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Research Article

CAPUTO FRACTIONAL DERIVATIVES IN NONLINEAR PROGRAMMING PROBLEMS

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Abstract: Convexity with respect to Caputo fractional derivative is introduced and applied in general nonlinear programming problems. The Karush Kuhn Tucker (KKT) conditions are developed and the sufficient optimality conditions are established. The consequences of the new findings are discussed and counterexamples are presented to validate the present investigation.

Keywords: Caputo fractional derivative, Convexity with fractional order derivatives, KKT conditions, Optimality.

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1. INTRODUCTION

Fractional order calculus has many uses in the field of science [1], economics [2], medical sciences [3] and modeling of many real-world phenomena [4], [5], [6]. The examples discussed in [7] shows that by using fractional order models the physical systems like ultra-capacitors and the beam heating problem could be described more precisely.

Defterli [8] presented Hamiltonian formulation to describe the multi-dimensional Fractional Optimal Control Problems (FOCPs), where the fractional derivatives were defined in the Riemann-Liouville (RL) sense. Almeida and Torres [9] proved the necessary and sufficient conditions as well as the isoperimetric problems of the calculus of variations using the RL fractional derivatives and integrals. The Ulam-Hyers and the Ulam-Hyers-Rassias stability theory was extended to fractional differential equations in Caputo fractional derivative sense in [10]. A single step modified one-parameter family of the Caputo type fractional iterative method was used by Shams et al. [11] for the convergence analysis of several real life problems encountered in civil and chemical engineering.

Odzijewicz et al. [12] considered fractional variational by using a combination of the Caputo fractional derivatives (CFD) and the classical derivatives to generalize the classical calculus of variations. Kiymaz et al. [13] introduced an extension of Caputo fractional derivatives as well as the extensions of hypergeometric function and Mellin transforms. Choi and Agarwal [14] formulated a fractional integral formula involving the generalised multi-index Mittag-Leffler function. Several authors [15], [16], [17], [18], [19], [20], [21] have discussed the necessary and sufficient optimality conditions of variational problems with the Lagrangian depending on different fractional derivatives.

Khan et al. [22] have suggested a fractional order initial and boundary condition to solve the linear and nonlinear fractional order partial differential equations, by means of the modified RL fractional derivative. Tariboon et al. [23] established a concept of fractional quantum calculus by defining q-shifting operator, using RL fractional derivatives and also proved the existence and uniqueness of first and second order initial value problems for impulsive fractional q-difference equation. Recently, the Leibnitz chain-rule property using the RL fractional derivative was discussed by Cresson and Szafranska [24]. Also, Usero [25] obtained the Taylor series expansion for the CFDs.

Convexity plays an important role in the study of extremum problems in variational calculus. Recently many researchers have used convexity as a tool to determine the extremum of a functional described by fractional derivatives. In the work by Moze [26], the relation between linear matrix inequalities (LMI) and convexity was established using fractional systems. In [27] the existence and uniqueness of a fractional thermostat model involving ψ-caputo fractional derivatives were considered for both convex and concave sources. The concavity property in fractional RL sense was developed in [28]. Pankaj and Joshi [29] introduced a generalized hybrid convex function for differentiable real valued functions to establish duality in multi objective fractional programming problem. The geometric properties of a function through its Caputo fractional derivative was discussed in [30] and convexity and monotonicity results were also discussed.

The study of fractional calculus generalizes the traditional calculus to non-integer differential and integral orders. Various types of fractional derivatives were established by different authors; Liouville, Abel, Leibniz, Riemann, Lagrange, Hadamard, Caputo etc., to mention a few. The fractional derivative formulated by Caputo and RL is yet to be explored in general nonlinear programming problems (NLPP). The present research work endeavors to explore the idea of CFDs and integrals to solve optimization problems.

The following are focused in the present investigation: At the out set, we introduce convexity via fractional derivative and apply it to mathematical programming problems. Also, we prove that the KKT conditions are sufficient for optimality of mathematical programming problems, where the objective as well as constraint functions are convex in the CFD sense. Further, we provide counterexamples to demonstrate the efficacy of our work.

2. PRELIMINARIES

Several generalizations of fractional order derivatives and in particular CFDs are mentioned by many authors [31], [32], [33]. Baleanu and Trujillo [34], Cruz et al. [35], and Jarad et al. [36] are few of them. Here, some definitions are presented and used in the rest of the paper.

Definition 1. [37]. Let $h : [a,b] \to \mathbb{R}$ be a function, $\alpha \in \mathbb{R}_+$, $n \in \mathbb{Z}$ satisfies $(n-1) \leq \alpha < n$, and Γ be the Euler gamma function. Then, the left and the right CFDs of order α are defined by

$$(i)\ {}_a^C D_r^\alpha h(r) = \frac{1}{\Gamma(n-\alpha)} \int_a^r \frac{h^{(n)}(\omega)}{(r-\omega)^{\alpha+1-n}} d\omega, \ \ and$$

$$(ii)\ {}_r^C D_b^\alpha h(r) = \frac{1}{\Gamma(n-\alpha)} \int_r^b (-1)^n \frac{h^{(n)}(\omega)}{(\omega-r)^{\alpha+1-n}} d\omega,$$

respectively.

Remark 2. *If $\alpha \in \mathbb{N}$, then the left and right CFDs generalizes to the classical derivatives ${}_a^C D_r^\alpha h(r) = (\frac{d}{dr})^\alpha h(r)$ and ${}_r^C D_b^\alpha h(r) = (-\frac{d}{dr})^\alpha h(r)$, respectively.*

Remark 3. *The derivative of a constant in Caputo sense is equal to zero, which is similar to the classical derivatives.*

3. MAIN RESULTS

In this section, fractional order convexity is defined and is followed by the discussion on the sufficient optimality conditions.

3.1. Fractional order convexity

We now define convexity *w.r.t.* the left and right CFDs. These definitions are then compared to the well known notion of classical convexity. Again counterexamples are provided to show the potential strength of the new definitions.

Definition 4. Let $h : [a,b] \to \mathbb{R}$, having left CFD of order α, $0 \leq \alpha < 1$. For $(x-a)^\alpha - (r-a)^\alpha \geq 0$, h is said to be convex *w.r.t.* the left CFD if

$$h(x) - h(r) \geq \frac{{}_a^C D_r^\alpha h(r)}{\Gamma(\alpha+1)} [(x-a)^\alpha - (r-a)^\alpha], \ \forall\, x, r \in [a,b]. \tag{1}$$

Remark 5. *Taking the Taylor's series expansion w.r.t. the left CFD [25] up to three terms, it can be easily observed that* ${}_a^C D_r^{2\alpha} h(r) \geq 0$ *is the sufficient condition for convexity. When* $\alpha = 1$, ${}_a^C D_r^{2\alpha} h(r) \geq 0$ *generalizes to* $h''(r) \geq 0$*, which suffices the condition for convexity in the classical sense.*

Definition 6. Let $h : E \to \mathbb{R}^n$ where $E \subset \mathbb{R}^n$, and have the left CFDs of order $\alpha,\ 0 \leq \alpha < 1$. $\big(E = I_1 \times I_2 \times ... \times I_n,\ I_i = [a_i, b_i],\ i = 1, 2, ..., n$ (through out the paper)$\big)$. For $(x-a)^\alpha - (r-a)^\alpha \geq 0$, h is said to be convex *w.r.t.* the left CFD if

$$h(x) - h(r) \geq \tfrac{{}_a\nabla_r^{(\alpha)} h(r)}{\Gamma(\alpha+1)} [(x-a)^\alpha - (r-a)^\alpha],\ \forall\, x, r \in E \tag{2}$$

where ${}_a\nabla_r^{(\alpha)} h(r) = \big({}_{a_1}^C D_{r_1}^\alpha h(r),\ {}_{a_2}^C D_{r_2}^\alpha h(r), ...,\ {}_{a_n}^C D_{r_n}^\alpha h(r)\big)$.

The following example shows that a classically non-convex function can be convex *w.r.t.* the CFD.

Example 7. Let $h : \left[-\frac{1}{6}, \frac{1}{6}\right] \to \mathbb{R}$ defined by $h(x) = x^3 + x$.
It is clearly seen that the function is not convex in the classical sense. Let us verify that h is convex in the sense of CFDs. Here, we are considering a univariate function, therefore, ${}_a\nabla_r^{(\alpha)} h(r) = {}_a^C D_r^\alpha h(r)$.
Consider $a = -\frac{1}{6}$ and $\alpha = \frac{1}{2}$.
Now

$$\begin{aligned} {}_{-\frac{1}{6}}^{\ C} D_r^{\frac{1}{2}} (r^3 + r) &= \frac{1}{\Gamma\left(1 - \frac{1}{2}\right)} \int_{-\frac{1}{6}}^{r} \frac{h'(\omega)}{(r-\omega)^{\frac{1}{2}}} d\omega \\ &= \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{6}}^{r} \frac{3\omega^2 + 1}{(r-\omega)^{\frac{1}{2}}} d\omega \\ &= -\left(1.81r^2 - 0.15045r + 1.1471855\right) \left(r + \frac{1}{6}\right)^{\frac{1}{2}} \end{aligned}$$

By using the definition(2) for $\left(x + \frac{1}{6}\right)^{\frac{1}{2}} \geq \left(r + \frac{1}{6}\right)^{\frac{1}{2}}$,

$$\begin{aligned} &h(x) - h(r) - \tfrac{1}{\Gamma(\frac{3}{2})}\, {}_{-\frac{1}{6}}^{\ C} D_r^{\frac{1}{2}} (r^3 + r) \left[\left(x + \tfrac{1}{6}\right)^{\frac{1}{2}} - \left(r + \tfrac{1}{6}\right)^{\frac{1}{2}}\right] \\ &= x^3 + x - r^3 - r + \tfrac{2}{\sqrt{\pi}} \left(1.81r^2 - 0.15045r + 1.1471855\right) \left(r + \tfrac{1}{6}\right)^{\frac{1}{2}} \left[\left(x + \tfrac{1}{6}\right)^{\frac{1}{2}} - \left(r + \tfrac{1}{6}\right)^{\frac{1}{2}}\right] \\ &\geq 0 \end{aligned}$$

So, $h(x) = x^3 + x$ is convex in the interval $\left[-\frac{1}{6}, \frac{1}{6}\right]$ *w.r.t.* the left Caputo derivative of order $\dfrac{1}{2}$.

Definition 8. Let $h : [a, b] \to \mathbb{R}$, with right CFD of order $\alpha,\ 0 \leq \alpha < 1$.
For $(b-x)^\alpha - (b-r)^\alpha \geq 0$, h is said to be convex *w.r.t.* the right CFD if

$$h(x) - h(r) \geq \tfrac{{}_r^C D_b^\alpha h(r)}{\Gamma(\alpha+1)} [(b-x)^\alpha - (b-r)^\alpha],\ \forall\, x, r \in [a, b]. \tag{3}$$

Remark 9. *Taking the Taylor's series expansion w.r.t. the right CFD [25] up to three terms, it can be easily observed that* ${}^{C}_{r}D^{2\alpha}_{b}h(r) \geq 0$ *is a sufficient condition for convexity. When* $\alpha = 1$, ${}^{C}_{r}D^{2\alpha}_{b}h(r) \geq 0$ *generalizes to* $h''(r) \geq 0$, *which suffices the condition for convexity in the classical sense.*

Definition 10. Let $h : E \to \mathbb{R}^n$ where $E \subset \mathbb{R}^n$ and have right CFDs of order α, $0 \leq \alpha < 1$. For $(b-x)^{\alpha} - (b-r)^{\alpha} \geq 0$, h is said to be convex w.r.t. the right CFD if

$$h(x) - h(r) \geq \frac{{}_{r}\nabla^{(\alpha)}_{b} h(r)}{\Gamma(\alpha+1)}[(b-x)^{\alpha} - (b-r)^{\alpha}],\ \forall\, x, r \in E \tag{4}$$

where ${}_{r}\nabla^{(\alpha)}_{b} h(r) = \left({}^{C}_{r_1}D^{\alpha}_{b_1}h(r),\ {}^{C}_{r_2}D^{\alpha}_{b_2}h(r), \ldots,\ {}^{C}_{r_n}D^{\alpha}_{b_n}h(r)\right)$.

Convexity w.r.t. the CFD is more general than the classical convexity. It can be easily seen from the following example.

Example 11. Let $h : \left[-\frac{1}{6}, \frac{1}{6}\right] \to \mathbb{R}$ defined by $h(x) = (x+1)^3$.
It is clearly seen that the function is not convex in classical sense. Let us verify that h is convex in the sense of CFDs. Here, we are considering a univariate function; therefore ${}_{r}\nabla^{(\alpha)}_{b} h(r) = {}^{C}_{r}D^{\alpha}_{b}h(r)$.
Consider $b = \frac{1}{6}$ and $\alpha = \frac{1}{2}$.
Now

$$\begin{aligned}
{}^{C}_{r}D^{\frac{1}{2}}_{\frac{1}{6}}((r+1)^3) &= -\frac{1}{\Gamma\left(1-\frac{1}{2}\right)}\int_{r}^{\frac{1}{6}} \frac{h'(\omega)}{(\omega-r)^{\frac{1}{2}}}d\omega \\
&= -\frac{1}{\sqrt{\pi}}\int_{r}^{\frac{1}{6}} \frac{3(\omega+1)^2}{(\omega-r)^{\frac{1}{2}}}d\omega \\
&= -\left(1.81r^2 - 6.619824r + 3.78007\right)\left(\frac{1}{6}-r\right)^{\frac{1}{2}}
\end{aligned}$$

By the definition(4) for $\left(\frac{1}{6}-x\right)^{\frac{1}{2}} \geq \left(\frac{1}{6}-r\right)^{\frac{1}{2}}$,

$$\begin{aligned}
&h(x) - h(r) - \tfrac{1}{\Gamma(\frac{3}{2})}\, {}^{C}_{r}D^{\frac{1}{2}}_{\frac{1}{6}}(-r^3 - r)\left[\left(\tfrac{1}{6}-x\right)^{\frac{1}{2}} - \left(\tfrac{1}{6}-r\right)^{\frac{1}{2}}\right] \\
&= -x^3 - x + r^3 + r - \tfrac{2}{\sqrt{\pi}}\left(1.81r^2 + 0.15045r + 1.1471855\right)\left(\tfrac{1}{6}-r\right)^{\frac{1}{2}}\left[\left(\tfrac{1}{6}-x\right)^{\frac{1}{2}} - \left(\tfrac{1}{6}-r\right)^{\frac{1}{2}}\right] \\
&\geq 0
\end{aligned}$$

So, $h(x) = (x+1)^3$ is convex in the interval $\left[-\frac{1}{6}, \frac{1}{6}\right]$ *w.r.t.* the right Caputo derivative of order $\frac{1}{2}$.

3.2. Result and Discussion

Consider the NLPP

$$\begin{aligned}
&(P)\ \min_{x \in C}\ h(x) \\
&\text{s.t.}\ \ k(x) \preceq 0,
\end{aligned} \tag{5}$$

(where "$\preceq$" refers that each component of the vector is less than or equal to 0 and it is considered in that sense throughout the paper), where $h(x), k_1(x), ..., k_m(x)$ are left CF differentiable functions of order α defined on a set $E \subset \mathbb{R}^n$. Further, let $k(x)$ denote the vector $(k_1(x), ..., k_m(x))^T$. According to the KKT theorem it is necessary, that for x_0 to be the minimal point of (P), there exists a vector $\lambda_0 \in \mathbb{R}^m$ such that for $0 \leq \alpha < 1$, the Lagrangian function

$$L(x, \lambda_0) = h(x) + \lambda_0^T k(x)$$

satisfies the following conditions at x_0:

$$\begin{aligned} {}_a\nabla_r^{(\alpha)} h(x_0) + {}_a\nabla_r^{(\alpha)} v_0^T k(x_0) = 0, \quad & (6) \\ \lambda_0^T k(x_0) = 0, \quad & (7) \\ \textit{and} \quad \lambda_0 \succeq 0. \quad & (8) \end{aligned}$$

(where "$\succeq$" refers that each component of the vector is greater than or equal to 0 and it is considered throughout the paper.)

Theorem 12. *Let $h : \mathbb{R}^n \to \mathbb{R}$ and $k : \mathbb{R}^n \to \mathbb{R}^m$ be left Caputo fractionally differentiable functions of order α on $E \subset \mathbb{R}^n$ satisfying the condition (2). If there exist $x_0 \in E$ and $\lambda_0 \in \mathbb{R}^m$ satisfying the KKT conditions (6)-(8) then $h(x_0)$ is the optimal solution of the problem (P).*

***Proof*:** For any $x \in C$ satisfying $h(x) \leq 0$, we have

$$\begin{aligned} h(x) - h(x_0) &\geq \tfrac{{}_a\nabla_{x_0}^{(\alpha)} h(x_0)}{\Gamma(\alpha+1)} [(x-a)^\alpha - (x_0-a)^\alpha] \\ &= -\tfrac{{}_a\nabla_{x_0}^{(\alpha)} (\lambda_0^T k(x_0))}{\Gamma(\alpha+1)} [(x-a)^\alpha - (x_0-a)^\alpha] \quad \text{(by (6))} \\ &\geq -\lambda_0^T (k(x) - k(x_0)) = -\lambda_0^T k(x) \ \big(\text{by convexity property of } \lambda_0^T k \text{ \& by (7)}\big) \\ &\geq 0, \qquad \text{(by (5) and (8))}. \end{aligned}$$

So, $h(x_0)$ is the optimal solution of the problem (P). ■

Again, consider the same NLPP

$$\begin{aligned} (P) \ \min_{x \in C} \ & h(x) \\ \text{s.t.} \ & k(x) \preceq 0, \quad (9) \end{aligned}$$

where $h(x), k_1(x), ..., k_m(x)$ are right CF differentiable functions of order α defined on a set $E \subset \mathbb{R}^n$, and let $k(x)$ denote the vector $(k_1(x), ..., k_m(x))^T$. According to the KKT theorem it is necessary, under certain constraint qualifications, that for x_0 to be the minimal point of (P), there exists a vector $\lambda_0 \in \mathbb{R}^m$ such that for $0 \leq \alpha < 1$, the Lagrangian function

$$L(x, \lambda_0) = h(x) + \lambda_0^T k(x)$$

satisfies the following conditions at x_0.

$$\begin{aligned} {}_r\nabla_b^{(\alpha)} h(x_0) + {}_r\nabla_b^{(\alpha)} \lambda_0^T k(x_0) = 0, \quad & (10) \\ \lambda_0^T k(x_0) = 0, \quad & (11) \\ \textit{and} \ \lambda_0 \succeq 0. \quad & (12) \end{aligned}$$

The next theorem provides sufficient conditions for the optimal solution for the problem P under the KKT condition.

Theorem 13. *Let $h : \mathbb{R}^n \to \mathbb{R}$ and $k : \mathbb{R}^n \to \mathbb{R}^m$ be right CF differentiable functions of order α on $E \subset \mathbb{R}^n$ satisfying (4). If there exists $x_0 \in E$ and $\lambda_0 \in \mathbb{R}^m$ satisfying the KKT conditions (10)-(12) then $h(x_0)$ is the optimal solution of the problem (P).*

***Proof*:** The prove is similar to that of Theorem (12) and hence omitted. ■

4. IMPORTANCE OF CAPUTO FRACTIONAL ORDER DERIVATIVES

We mention the importance of CFDs are explained in Examples 14 and 15.

The objective and constraint functions of the NLPP described in Example 14 are not convex *w.r.t.* the classical derivative, but convex *w.r.t.* the left CFD. Hence, it fails to satisfy the sufficient optimality conditions *w.r.t.* the classical derivative, but is able to satisfy sufficient optimality conditions *w.r.t.* the left CFD.

Example 14. Suppose $E = [\frac{1}{2}, \frac{3}{2}]$. Let $h,\ k$ be functions from E to $\mathbb{R}$, defined by $h(x) = x^4 - 2x^3 - 3x^2 + 10x + 5$ and $k(x) = x^3 - 6x^2 + 11x - 6$.
Now consider the NLPP

$$\begin{array}{ll} Minimize & h(x) = x^4 - 2x^3 - 3x^2 + 10x + 5,\ \forall x \in C(convex) \\ subject\ to & k(x) = x^3 - 6x^2 + 11x - 6 \leq 0, \\ & \qquad\qquad\qquad\qquad \forall x \geq 0. \end{array}$$

At the point $x = 1 \in [\frac{1}{2}, \frac{3}{2}]$, the objective and constraint functions considered in the above problem are not convex in the classical sense. But these functions are convex *w.r.t.* the left CFD in $[\frac{1}{2}, \frac{3}{2}]$. Now the corresponding Lagrangian function is given by,

$$\begin{array}{l} L(x,\lambda) = h(x) + \lambda k(x) \\ \Rightarrow L(x,\lambda) = x^4 - 2x^3 - 3x^2 + 10x + 5 + \lambda(x^3 - 6x^2 + 11x - 6). \end{array}$$

Firstly, consider the KKT conditions in the sense of the classical derivative,

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 4x^3 - 6x^2 - 6x + 10 + \lambda(3x^2 - 12x + 11) = 0,\ and \tag{13}$$

$$\lambda(x^3 - 6x^2 + 11x - 6) = 0 \tag{14}$$

Solving the system of equations (13) and (14) we have, for $x = 1, 2, 3$ the values of λ are given by $\lambda = -1, 6, -23$, respectively. Of course $x = 2$ is a KKT point, but not the minimum point, because the objective and constraint functions are nonconvex.

Now consider the KKT conditions *w.r.t.* the left CFD,

$${}_a\nabla_x^{(\alpha)} L(x,\lambda) = {}_a\nabla_x^{(\alpha)} h(x) + {}_a\nabla_x^{(\alpha)} \lambda k(x) = 0.$$

This implies,

$${}_a\nabla_x^{(\alpha)} L(x,\lambda) = {}_a\nabla_x^{(\alpha)} (x^4 - 2x^3 - 3x^2 + 10x + 5) + {}_a\nabla_x^{(\alpha)} (\lambda(x^3 - 6x^2 + 11x - 6)) = 0.$$

Let $x \in [\frac{1}{2}, \frac{3}{2}]$, $n = 1$ and $\alpha = \frac{1}{2}$.

$$
\begin{aligned}
&{}_{\frac{1}{2}}\nabla_x^{(\frac{1}{2})} L(x,\lambda) = 0 \\
\Rightarrow\ & {}_{\frac{1}{2}}\nabla_x^{(\frac{1}{2})}(x^4 - 2x^3 - 3x^2 + 10x + 5) + {}_{\frac{1}{2}}\nabla_x^{(\frac{1}{2})}(\lambda(x^3 - 6x^2 + 11x - 6)) = 0 \\
\Rightarrow\ & \frac{1}{\Gamma(\frac{1}{2})}\left[\int_{\frac{1}{2}}^{x} \frac{h'(\omega^4-2\omega^3-3\omega^2+10\omega+5)}{(x-\omega)^{\frac{1}{2}}} d\omega + \int_{\frac{1}{2}}^{x} \frac{h'(\lambda(\omega^3-6\omega^2+11\omega-6))}{(x-\omega)^{\frac{1}{2}}} d\omega\right] = 0 \\
\Rightarrow\ & \int_{\frac{1}{2}}^{x} \frac{(4\omega^3-6\omega^2-6\omega+10)}{(x-\omega)^{\frac{1}{2}}} d\omega + \int_{\frac{1}{2}}^{x} \frac{\lambda(3\omega^2-12\omega+11)}{(x-\omega)^{\frac{1}{2}}} d\omega = 0 \\
\Rightarrow\ & \frac{-8}{7}\left(x-\frac{1}{2}\right)^3 + \frac{24}{5}x\left(x-\frac{1}{2}\right)^2 - 8x^2\left(x-\frac{1}{2}\right) + 8x^3 - \frac{12}{5}\left(x-\frac{1}{2}\right)^2 \\
& +8x\left(x-\frac{1}{2}\right) - 12x^2 + 4\left(x-\frac{1}{2}\right) - 12x + 10 \\
& +\lambda\left[\frac{6}{5}\left(x-\frac{1}{2}\right)^2 - 4x\left(x-\frac{1}{2}\right) + 6x^2 + 8\left(x-\frac{1}{2}\right) - 24x + 22\right] = 0,
\end{aligned}
\tag{15}
$$

$$
\textit{and},\ \lambda(x^3 - 6x^2 + 11x - 6) = 0. \tag{16}
$$

Solving equations (15) and (16), we obtain, for $x = 1, 2, 3$ the values of are $\lambda = 0.562358, 4.081632, -86.095239$, respectively. It is easily seen that $x = 1$ is the KKT point *w.r.t.* the left CFD and the minimum solution of the problem is attained at $x = 1$ and is 11.
However, the point $x = 2$ does not lie in the feasible region.
Therefore, it is clearly seen that the KKT conditions *w.r.t.* the left CFD are the sufficient conditions for optimality, even when the classical derivatives fails to give any solution.

In the light of Example 14, another NLPP is constructed for the right CFD.

Example 15. Suppose h, k are two real valued functions define on $[\frac{1}{2}, \frac{3}{2}]$ and defined by $h(x) = x^4 - 4x^3 + 5x^2 - 3x + 5$, $k(x) = x^3 - 9x^2 + 15x - 7$.
Now consider the following mathematical programming problem

$$
\begin{aligned}
&Minimize && h(x) = x^4 - 4x^3 + 5x^2 - 3x + 5\ \forall x \in C \\
&subject\ to && k(x) = x^3 - 9x^2 + 15x - 7 \leq 0, \\
&&& \forall x \geq 0.
\end{aligned}
$$

At the point $x = 1 \in [\frac{1}{2}, \frac{3}{2}]$, the objective and constraint functions considered here are not convex in the classical sense. But these functions are convex *w.r.t.* the right CFD in $[\frac{1}{2}, \frac{3}{2}]$.
Consider the Lagrangian function,

$$
\begin{aligned}
& L(x,\lambda) = h(x) + \lambda^T k(x) \\
\Rightarrow\ & L(x,\lambda) = x^4 - 4x^3 + 5x^2 - 3x + 5 + \lambda(x^3 - 9x^2 + 15x - 7)
\end{aligned}
$$

Let us discuss the KKT conditions for the classical derivative case,

$$
\frac{\partial L}{\partial x} = 0
$$

$$
\Rightarrow 4x^3 - 12x^2 + 10x - 3 + \lambda(3x^2 - 18x + 15) = 0,\ \textit{and} \tag{17}
$$

$$
\lambda(x^3 - 9x^2 + 15x - 7) = 0. \tag{18}
$$

Solving equations (17) and (18) we have, for $x = 1, 1$ and $x = 7$ the value $\lambda =$ *any finite value* and -23.638888, respectively.

Now, consider the KKT conditions *w.r.t.* the right CFD,

$$\begin{aligned}
&{}_x\nabla_b^{(\beta)} L(x,\lambda) = {}_x\nabla_b^{(\beta)} h(x) + {}_x\nabla_b^{(\beta)} \lambda^T k(x) = 0,\\
\Rightarrow\ &{}_x\nabla_b^{(\beta)} L(x,\lambda) = {}_x\nabla_b^{(\beta)} (x^4 - 4x^3 + 5x^2 - 3x + 5) + {}_x\nabla_b^{(\beta)} (\lambda(x^3 - 9x^2 + 15x - 7)) = 0.
\end{aligned}$$

Let $x \in [\frac{1}{2}, \frac{3}{2}]$, $n = 1$ and $\beta = \frac{1}{2}$.

$$\begin{aligned}
{}_x\nabla_{\frac{3}{2}}^{(\frac{1}{2})} L(x,\lambda) &= {}_x\nabla_{\frac{3}{2}}^{(\frac{1}{2})} (x^4 - 4x^3 + 5x^2 - 3x + 5) + {}_x\nabla_{\frac{3}{2}}^{(\frac{1}{2})} (\lambda(x^3 - 9x^2 + 15x - 7)) = 0\\
\Rightarrow \frac{(-1)}{\Gamma(\frac{1}{2})} &\left[\int_x^{\frac{3}{2}} \frac{h'(\omega^4 - 4\omega^3 + 5\omega^2 - 3\omega + 5)}{(\omega - x)^{\frac{1}{2}}} d\omega + \int_x^{\frac{3}{2}} \frac{h'(\lambda(\omega^3 - 9\omega^2 + 15\omega - 7))}{(\omega - x)^{\frac{1}{2}}} d\omega\right] = 0\\
\Rightarrow &\int_x^{\frac{3}{2}} \frac{4\omega^3 - 12\omega^2 + 10\omega - 3}{(\omega - x)^{\frac{1}{2}}} d\omega + \int_x^{\frac{3}{2}} \frac{\lambda(3\omega^2 - 18\omega + 15)}{(\omega - x)^{\frac{1}{2}}} d\omega = 0\\
\Rightarrow &\frac{8}{7}\left(\frac{3}{2} - x\right)^3 + \frac{24}{5}x\left(\frac{3}{2} - x\right)^2 + 8x^2\left(\frac{3}{2} - x\right) + 8x^3 - \frac{24}{5}\left(\frac{3}{2} - x\right)^2\\
&-16x\left(\frac{3}{2} - x\right) - 24x^2 + \frac{20}{3}\left(\frac{3}{2} - x\right) + 20x - 6\\
&+\lambda\left[\frac{6}{5}\left(\frac{3}{2} - x\right)^2 + 4x\left(\frac{3}{2} - x\right) + 6x^2 - 12\left(\frac{3}{2} - x\right) - 36x + 30\right] = 0, \textit{ and}
\end{aligned} \tag{19}$$

$$\lambda(x^3 - 9x^2 + 15x - 7) = 0. \tag{20}$$

Solving equations (19) and (20) we get, for $x = 1, 1$ and $x = 7$ the value of $\lambda =$ 0.083455 *and* 2.301488, respectively. The point $x = 7$ does not lie in the feasible region. So, $x = 1$ is the only KKT point. Hence, the minimum solution is attained with minimum value 4.

From Example 15, it is observed that although the classical derivative fails to give any KKT points, the right CFD gives the KKT point as well as the minimum solution. This justify the use of CFDs.

5. CONCLUSION

A new type of convexity is described in this paper with the use of Caputo derivatives. We have defined KKT conditions *w.r.t.* the CFDs are defined. We have shown that these conditions are sufficient conditions for optimality. Several counterexamples are presented in order to justify our findings. The definitions, theorems and their proofs presented here are very similar to those used to prove classical convexity. Many notions of convexity can be extended with minor variations using fractional calculus. The usefulness of fractional derivatives will be helpful to describe many systems more accurately. It is expected that more research will continue in this field.

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