idempotency)*** *Let $\kappa_j = \langle (a_j, b_j, c_j, d_j); (\alpha^j, \beta^j) \rangle$, $j = 1, 2, \ldots, n$, be a collection of THIF-numbers defined on $[0,1] \subseteq \mathbb{R}$, and let $G = \{G_j : j \in \{1, 2, \ldots, n\}\}$ be a set of attributes with a prioritization as described in Definition 5.*

If $\kappa_j = \kappa$ for all $j \in \{1, 2, \ldots, n\}$, then the aggregated value satisfies:

$$\Delta_G(\kappa_1, \kappa_2, ..., \kappa_n) = \kappa \tag{17}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (j = 2, ..., n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (i = 2, ..., n).$$

Theorem 19. ***(Boundedness)*** *Let $\kappa_j = \langle (a_j, b_j, c_j, d_j); (\alpha^j, \beta^j) \rangle$, $j = 1, 2, \ldots, n$, be a collection of THIF-numbers defined on $[0,1] \subseteq \mathbb{R}$. Let $G = \{G_j : j \in \{1, 2, \ldots, n\}\}$ be a set of attributes with a prioritization as described in Definition 5. Let*

$$\kappa^- = \langle (\min_{j\in\{1,2,...,n\}}\{a_j\}, \min_{j\in\{1,2,...,n\}}\{b_j\}, \min_{j\in\{1,2,...,n\}}\{c_j\}, \min_{j\in\{1,2,...,n\}}\{d_j\}); \min_{j\in\{1,2,...,n\}} \{ \min_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \{(\alpha_1^j, \beta_1^j)\}\}\rangle$$

and

$$\kappa^+ = \langle (\max_{j\in\{1,2,...,n\}}\{a_j\}, \max_{j\in\{1,2,...,n\}}\{b_j\}, \max_{j\in\{1,2,...,n\}}\{c_j\}, \max_{j\in\{1,2,...,n\}}\{d_j\}); \max_{j\in\{1,2,...,n\}} \{ \max_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \{(\alpha_1^j, \beta_1^j)\}\}\rangle$$

If

"$a_j \leq a_i, b_j \leq b_i, c_j \leq c_i, d_j \leq d_i$ and $\alpha_1^j \leq \alpha_1^i$ and $\beta_1^j \geq \beta_1^i$ for all $\alpha_1^j \in \alpha^j, \alpha_1^i \in \alpha^i$ and $\beta_1^j \in \beta^j, \beta_1^i \in \beta^i$" $\Rightarrow$ "$\kappa_j = \langle (a_j, b_j, c_j, d_j); (\alpha^j, \beta^j) \rangle \leq \kappa_i = \langle (a_i, b_i, c_i, d_i); (\alpha^i, \beta^i) \rangle$"

then

$$\kappa^- \leq \Delta_G(\kappa_1, \kappa_2, ..., \kappa_n) \leq \kappa^+. \tag{18}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (j = 2, ..., n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),...,(\alpha_1^n,\beta_1^n)\in(\alpha^n,\beta^n)} \frac{(\alpha_1^k-\beta_1^k)(d_k^2-2c_k^2+2b_k^2-a_k^2+d_kc_k-a_kb_k)+3(c_k^2-b_k^2)}{3(\alpha_1^k-\beta_1^k)(d_k-c_k+b_k-a_k)+6(c_k-b_k)} (i = 2, ..., n).$$

Theorem 20. ***(Monotonicity)*** *Let* $\kappa_j = \langle (a_j, b_j, c_j, d_j); (\alpha^j, \beta^j) \rangle$ *and* $\acute{\kappa}_j = \langle (\acute{a}_j, \acute{b}_j, \acute{c}_j, \acute{d}_j); (\acute{\alpha}^j, \acute{\beta}^j) \rangle$*, for* $j = 1, 2, \ldots, n$*, be two collections of THIF-numbers on* $[0,1] \subseteq \mathbb{R}$*. Let* $G = \{G_j : j \in \{1, 2, \ldots, n\}\}$ *be a set of attributes with a prioritization as described in Definition 5.*

If $\kappa_j \leq \acute{\kappa}_j$ *for all* $j \in \{1, 2, \ldots, n\}$*, based on Equation 19, then the prioritized geometric aggregation satisfies:*

$$\Delta_G(\kappa_1, \kappa_2, ..., \kappa_n) \leq \Delta_G(\acute{\kappa}_1, \acute{\kappa}_2, ..., \acute{\kappa}_n). \tag{19}$$

where

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\acute{\alpha}_1^1, \acute{\beta}_1^1) \in (\acute{\alpha}^1, \acute{\beta}^1), (\acute{\alpha}_1^2, \acute{\beta}_1^2) \in (\acute{\alpha}^2, \acute{\beta}^2), \ldots, (\acute{\alpha}_1^n, \acute{\beta}_1^n) \in (\acute{\alpha}^n, \acute{\beta}^n)} \frac{(\acute{\alpha}_1^k - \acute{\beta}_1^k)(\acute{d}_k^2 - 2\acute{c}_k^2 + 2\acute{b}_k^2 - \acute{a}_k^2 + \acute{d}_k\acute{c}_k - \acute{a}_k\acute{b}_k) + 3(\acute{c}_k^2 - \acute{b}_k^2)}{3(\acute{\alpha}_1^k - \acute{\beta}_1^k)(\acute{d}_k - \acute{c}_k + \acute{b}_k - \acute{a}_k) + 6(\acute{c}_k - \acute{b}_k)} (j = 2, ..., n)$$

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\acute{\alpha}_1^1, \acute{\beta}_1^1) \in (\acute{\alpha}^1, \acute{\beta}^1), (\acute{\alpha}_1^2, \acute{\beta}_1^2) \in (\acute{\alpha}^2, \acute{\beta}^2), \ldots, (\acute{\alpha}_1^n, \acute{\beta}_1^n) \in (\acute{\alpha}^n, \acute{\beta}^n)} \frac{(\acute{\alpha}_1^k - \acute{\beta}_1^k)(\acute{d}_k^2 - 2\acute{c}_k^2 + 2\acute{b}_k^2 - \acute{a}_k^2 + \acute{d}_k\acute{c}_k - \acute{a}_k\acute{b}_k) + 3(\acute{c}_k^2 - \acute{b}_k^2)}{3(\acute{\alpha}_1^k - \acute{\beta}_1^k)(\acute{d}_k - \acute{c}_k + \acute{b}_k - \acute{a}_k) + 6(\acute{c}_k - \acute{b}_k)} (i = 2, ..., n).$$

$$T_1 = 1, T_j = \prod_{k=1}^{j-1} \sum_{(\alpha_1^1, \beta_1^1) \in (\alpha^1, \beta^1), (\alpha_1^2, \beta_1^2) \in (\alpha^2, \beta^2), \ldots, (\alpha_1^n, \beta_1^n) \in (\alpha^n, \beta^n)} \frac{(\alpha_1^k - \beta_1^k)(d_k^2 - 2c_k^2 + 2b_k^2 - a_k^2 + d_k c_k - a_k b_k) + 3(c_k^2 - b_k^2)}{3(\alpha_1^k - \beta_1^k)(d_k - c_k + b_k - a_k) + 6(c_k - b_k)} (j = 2, ..., n)$$

and

$$T_1 = 1, T_i = \prod_{k=1}^{i-1} \sum_{(\alpha_1^1, \beta_1^1) \in (\alpha^1, \beta^1), (\alpha_1^2, \beta_1^2) \in (\alpha^2, \beta^2), \ldots, (\alpha_1^n, \beta_1^n) \in (\alpha^n, \beta^n)} \frac{(\alpha_1^k - \beta_1^k)(d_k^2 - 2c_k^2 + 2b_k^2 - a_k^2 + d_k c_k - a_k b_k) + 3(c_k^2 - b_k^2)}{3(\alpha_1^k - \beta_1^k)(d_k - c_k + b_k - a_k) + 6(c_k - b_k)} (i = 2, ..., n).$$

4. AN APPROACH MADM BASED ON THE PROPOSED OPERATORS

4.1. Description of Prioritized MADM Problems Using THIF-numbers

This section develops a novel decision-making method, described in Fiqure 1, designed to address prioritized MADM problems by employing the Δ_A operator based on THIF-numbers.

Definition 21. *Let* $\mathscr{P} = \{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_m\}$ *be a set of alternatives, and let* $\mathscr{L} = \left\{ \mathscr{L}_1 = \bigcup_{r=1}^{k_1} \{l_{1r}\}, \mathscr{L}_2 = \bigcup_{r=1}^{k_2} \{l_{2r}\}, \ldots, \mathscr{L}_n = \bigcup_{r=1}^{k_n} \{l_{nr}\} : k_1, k_2, \ldots, k_n \in \mathbb{Z}^+ \right\}$ *be a set of attributes, where each attribute* $\mathscr{L}_j$ *is composed of* k_j *sub-attributes. A prioritization exists among the attributes, represented by the linear ordering* $\mathscr{L}_1 \succ \mathscr{L}_2 \succ \cdots \succ \mathscr{L}_n$*, indicating that attribute* $\mathscr{L}_j$ *has a higher priority than attribute* $\mathscr{L}_k$ *if* $j < k$*. Let the THIF sub-decision matrix corresponding to the sub-attribute set* $\mathscr{L}_j = \bigcup_{r=1}^{k_j} \{l_{jr}\}$ *be denoted as* $H^j = (x_{ir}^j)_{m \times k_j}$*, where* $i = 1, 2, \ldots, m$ *and* $r = 1, 2, \ldots, k_j$*, and each element* x_{ir}^j *is a THIF-number representing the evaluation of alternative* $\mathscr{P}_i$ *with respect to sub-attribute* l_{jr}*.*

An approach for solving MADM problems using the Δ_A (or Δ_G) operator under THIF-numbers is developed and presented in the following algorithm.

Algorithm

Step 1. Build the THIF sub-decision matrix associated with the sub-attribute set $\mathscr{L}_j = \bigcup_{r=1}^{k_j}\{l_{jr}\}$ as

$$H^j = (x_{ir}^j)_{m\times k_j} = \left(\left\langle (a_{ir}^j, b_{ir}^j, c_{ir}^j, d_{ir}^j);\ (\alpha_{ir}^j, \beta_{ir}^j)\right\rangle\right)_{m\times k_j},$$

for $i = 1,2,\ldots,m$ and $r = 1,2,\ldots,k_j$, to be used in the decision-making process.

Step 2. Calculate the overall evaluation matrix

$$\begin{aligned} A_{ij} &= \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); (\alpha^{ij}, \beta^{ij})\rangle = \Delta_A(x_{i1}^j, x_{i2}^j, ..., x_{ik_j}^j) \\ &= \frac{T_{i1}^J}{\sum_{r=1}^{k_j} T_{ir}^J} x_{i1}^j \oplus \frac{T_{i2}^J}{\sum_{r=1}^{k_j} T_{ir}^J} x_{i2}^j \oplus \cdots \oplus \frac{T_{ik_j}^J}{\sum_{r=1}^{k_j} T_{ir}^J} x_{ik_j}^j \end{aligned} \tag{20}$$

for $i \in \{1,2,...,m\}$ and $j \in \{1,2,...,n\}$ where A_{ij} denotes evaluation of the alternative $\mathscr{P}_i$ with respect to the attribute $\mathscr{L}_j$

where

$$T_{ir}^j = \prod_{p=1}^{r-1} \sum_{(\alpha_{ip},\beta_{ip})\in(\alpha_{ir}^j,\beta_{ir}^j)} \frac{(\alpha_{ip}-\beta_{ip})(d_{ip}^2 - 2c_{ip}^2 + 2b_{ip}^2 - a_{ip}^2 + d_{ip}c_{ip} - a_{ip}b_{ip}) + 3(c_{ip}^2 - b_{ip}^2)}{3(\alpha_{ip}-\beta_{ip})(d_{ip} - c_{ip} + b_{ip} - a_{ip}) + 6(c_{ip} - b_{ip})}$$
$$(i = 1,2,\ldots,m,\ r = 2,\ldots,k_j) \tag{21}$$

and

$$T_{i1} = 1, \quad (i = 1,2,...,m). \tag{22}$$

Step 3. Find $A_i (i = 1,2,...,m)$ by agregate all THIF-numbers under $A_{ij} (j = 1,2,...,n)$ by using the Δ_A operator as:

$$\begin{aligned} A_i = &\ \Delta_A(A_{i1}, A_{i2}, ...A_{in}) = \langle (a_i, b_i, c_i, d_i); (\alpha^i, \beta^i)\rangle \\ = &\ \frac{T_{i1}}{\sum_{j=1}^n T_{ij}} A_{i1} \oplus \frac{T_{i2}}{\sum_{j=1}^n T_{ij}} A_{i2} \oplus \cdots \oplus \frac{T_{in}}{\sum_{j=1}^n T_{ij}} A_{in} \end{aligned} \tag{23}$$

where

$$T_{ij} = \prod_{p=1}^{j-1} \sum_{(\alpha_{ip},\beta_{ip})\in(\alpha^{ij},\beta^{ij})} \frac{(\alpha_{ip}-\beta_{ip})(d_{ip}^2 - 2c_{ip}^2 + 2b_{ip}^2 - a_{ip}^2 + d_{ip}c_{ip} - a_{ip}b_{ip}) + 3(c_{ip}^2 - b_{ip}^2)}{3(\alpha_{ip}-\beta_{ip})(d_{ip} - c_{ip} + b_{ip} - a_{ip}) + 6(c_{ip} - b_{ip})}$$
$$(i = 1,2,\ldots,m,\ j = 2,\ldots,n) \tag{24}$$

and

$$T_{i1} = 1, \quad (i = 1,2,...,m). \tag{25}$$

Step 4. Calculate the centroid values $s(A_i)$ of $A_i(i=1,2,...,m)$ with Definition 3

$$s(A_i)=\sum_{\substack{(\alpha_1^1,\beta_1^1)\in(\alpha^1,\beta^1),\\(\alpha_1^2,\beta_1^2)\in(\alpha^2,\beta^2),\\ \ldots,\\(\alpha_1^m,\beta_1^m)}}\frac{(\alpha_1^i-\beta_1^i)(d_i^2-2c_i^2+2b_i^2-a_i^2+d_ic_i-a_ib_i)+3(c_i^2-b_i^2)}{3(\alpha_1^i-\beta_1^i)(d_i-c_i+b_i-a_i)+6(c_i-b_i)} \tag{26}$$

Step 5. Rank all the alternatives $\mathscr{P}_i(i=1,2,,...,m)$ and select the best one(s). In here if $s(A_s)>s(A_t)\Rightarrow\mathscr{P}_s>\mathscr{P}_t,(s,t\in\{1,2,,...,m\})$.

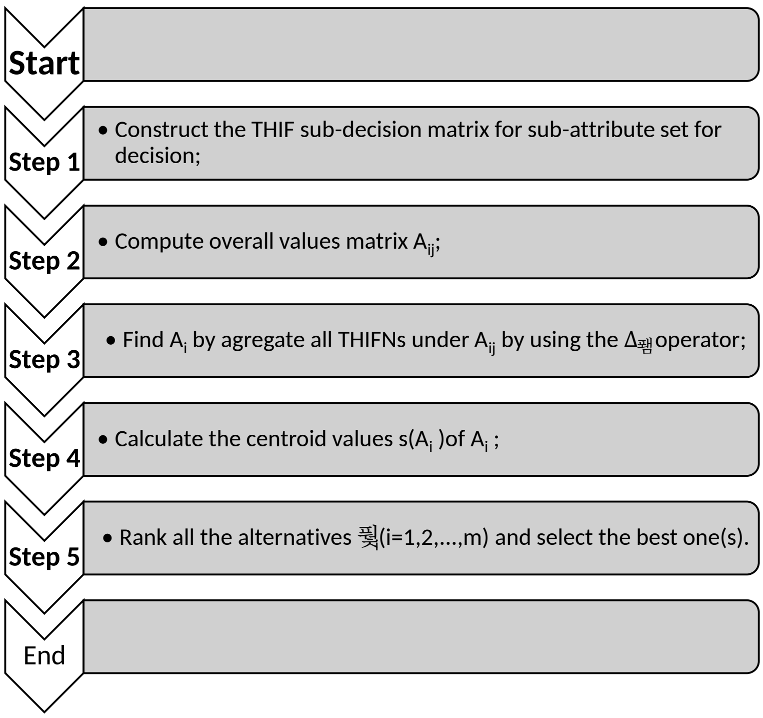

Figure 1: Algorithm

4.2. Numerical example

In this section, we present a MADM problem concerning the selection of an e-learning platform, adapted and inspired by the works of [36, 37]. The proposed approach is applied to identify the most suitable e-learning platform from a subjective perspective, utilizing linguistic scales for evaluation.

In this study, four e-learning platforms are evaluated, as shown in Table 1: Platform 1 ($\mathscr{P}_1$), Platform 2 ($\mathscr{P}_2$), Platform 3 ($\mathscr{P}_3$), and Platform 4 ($\mathscr{P}_4$). The assessment is based on three key attributes that are commonly considered in the selection of e-learning platforms:

- $\mathscr{L}_1$: Clear and understandable content,
- $\mathscr{L}_2$: Interactivity,
- $\mathscr{L}_3$: Variety of educational levels.

With the rapid advancement of digital technologies, e-learning has become an integral part of modern education systems. It is increasingly preferred by both individuals and institutions due to its advantages such as flexibility, accessibility, and efficiency.

Considering these benefits, the attribute **variety of educational levels** ($\mathscr{L}_3$) is regarded as the most critical criterion, as it reflects the platform's capability to serve a wide spectrum of learners. This is followed by **clear and understandable content** ($\mathscr{L}_1$), and finally **interactivity** ($\mathscr{L}_2$).

Accordingly, the prioritization of the attributes is strictly defined as:

$$\mathscr{L}_3 > \mathscr{L}_1 > \mathscr{L}_2$$

where the symbol "$>$" denotes a higher level of preference in the decision-making process.

Table 1: e-learning platform alternatives.

Symbol	The e-learning platform
$\mathscr{P}_1$	e-learning platform 1
$\mathscr{P}_2$	e-learning platform 2
$\mathscr{P}_3$	e-learning platform 3
$\mathscr{P}_4$	e-learning platform 4

Table 3 provides a detailed description of the sub-attributes associated with the main criteria $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$, along with the linguistic evaluations of the e-learning platform alternatives. These evaluations are expressed using THIF-numbers, is given as:

$\mathscr{L}_1$: Clear and understandable content attribute contains l_{11} = Simplicity of the language of the content, l_{12}= Level of conceptual clarity and l_{13}= Regularity and integrity of content structure. There are strict prioritization between parameters such as such as $l_{11} > l_{12} > l_{13}$, here $>$ indicates preferred to.

$\mathscr{L}_2$: Interactivity attribute contains l_{21}= Feedback Mechanism, l_{22}= Live Sessions and Q – A, l_{23}= Discussion Forums and l_{24}= Group Work and Collaboration Tools. There are strict prioritization between parameters such as such as $l_{21} > l_{22} > l_{23} > l_{24}$, here $>$ indicates preferred to.

$\mathscr{L}_3$: Variety of educational levels attribute contains l_{31}= Program Levels and l_{32}= Diversity of Target Learners. There are strict prioritization between parameters such as $l_{31} > l_{32}$, here $>$ indicates preferred to.

Furthermore, to enhance the objectivity of the results, the experts were selected from diverse departments within the Faculty of Languages. These experts provided their evaluations using THIF-numbers corresponding to the linguistic terms presented in Table 2.

Table 2: THIF-numbers for linguistic terms

Linguistic terms	Linguistic values of THIF-numbers	Score
Absolutely low(AL)	$\langle(0.1,0.1,0.2,0.3);\{(0.1,0.8),(0.2,0.8),(0.4,0.9)\}\rangle$	0,0338
Very very Low(L)	$\langle(0.1,0.2,0.4,0.4);\{(0.1,0.8),(0.3,0.7),(0.5,0.8)\}\rangle$	0,0585
Very Low(VL)	$\langle(0.2,0.3,0.5,0.5);\{(0.2,0.6),(0.4,0.7),(0.7,0.6)\}\rangle$	0,0780
Fairly low(FL)	$\langle(0.2,0.4,0.5,0.6);\{(0.2,0.6),(0.2,0.4),(0.5,0.6)\}\rangle$	0,0880
low(L)	$\langle(0.4,0.5,0.7,0.8);\{(0.2,0.6),(0.2,0.4),(0.5,0.5)\}\rangle$	0,1560
Medium(M)	$\langle(0.6,0.7,0.7,0.9);\{(0.3,0.4),(0.4,0.5),(0.6,0.4)\}\rangle$	0,1657
Fairly high(FH)	$\langle(0.3,0.5,0.6,0.8);\{(0.6,0.3),(0.5,0.2),(0.6,0.3)\}\rangle$	0,1760
High(H)	$\langle(0.2,0.6,0.8,0.9);\{(0.6,0.4),(0.7,0.5),(0.8,0.3)\}\rangle$	0,1920
Very High(VH)	$\langle(0.2,0.5,0.6,0.7);\{(0.7,0.3),(0.8,0.1),(0.9,0.1)\}\rangle$	0,2240
Very Very High(VVH)	$\langle(0.6,0.7,0.8,0.9);\{(0.8,0.2),(0.8,0.1),(0.9,0.1)\}\rangle$	0,2300
Absolutely high(AH)	$\langle(0.5,0.6,0.7,0.9);\{(0.9,0.1),(0.9,0.1),(0.9,0.3)\}\rangle$	0,2818

Table 3: Linguistic assessment of e-learning platform alternatives using THIF-numbers

Main criteria	Sub-criteria	Alternatives	Linguistic labels
L_1-Clear and understandable content criterion	l_{11}-Simplicity of the language of the content	$\mathscr{P}_1$	{(L)}
		$\mathscr{P}_2$	{(FL)}
		$\mathscr{P}_3$	{(H)}
		$\mathscr{P}_4$	{(AH)}
	l_{12}-Level of conceptual clarity	$\mathscr{P}_1$	{(H)}
		$\mathscr{P}_2$	{(FH)}
		$\mathscr{P}_3$	{(VH)}
		$\mathscr{P}_4$	{(AH)}
	l_{13}-Regularity and integrity of content structure	$\mathscr{P}_1$	{(AH)}
		$\mathscr{P}_2$	{(L)}
		$\mathscr{P}_3$	{(VVH)}
		$\mathscr{P}_4$	{(FL)}
L_2-Interactivity criterion	l_{21}-Feedback Mechanism	$\mathscr{P}_1$	{(M)}
		$\mathscr{P}_2$	{(VVH)}
		$\mathscr{P}_3$	{(H)}
		$\mathscr{P}_4$	{(L)}
	l_{22}-Live sessions and Q – A	$\mathscr{P}_1$	{(M)}
		$\mathscr{P}_2$	{(L)}
		$\mathscr{P}_3$	{(H)}
		$\mathscr{P}_4$	{(VH)}
	l_{23}-Discussion forums	$\mathscr{P}_1$	{(FH)}
		$\mathscr{P}_2$	{(H)}

Continued on next page

Table 3 (continued)

Main criteria	Sub-criteria	Alternatives	Linguistic labels
		$\mathscr{P}_3$	$\{(\text{VH})\}$
		$\mathscr{P}_4$	$\{(\text{VVH})\}$
	l_{24}-Group work and collaboration tools	$\mathscr{P}_1$	$\{(\text{AL})\}$
		$\mathscr{P}_2$	$\{(\text{VL})\}$
		$\mathscr{P}_3$	$\{(\text{FH})\}$
		$\mathscr{P}_4$	$\{(\text{VVH})\}$
L_3-Variety of educational levels criterion	l_{31}- Program levels	$\mathscr{P}_1$	$\{(\text{AL})\}$
		$\mathscr{P}_2$	$\{(\text{AH})\}$
		$\mathscr{P}_3$	$\{(\text{H})\}$
		$\mathscr{P}_4$	$\{(\text{VH})\}$
	l_{32}-Diversity of target learners	$\mathscr{P}_1$	$\{(\text{M})\}$
		$\mathscr{P}_2$	$\{(\text{VL})\}$
		$\mathscr{P}_3$	$\{(\text{AL})\}$
		$\mathscr{P}_4$	$\{(\text{VVH})\}$

Subsequently, to select / rank the e-learning platforms, the Δ_A operator is employed to construct a solution approach for MADM problems under THIF information. The proposed method can be described through the following algorithm:

Proposed operators' computational algorithm:

Step 1. Three THIF sub-decision matrices corresponding to the sub-attribute sets were formed $\mathscr{L}_j = \cup_{r=1}^{k_j}\{l_{jr}\}$ as $H^j = (x_{ir}^j)_{4\times k_j}(i = 1,2,3,4; j = 1,2,3; r = k_1,k_2,k_3; k_1 = 1,2,3; k_2 = 1,2,3,4; k_3 = 1,2)$ for decision in Tables 4-6;

Table 4: Sub-decision matrices based on THIF $(x_{ir}^1)_{5\times 3}$

$\mathscr{L}_1$	Simplicity of the language of the content	Level of conceptual clarity conceptual clarity
$\mathscr{P}_1$	$\langle(0.4,0.5,0.7,0.8);\{(0.2,0.6),(0.2,0.4),(0.5,0.5)\}\rangle$	$\langle(0.2,0.6,0.8,0.9);\{(0.6,0.4),(0.7,0.5),(0.8,0.3)\}\rangle$
$\mathscr{P}_2$	$\langle(0.2,0.4,0.5,0.6);\{(0.2,0.6),(0.2,0.4),(0.5,0.6)\}\rangle$	$\langle(0.3,0.5,0.6,0.8);\{(0.6,0.3),(0.5,0.2),(0.6,0.3)\}\rangle$
$\mathscr{P}_3$	$\langle(0.2,0.6,0.8,0.9);\{(0.6,0.4),(0.7,0.5),(0.8,0.3)\}\rangle$	$\langle(0.2,0.5,0.6,0.7);\{(0.7,0.3),(0.8,0.1),(0.9,0.1)\}\rangle$
$\mathscr{P}_4$	$\langle(0.5,0.6,0.7,0.9);\{(0.9,0.1),(0.9,0.1),(0.9,0.3)\}\rangle$	$\langle(0.5,0.6,0.7,0.9);\{(0.9,0.1),(0.9,0.1),(0.9,0.3)\}\rangle$
	Regularity and integrity of content structure	
$\mathscr{P}_1$	$\langle(0.5,0.6,0.7,0.9);\{(0.9,0.1),(0.9,0.1),(0.9,0.3)\}\rangle$	
$\mathscr{P}_2$	$\langle(0.4,0.5,0.7,0.8);\{(0.2,0.6),(0.2,0.4),(0.5,0.5)\}\rangle$	
$\mathscr{P}_3$	$\langle(0.6,0.7,0.8,0.9);\{(0.8,0.2),(0.8,0.1),(0.9,0.1)\}\rangle$	
$\mathscr{P}_4$	$\langle(0.2,0.4,0.5,0.6);\{(0.2,0.6),(0.2,0.4),(0.5,0.6)\}\rangle$	

Table 5: THIF sub-decision matrices $(x_{ir}^2)_{5\times4}$

$\mathscr{L}_2$	Feedback Mechanism	Live sessions and Q–A
$\mathscr{P}_1$	$\langle(0.6,0.7,0.7,0.9);\{(0.3,0.4),(0.4,0.5),(0.6,0.4)\}\rangle$	$\langle(0.6,0.7,0.7,0.9);\{(0.3,0.4),(0.4,0.5),(0.6,0.4)\}\rangle$
$\mathscr{P}_2$	$\langle(0.6,0.7,0.8,0.9);\{(0.8,0.2),(0.8,0.1),(0.9,0.1)\}\rangle$	$\langle(0.4,0.5,0.7,0.8);\{(0.2,0.6),(0.2,0.4),(0.5,0.5)\}\rangle$
$\mathscr{P}_3$	$\langle(0.2,0.6,0.8,0.9);\{(0.6,0.4),(0.7,0.5),(0.8,0.3)\}\rangle$	$\langle(0.2,0.6,0.8,0.9);\{(0.6,0.4),(0.7,0.5),(0.8,0.3)\}\rangle$
$\mathscr{P}_4$	$\langle(0.4,0.5,0.7,0.8);\{(0.2,0.6),(0.2,0.4),(0.5,0.5)\}\rangle$	$\langle(0.2,0.5,0.6,0.7);\{(0.7,0.3),(0.8,0.1),(0.9,0.1)\}\rangle$
	Discussion forums	Group work and Collaboration Tools
$\mathscr{P}_1$	$\langle(0.3,0.5,0.6,0.8);\{(0.6,0.3),(0.5,0.2),(0.6,0.3)\}\rangle$	$\langle(0.1,0.1,0.2,0.3);\{(0.1,0.8),(0.2,0.8),(0.4,0.9)\}\rangle$
$\mathscr{P}_2$	$\langle(0.2,0.6,0.8,0.9);\{(0.6,0.4),(0.7,0.5),(0.8,0.3)\}\rangle$	$\langle(0.2,0.3,0.5,0.5);\{(0.2,0.6),(0.4,0.7),(0.7,0.6)\}\rangle$
$\mathscr{P}_3$	$\langle(0.2,0.5,0.6,0.7);\{(0.7,0.3),(0.8,0.1),(0.9,0.1)\}\rangle$	$\langle(0.3,0.5,0.6,0.8);\{(0.6,0.3),(0.5,0.2),(0.6,0.3)\}\rangle$
$\mathscr{P}_4$	$\langle(0.6,0.7,0.8,0.9);\{(0.8,0.2),(0.8,0.1),(0.9,0.1)\}\rangle$	$\langle(0.6,0.7,0.8,0.9);\{(0.8,0.2),(0.8,0.1),(0.9,0.1)\}\rangle$

Table 6: THIF sub-decision matrices $(x_{ir}^3)_{5\times2}$

$\mathscr{L}_3$	Program levels	Diversity of Target Learners
$\mathscr{P}_1$	$\langle(0.1,0.1,0.2,0.3);\{(0.1,0.8),(0.2,0.8),(0.4,0.9)\}\rangle$	$\langle(0.6,0.7,0.7,0.9);\{(0.3,0.4),(0.4,0.5),(0.6,0.4)\}\rangle$
$\mathscr{P}_2$	$\langle(0.5,0.6,0.7,0.9);\{(0.9,0.1),(0.9,0.1),(0.9,0.3)\}\rangle$	$\langle(0.2,0.3,0.5,0.5);\{(0.2,0.6),(0.4,0.7),(0.7,0.6)\}\rangle$
$\mathscr{P}_3$	$\langle(0.2,0.6,0.8,0.9);\{(0.6,0.4),(0.7,0.5),(0.8,0.3)\}\rangle$	$\langle(0.1,0.1,0.2,0.3);\{(0.1,0.8),(0.2,0.8),(0.4,0.9)\}\rangle$
$\mathscr{P}_4$	$\langle(0.2,0.5,0.6,0.7);\{(0.7,0.3),(0.8,0.1),(0.9,0.1)\}\rangle$	$\langle(0.6,0.7,0.8,0.9);\{(0.8,0.2),(0.8,0.1),(0.9,0.1)\}\rangle$

Step 2. Based on Equations 20–21, we calculated the overall values matrix presented in Table 7;

Table 7: The assessment of alternatives by decision makers concerning the criteria.

	Clear and understandable content
$\mathscr{P}_1$	$\langle(0.38,0.52,0.71,0.81);\{(0.33,,0.53),(0.36,0.53),(0.59,0.41)\}\rangle$
$\mathscr{P}_2$	$\langle(0.22,0.42,0.52,0.64);\{(0.28,0.54),(0.26,0.36),(0.44,0.53)\}\rangle$
$\mathscr{P}_3$	$\langle(0.23,0.59,0.76,0.86);\{(0.65,0.35),(0.74,0.21),(0.84,0.16)\}\rangle$
$\mathscr{P}_4$	$\langle(0.44,0.56,0.66,0.85);\{(0.85,0.14),(0.85,0.13),(0.86,0.14)\}\rangle$
	Interactivity
$\mathscr{P}_1$	$\langle(0.53,0.64,0.69,0.90);\{(0.38,0.39),(0.42,0.42),(0.58,0.39)\}\rangle$
$\mathscr{P}_2$	$\langle(0.51,0.63,0.78,0.88);\{(0.65,0.31),(0.66,0.18),(0.81,0.19)\}\rangle$
$\mathscr{P}_3$	$\langle(0.22,0.62,0.81,0.93);\{(0.61,0.39),(0.71,0.29),(0.81,0.19)\}\rangle$
$\mathscr{P}_4$	$\langle(0.41,0.53,0.72,0.82);\{(0.36,0.51),(0.39,0.30),(0.64,0.36)\}\rangle$
	Variety of educational levels
$\mathscr{P}_1$	$\langle(0.15,0.16,0.25,0.36);\{(0.12,0.75),(0.22,0.77),(0.42,0.58)\}\rangle$
$\mathscr{P}_2$	$\langle(0.39,0.49,0.62,0.75);\{(0.64,0.29),(0.71,0.29),(0.82,0.18)\}\rangle$
$\mathscr{P}_3$	$\langle(0.17,0.47,0.64,0.75);\{(0.51,0.48),(0.61,0.39),(0.73,0.27)\}\rangle$
$\mathscr{P}_4$	$\langle(0.31,0.56,0.66,0.76);\{(0.74,0.26),(0.80,0.10),(0.90,0.10)\}\rangle$

Step 3. We found $A_i(i=1,2,...,m)$ by aggregated all THIF-numbers under $A_{ij}(j=1,2,...,n)$ by using Eq. 23-25 in Table 8;

Table 8: $A_i(i=1,2,3,4)$ through the aggregation of all THIF-numbers

A_1	$\langle(0.39,0.52,0.70,0.82);\{(0.34,0.51),(0.36,0.37),(0.58,0.41)\}\rangle$
A_2	$\langle(0.26,0.45,0.56,0.67);\{(0.36,0.49),(0.35,0.33),(0.53,0.45)\}\rangle$
A_3	$\langle(0.23,0.58,0.76,0.86);\{(0.63,0.37),(0.72,0.24),(0.82,0.18)\}\rangle$
A_4	$\langle(0.42,0.55,0.68,0.83);\{(0.75,0.22),(0.76,0.17),(0.81,0.19)\}\rangle$

Step 4. The centroid values were computed $s(A_i)$ of A_i by using Eq.26 in Table 9;

Table 9: Centroid values $s(A_i)$ of A_i

$s(A_1) =$	1.84094
$s(A_2) =$	1.49675
$s(A_3) =$	1.86517
$s(A_4) =$	1.86316

Step 5. Ranking all the alternatives $\mathscr{P}_i (i = 1, 2, 3, 4)$ and select the best one(s). In here if $s(A_3) > s(A_4) > s(A_1) > s(A_2) \Rightarrow \mathscr{P}_3 > \mathscr{P}_4 > \mathscr{P}_1 > \mathscr{P}_2$ and thus the most suitable e-learning platform $\mathscr{P}_3$.

4.3. Comparison and Analysis Discussion

When decision analysts work with data expressed as THIF-numbers, they face a challenge due to the lack of existing prioritization operators that can effectively handle this type of information. The THIFN-prioritized aggregation operators introduced in this study are specifically designed to address this gap, offering a practical solution to support decision-making processes involving THIF data. A comparative analysis with several existing methods was conducted to validate the effectiveness of the proposed MADM approach using prioritized aggregation operators, including the methods of Wei [13], Wan [38], Lia et al. [39], and Ulucay et al. [36]. The comparison results are summarized in Table 10. A brief critical analysis of the results shows that the proposed operators yield rankings consistent with existing methods while capturing subtle differences among alternatives more effectively, demonstrating the robustness and practical relevance of the approach.

Table 10: Comparative analysis of different techniques and their ranking results

Methods	Preference Order	The optimal alternative
The method in [39]	$\mathscr{P}_4 > \mathscr{P}_3 > \mathscr{P}_2 > \mathscr{P}_1$,	$\mathscr{P}_4$
The method in [13]	$\mathscr{P}_4 > \mathscr{P}_2 > \mathscr{P}_3 > \mathscr{P}_1$,	$\mathscr{P}_4$
The method in [38]	$\mathscr{P}_4 > \mathscr{P}_3 > \mathscr{P}_1 > \mathscr{P}_2$,	$\mathscr{P}_4$
The method in [36]	$\mathscr{P}_4 > \mathscr{P}_3 > \mathscr{P}_2 > \mathscr{P}_1$,	$\mathscr{P}_4$
The method in [36]	$\mathscr{P}_4 > \mathscr{P}_3 > \mathscr{P}_2 > \mathscr{P}_1$,	$\mathscr{P}_4$
Proposed Δ_A Method	$\mathscr{P}_3 > \mathscr{P}_4 > \mathscr{P}_1 > \mathscr{P}_2$,	$\mathscr{P}_3$
Proposed Δ_G Method	$\mathscr{P}_3 > \mathscr{P}_4 > \mathscr{P}_1 > \mathscr{P}_2$,	$\mathscr{P}_3$

5. CONCLUSION

THIF-numbers are highly effective for representing ill-known or imprecise quantities. Given the critical role of aggregation operators in decision-making processes, this study addresses prioritized MADM problems in which the attribute values are expressed using THIF-numbers. To this end, we proposed two novel aggregation operators: the

Trapezoidal Hesitant Intuitionistic Fuzzy Prioritized Weighted Average Δ_A operator and the Trapezoidal Hesitant Intuitionistic Fuzzy Prioritized Weighted Geometric Δ_G operator, both designed to aggregate THIF information effectively. We also examined several fundamental properties of the proposed operators, including idempotency, boundary, and monotonicity. Furthermore, two MADM approaches were developed based on the proposed operators under the THIF environment to select the best e-learning platform. A practical case study was presented to illustrate the applicability and effectiveness of the proposed methods. The results show that incorporating attribute prioritization significantly enhances decision quality, making the proposed approach both practical and robust for real-world applications. Additionally, the comparison of our method with several existing approaches from the literature highlights its superior performance and effectiveness in handling prioritized and uncertain information. Moreover, the introduction of prioritized operators for THIF-numbers provides a new avenue for information fusion in uncertain and hesitant decision-making environments.

5.1. Shortcomings and future research directions

While the proposed MADM method offers several advantages, it is not without limitations. Specifically, our approach is designed to work with THIF-numbers and is applicable to various subclasses within the THIF framework. However, it does not currently support other generalized fuzzy number types, such as Pythagorean hesitant fuzzy sets, interval-valued hesitant Fermatean hesitant fuzzy sets, or spherical hesitant fuzzy sets. In future research, we plan to extend our aggregation approach by developing aggregation based operators suitable for these broader classes of fuzzy sets and their extensions. Additionally, further efforts will focus on integrating the proposed aggregation framework into other fuzzy MADM techniques, including VIKOR, ARAS, WASPAS, TOPSIS, EDAS, and related methods. Beyond methodological development, we also envision applying this approach across various real-world domains such as marketing, engineering, information technology, human resource management, and energy management, where decision-making under uncertainty plays a critical role.

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REFERENCES

[1] M. Sarfraz and R. Gul, "Evaluating medical college projects with hamacher aggregation operators under the interval-valued complex t-spherical fuzzy environment," *Management Science Advances*, vol. 2, no. 1, pp. 69–90, 2025.

[2] W. Wang, W. Jiang, X. Han, and S. Liu, "An extended gained and lost dominance score method based risk prioritization for fine-kinney model with interval type-2 fuzzy information," *Human and Ecological Risk Assessment: An International Journal*, pp. 1–30, 2022.

[3] P. Liu and H. Gao, "A novel green supplier selection method based on the interval type-2 fuzzy prioritized choquet bonferroni means," *IEEE/CAA Journal of Automatica Sinica*, vol. 8, no. 9, pp. 1549–1566, 2020.

[4] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[5] K. T. Atanassov, "Intuitionistic fuzzy sets," in *Intuitionistic fuzzy sets: theory and applications*. Springer, 1999, pp. 1–137.

[6] T. Fujita, A. Mehmood, and A. A. Ghaib, "Hyperfuzzy control system and superhyperfuzzy control system," *Smart Multi-Criteria Analytics and Reasoning Technologies*, vol. 1, no. 1, pp. 1–21, 2025.

[7] T. Fujita, "Hyperfuzzy and superhyperfuzzy group decision-making," *Spectrum of Decision Making and Applications*, pp. 1–18, 2027.

[8] M. Mizumoto and K. Tanaka, "Some properties of fuzzy sets of type 2," *Information and control*, vol. 31, no. 4, pp. 312–340, 1976.

[9] Z. Başer and V. Uluçay, "Effective q–fuzzy soft expert sets and its some properties," *Uncertainty Discourse and Applications*, vol. 1, no. 2, pp. 195–209, 2024.

[10] ——, "Effective q-neutrosophic soft expert sets and its application in decision making," *Algebraic structures in the universe of neutrosophic: analysis with innovative algorithmic approaches*, p. 147, 2024.

[11] V. Torra, "Hesitant fuzzy sets," *International journal of intelligent systems*, vol. 25, no. 6, pp. 529–539, 2010.

[12] Z. Xu and M. Xia, "On distance and correlation measures of hesitant fuzzy information," *International Journal of Intelligent Systems*, vol. 26, no. 5, pp. 410–425, 2011.

[13] G. Wei, "Hesitant fuzzy prioritized operators and their application to multiple attribute decision making," *Knowledge-Based Systems*, vol. 31, pp. 176–182, 2012.

[14] I. Deli, "Trapezoidal hesitant intuitionistic fuzzy numbers and their applications to multiple-criteria decision making problems," *International Journal of Information Technology*, pp. 1–12, 2025.

[15] W. Ali, T. Shaheen, I. U. Haq, H. Toor, F. Akram, H. Garg, M. Z. Uddin, and M. M. Hassan, "Aczel-alsina-based aggregation operators for intuitionistic hesitant fuzzy set environment and their application to multiple attribute decision-making process," *Aims Math*, vol. 8, no. 8, pp. 18 021–18 039, 2023.

[16] K. T. Atanassov, "Two theorems for intuitionistic fuzzy sets," *Fuzzy sets and systems*, vol. 110, no. 2, pp. 267–269, 2000.

[17] R. R. Yager, "Prioritized aggregation operators," *International Journal of Approximate Reasoning*, vol. 48, no. 1, pp. 263–274, 2008.

[18] I. Deli and F. Karaaslan, "Generalized trapezoidal hesitant fuzzy numbers and their applications to multi criteria decision-making problems," *Soft Computing*, vol. 25, no. 2, pp. 1017–1032, 2021.

[19] I. Deli, "Bonferroni mean operators of generalized trapezoidal hesitant fuzzy numbers and their application to decision-making problems," *Soft Computing*, vol. 25, no. 6, pp. 4925–4949, 2021.

[20] ——, "A topsis method by using generalized trapezoidal hesitant fuzzy numbers and application to a robot selection problem," *Journal of intelligent & fuzzy systems*, vol. 38, no. 1, pp. 779–793, 2020.

[21] G. Anusha, P. V. Ramana, and R. Sarkar, "Hybridizations of archimedean copula and generalized msm operators and their applications in interactive decision-making with q-rung probabilistic dual hesitant fuzzy environment," *Decision making: applications in management and engineering*, vol. 6, no. 1, pp. 646–678, 2023.

[22] H. Liao, Z. Xu, and X.-J. Zeng, "Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making," *Information sciences*, vol. 271, pp. 125–142, 2014.

[23] T. Karamat and M. Sarfraz, "Applied multi-attribute decision-making with complex

pythagorean fuzzy data based on prioritized aczel–alsina aggregation operators: A case for a software company," *Applied Research Advances*, vol. 1, no. 1, pp. 14–27, 2025.

[24] S. P. Wan, "Power average operators of trapezoidal intuitionistic fuzzy numbers and application to multi-attribute group decision making," *Applied Mathematical Modelling*, vol. 37, no. 6, pp. 4112–4126, 2013. doi: 10.1016/j.apm.2012.09.030

[25] S. B. Aydemir and S. Yilmaz Gündüz, "Extension of multi-moora method with some q-rung orthopair fuzzy dombi prioritized weighted aggregation operators for multi-attribute decision making," *Soft Computing*, vol. 24, no. 24, pp. 18 545–18 563, 2020.

[26] X. Zhao and G. Wei, "Some intuitionistic fuzzy einstein hybrid aggregation operators and their application to multiple attribute decision making," *Knowledge-Based Systems*, vol. 37, pp. 472–479, 2013.

[27] R. Verma and B. D. Sharma, "Trapezoid fuzzy linguistic prioritized weighted average operators and their application to multiple attribute group decision-making," *Journal of Uncertainty Analysis and Applications*, vol. 2, no. 1, p. 10, 2014.

[28] P. Liu, Y. Li, and J. Antuchevičienė, "Multi-criteria decision-making method based on intuitionistic trapezoidal fuzzy prioritised owa operator," *Technological and Economic Development of Economy*, vol. 22, no. 3, pp. 453–469, 2016.

[29] M. Akram, S. Alsulami, A. Khan, and F. Karaaslan, "Multi-criteria group decision-making using spherical fuzzy prioritized weighted aggregation operators," *International Journal of Computational Intelligence Systems*, vol. 13, no. 1, pp. 1429–1446, 2020.

[30] H. Garg and D. Rani, "New prioritized aggregation operators with priority degrees among priority orders for complex intuitionistic fuzzy information," *Journal of Ambient Intelligence and Humanized Computing*, vol. 14, no. 3, pp. 1373–1399, 2023.

[31] K. Kumar and S.-M. Chen, "Multiple attribute group decision making based on advanced linguistic intuitionistic fuzzy weighted averaging aggregation operator of linguistic intuitionistic fuzzy numbers," *Information Sciences*, vol. 587, pp. 813–824, 2022.

[32] R. R. Yager, "Prioritized aggregation operators," *International Journal of Approximate Reasoning*, vol. 48, no. 1, pp. 263–274, 2008.

[33] J. Wang and Z. Zhang, "Multi-criteria decision-making method with incomplete certain information based on intuitionistic fuzzy number," *Control and Decision*, vol. 24, no. 2, pp. 226–230, 2009.

[34] Y. M. Wang, J. B. Yang, D. L. Xu, and K. S. Chin, "On the centroids of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 157, pp. 919–926, 2006. doi: 10.1016/j.fss.2005.11.007

[35] V. Torra, "Hesitant fuzzy sets," *International journal of intelligent systems*, vol. 25, no. 6, pp. 529–539, 2010.

[36] V. Ulucay, I. Deli, and S. A. Edalatpanah, "Prioritized aggregation operators of gthfns madm approach for the evaluation of renewable energy sources," *Informatica*, vol. 35, no. 4, pp. 859–882, 2024.

[37] B. B. Meher and S. Jeevaraj, "Trapezoidal-valued intuitionistic fuzzy aczel-alsina weighted geometric aggregation operator and its applications to selection of e-learning platform," *Computational and Applied Mathematics*, vol. 44, no. 6, p. 285, 2025.

[38] S. Wan, J. Dong, and D. Yang, "Trapezoidal intuitionistic fuzzy prioritized aggregation operators and application to multi-attribute decision making," *Iranian Journal of Fuzzy Systems*, vol. 12, no. 4, pp. 1–32, 2015.

[39] C. Liang, S. Zhao, and J. Zhang, "Multi-criteria group decision making method based on generalized intuitionistic trapezoidal fuzzy prioritized aggregation operators," *International Journal of Machine Learning and Cybernetics*, vol. 8, no. 2, pp. 597–610, 2017.