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Research Article

AUGMENTED FUZZY LAGRANGE PENALTY METHOD FOR FUZZY NONLINEAR PROGRAMMING PROBLEMS

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Abstract: This article introduces the Augmented Fuzzy Lagrange Multiplier Method (AFLM) as an advanced technique for solving fuzzy nonlinear programming problems (FNLPP) without requiring conversion into crisp equivalents. Unlike traditional methods, which may lead to information loss, AFLM directly transforms constrained fuzzy optimization problems into unconstrained ones while preserving the fuzzy characteristics of the data. The proposed method employs novel fuzzy arithmetic operations and ranking techniques tailored to the parametric representation of Triangular Fuzzy Numbers (TFN), enabling precise handling of both equality and inequality constraints. A key contribution of this work is the development of a convergence theorem and a supporting lemma, ensuring the theoretical soundness of AFLM. The effectiveness of the approach is demonstrated through numerical comparisons, highlighting its ability to achieve more accurate and interpretable solutions than existing methods. The results underscore AFLMs significance as a powerful tool for addressing optimization problems in uncertain environments, making it a valuable asset for decision-makers in fuzzy nonlinear programming problems.

Keywords: Equality and inequality constraints, augmented fuzzy Lagrange multiplier, fuzzy constraints, fuzzy environment, penalty function, uncertain parameters.

MSC: 03B52,03E72,90C70,90C30.

1. INTRODUCTION

Al-Naemi et al. [1] explored an innovative conjugate gradient formula derived from the memory less, self-scaling DFP quasi-Newton (QN) technique. Burman et al. [2] provided an overview of recent developments in utilizing the ALM method, highlighting its effectiveness as a general technique for developing stabilized methods without requiring explicit multipliers. Cesarano et al. [3] presented a class of Bessel-type functions obtained through the application of an isomorphism associated with Laguerre-type exponentials. Additionally, they discussed potential extensions and generalizations to the multivariable case. Dattoli et al.[4] explored the theory of Lagrange polynomials along with their generalized extensions, employing two distinct methodologies: the integral transform method and Umbral Calculus. The proposed approaches serve as effective tools for analyzing the characteristics of this polynomial family. Dattoli and Cesarano [5] utilized the integral representation method to establish a connection between conventional Hermite polynomials and a newly introduced family of Hermite polynomials associated with parabolic cylinder functions. El-Morsy [6] presented a novel approach for modeling Zero-Based Budgeting (ZBB) in a fuzzy environment, utilizing triangular fuzzy numbers to represent uncertain budget data. Gupta and Hira [7] explained LPP and NLPP, by using scientific methods and machinery, operations research is an organized approach to solving problems relevant to system operations. This well-established discipline provides an extensive collection of methods that are frequently used to address various real-world problems.

A penalty function called the logarithmic penalty function and its convergence properties were assessed and proposed by Hassan et al. [8]. Jayswal [9] investigated new insights into multi-time first-order PDE-constrained control optimization problems (MCOPU) involving uncertain data and developed strong sufficient conditions to determine optimal solutions for these problems. Jia et al. [10] concentrated on the theoretical study and computational testing of an augmented Lagrangian method designed to address optimization problems with geometric constraints. Kanzow et al. [11] examined the potential of applying a standard safeguarded multiplier penalty method, without customization for specific problems, to solve reformulated optimization challenges. Li and Xu [12] proposed a novel first-order method based on the augmented Lagrangian (AL) approach, tailored for optimization problems with nonconvex objectives and convex constraints, developed within the proximal point (PP) method framework. Liu et al. [13] introduced a smooth augmented Lagrangian technique for solving nonlinear optimization problems with general inequality constraints, demonstrating that a strict local minimizer of the original problem corresponds to an approximate strict local solution of the augmented Lagrangian function. Michael and Rolf [14] presented an innovative, efficient, and accurate method for nonlinear constrained optimization, integrating the exterior penalty function (EPF) method with a backtracking line search and the adaptive ensemble-based optimization (EnOpt) technique. Ming Ma et al. [15] presented a new fuzzy arithmetic that has been defined and used to perform on fuzzy calculus and solve fuzzy linear equations. Na et al. [16] examined optimization problems that involve randomness in the objective function while having fixed equality and inequality constraints. These types of problems appear in various fields such as power systems, deep neural networks, manufacturing, more recently and finance.

Nagoorgani and Sudha [17] discussed the conditions for finding the best solutions in fuzzy nonlinear minimization problems that do not have constraints and use triangular fuzzy numbers. Palanivel and Cebi [18] presented a fuzzy-based mathematical framework utilizing LM conditions to solve Nonlinear Programming (NLP) problems with equality constraints.Palanivel and Muralikrishna [19] introduced a fuzzy mathematical model that applies Beale's condition to address nonlinear programming (NLP) problems with uncertainty in inequality constraints. The model further demonstrates the application of membership functions (MFs) in solving quadratic programming problems. Paryzad et al. [20] proposed a model to address the discrete time-cost-quality trade-off problem. For every activity, a specific execution mode can be chosen from several available options. While the time and cost associated with each mode are treated as precise values, the quality is represented as a linguistic variable. Purnima Raj and Ranjana [21] suggested technique is used to the fuzzy solution of completely FNLPPs with inequality constraints, especially for situations where inequality constraints occur in real-world scenarios. Rockafellar [22] expanded the augmented Lagrangian method (ALM) to a wide range of generalized NLPPs, including convex and non-convex optimization. Salman and Jilawi [23] introduced the exterior and interior penalty function methods for solving constrained optimization problems by transforming them into equivalent unconstrained problems. Sama et al. [24] introduced an innovative approach to minimize a fuzzy single-objective function while adhering to fuzzy constraints. Their method employs an algorithm that relies on the null set concept to address the optimization process effectively.

Torrealba et al. [25] introduced a set of algorithms designed to address the continuous nonlinear resource allocation problem, often referred to in the literature as the Knapsack problem. Vanaja et.al, [26, 27] developed an interior fuzzy penalty function approach for addressing FNLPP. They also introduced novel ranking methods and fuzzy arithmetic based on parametric representations of triangular fuzzy numbers have been employed, combined with the exterior penalty method using fuzzy-valued functions, to solve (FNLPP). Yan [28] introduced the augmented Lagrange multiplier method, detailing its application to nonlinear mathematical equations and providing a comprehensive summary of its principles and implementation. In 1965, Zadeh [29] introduced fuzzy sets because traditional mathematics struggled to handle real-life problems that are uncertain and imprecise. Fuzzy sets help us solve these problems more effectively and play a vital role in addressing real-world issues.

The significant contributions of this research are:

- A new approach based on the Fuzzy Augmented Lagrangian Multiplier (FALM) method is proposed to solve Fuzzy Nonlinear Programming Problems (FNLPP) directly without transforming them into crisp nonlinear programming problems, thereby preserving fuzzy information.
- The proposed method employs novel fuzzy arithmetic and ranking techniques based on the parametric representation of triangular fuzzy numbers (TFNs) to effectively handle both equality and inequality constraints.
- A convergence theorem and a supporting lemma are established to demonstrate the

theoretical validity and robustness of the proposed method.

- The effectiveness of the proposed approach is illustrated through numerical examples and comparisons with existing methods.

Nomenclature

$\tilde{x}$	Variables	$\tilde{\delta}$	Objective function
$\tilde{\theta}$	Equality constraint	$\tilde{\tau}$	Inequality constraint
$\approx$	Equivalent in fuzzy sense	$\succ$	Greater than in fuzzy sense
$\prec$	Less than in fuzzy sense	$\succeq$	Greater than or equal to in fuzzy sense
$\preceq$	Less than or equal to in fuzzy sense	λ	Fuzzy lagrange multiplier
μ	Penalty Parameter (Used in the final AFLF)	ρ	Iteration counter
$\tilde{\mathcal{L}}_A$	Augmented lagrange multiplier	ϑ	ϑ- cut of fuzzy number
α	Penalty Function	γ	Growth parameters
σ	Auxiliary Penalty Parameter (Used in the intermediate ALM formulation)		

2. PRELIMINARIES

Definition 1. *A fuzzy number $\tilde{C}$ is a fuzzy set defined on R, where its membership function $\tilde{C} : R \to [0,1]$ possesses the following properties:*

1. *Convexity: The function $\tilde{C}$ is convex, i.e.,* $\tilde{C}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\tilde{C}(x_1), \tilde{C}(x_2)\}$, $\lambda \in [0,1]$, *for all* $x_1, x_2 \in R$.
2. *Normality: The function $\tilde{C}$ is normal, implying that there is at least one point $x \in R$ where $\tilde{C}(x) = 1$.*
3. *Piecewise Continuity: The function $\tilde{C}$ is piecewise continuous, meaning it may have a finite number of discontinuities.*

The collection of all fuzzy numbers defined on the real line R is represented as $F(R)$.

Definition 2. *A triangular fuzzy number (TFN) $\tilde{C}$ is a fuzzy number $\tilde{C}$ on R, where $\tilde{C}$: $R \to [0,1]$ satisfies:*

$$\tilde{C}(x) = \begin{cases} \dfrac{x - c_1}{c_2 - c_1}, & s_1 \leq x \leq c_2 \\ \dfrac{c_3 - x}{c_3 - c_2}, & s_2 \leq x \leq c_3 \\ 0, & elsewhere \end{cases}$$

Let us denote this TFN by $\tilde{C} = (c_1, c_2, c_3)$.

Definition 3. *A fuzzy number $\tilde{C}$ can alternatively be described as a functions pair $(\underline{c};\ \overline{c})$ where $\underline{c}(\vartheta)$, $\overline{c}(\vartheta)$are defined for $0 \leq \vartheta \leq 1$ and meet the subsequent conditions:*

1. *$\underline{c}(\vartheta)$ is a left-continuous function that is bounded and increases monotonically.*
2. *$\overline{c}(\vartheta)$ is a left-continuous function that is bounded and decreases monotonically.*
3. *$\underline{c}(\vartheta) \leq \overline{c}(\vartheta), 0 \leq \vartheta \leq 1$.*

Definition 4. *Consider a triangular fuzzy number $\tilde{C} = (c_1, c_2, c_3) = (\underline{c}, \overline{c})$ where $\underline{c}(\vartheta) = c_1 + (c_2 - c_1)\vartheta$, $\overline{c}(\vartheta) = c_3 - (c_3 - c_2)\vartheta$, $\vartheta \in [0,1]$. The parametric form of the TFN is defined as $\tilde{C} = (c_0, c_*, c^*)$, where $c_* = c_0 - \underline{c}$ and $c^* = \overline{c} - c_0$ are the left and right fuzziness index functions respectively. The number $c_0 = \left(\dfrac{\underline{c}(1) + \overline{c}(1)}{2}\right)$ is called the location index number. When $\vartheta = 1$, we get $c_0 = c_2$.*

2.1. Arithmetic Operations on Fuzzy Numbers

Ming Ma et al. [15] introduced a parametric method to represent fuzzy numbers, defining them through a location index and a fuzziness index function. They suggested performing arithmetic operations on fuzzy numbers by applying standard arithmetic rules to the location indices and using the Lattice rule for the fuzziness index functions. In this rule, for elements c, d within a Lattice L, $c \vee d = \max\{c, d\}$ and $c \wedge d = \min\{c, d\}$. For any two fuzzy numbers $\tilde{C} = (c_0, c_*, c^*)$, $\tilde{D} = (d_0, d_*, d^*)$ and $* \in \{+, -, \times, \div\}$, the arithmetic operations are defined as

$$\begin{aligned}\tilde{C} * \tilde{D} &= (c_0, c_*, c^*) * (d_0, d_*, d^*) = (c_0 * d_0, c_* \vee d_*, c^* \vee d^*)\\ &= (c_0 * d_0, \max\{c_*, d_*\}, \max\{c^*, d^*\})\end{aligned}$$

In particular for $\tilde{C} = (c_0, c_*, c^*)$ and $\tilde{D} = (d_0, d_*, d^*)$ in $F(R)$, we have

1. Addition :
$$\begin{aligned}\tilde{C} + \tilde{D} &= (c_0, c_*, c^*) + (d_0, d_*, d^*)\\ &= (c_0 + d_0, \max\{c_*, d_*\}, \max\{c^*, d^*\})\end{aligned}$$
2. Subtraction :
$$\begin{aligned}\tilde{C} - \tilde{D} &= (c_0, c_*, c^*) - (d_0, d_*, d^*)\\ &= (c_0 - d_0, \max\{c_*, d_*\}, \max\{c^*, d^*\})\end{aligned}$$
3. Multiplication :
$$\begin{aligned}\tilde{C} \times \tilde{D} &= (c_0, c_*, c^*) \times (d_0, d_*, d^*)\\ &= (c_0 \times d_0, \max\{c_*, d_*\}, \max\{c^*, d^*\})\end{aligned}$$
4. Division :
$$\begin{aligned}\tilde{C} \div \tilde{D} &= (c_0, c_*, c^*) \div (d_0, d_*, d^*)\\ &= (c_0 \div d_0, \max\{c_*, d_*\}, \max\{c^*, d^*\}),\end{aligned}$$
provided $d_0 \neq 0$.

2.2. Ranking of Fuzzy Numbers

The process of ranking fuzzy numbers plays a vital role in decision-making within fuzzy environments. Numerous researchers have proposed various ranking methods, which are widely documented in the literature. This article applies an efficient ranking technique derived from the graded mean.

For $\tilde{C} = (c_0, c_*, c^*) \in F(R)$, define $\Re : F(R) \to R$ by $\Re(\tilde{C}) = \left(\dfrac{c_* + 4c_0 + c^*}{6}\right)$.

For any two TFN $\tilde{C} = (c_0, c_*, c^*)$ and $\tilde{D} = (d_0, d_*, d^*)$ in $F(R)$, we have the following comparison:

- If $\Re(\tilde{C}) < \Re(\tilde{D})$, then $\tilde{C} \prec \tilde{D}$
- If $\Re(\tilde{C}) > \Re(\tilde{D})$, then $\tilde{C} \succ \tilde{D}$
- If $\Re(\tilde{C}) = \Re(\tilde{D})$, then $\tilde{C} \approx \tilde{D}$.

3. FUZZY NON LINEAR PROGRAMMING PROBLEMS (FNLPP)

We consider Fuzzy Non-Linear Programming Problems (FNLPP) involve optimizing a nonlinear objective function under fuzzy equality and inequality constraints. These problems address uncertainties present in both the objective function and constraints, necessitating sophisticated methods to determine optimal solutions in fuzzy environments.

Let a general FNLPP be

$$\begin{aligned} &\min(or)\max \tilde{\delta}(\tilde{x}) \\ &\quad \text{sub. to } \tilde{\theta}_i(\tilde{x}) \approx \tilde{0} \text{ for } i = 1,2,\cdots,l \\ &\qquad \tilde{\tau}_i(\tilde{x}) \preceq \tilde{0} \text{ for } j = 1,2,\cdots,m, \\ &\qquad \tilde{x} \succeq \tilde{0} \end{aligned} \tag{1}$$

Here, $\tilde{\delta}, \tilde{\theta}_1, \cdots, \tilde{\theta}_l, \tilde{\tau}_1, \cdots, \tilde{\tau}_m$ represent continuous fuzzy-valued functions defined on R^n.

A vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \cdots, \tilde{x}_n)$ qualifies as a feasible solution to the FNLPP if it satisfies all constraints and fulfills the non-negativity condition specified in the problem. The set of all such feasible solutions forms the feasible region, defined by these conditions.

$$X = \{\tilde{x} \in R^n / \tilde{\theta}_i(\tilde{x}) \approx \tilde{0} \text{ for } i = 1,2,\cdots,l,\ \tilde{\tau}_j(\tilde{x}) \preceq \tilde{0} \text{ for } j = 1,2,\cdots,m \text{ and } \tilde{x} \succeq \tilde{0}\}.$$

4. AUGMENTED FUZZY LAGRANGE MULTIPLIER METHOD

The augmented fuzzy Lagrange Multiplier (AFLM) method is a technique used to solve constrained optimization problems. It combines the classical fuzzy Lagrange multiplier method with a penalty tern to handle inequality and equality constraints. This section provides the step-by-step procedures for the augmented fuzzy Lagrange multiplier method with equality and inequality constraints. Algorithms are developed to iteratively improve solutions by considering uncertainties and inconsistencies during the process. They represent the principles of augmented fuzzy Lagrange multiplier insights into managing uncertainty during optimization.

4.1. Nonlinear Optimization Problem with equality constraints

Consider a general FNLPP with equality constraint constraint

$$\begin{aligned} &\min \tilde{\delta}(\tilde{x}) \\ \text{sub. to } &\tilde{\theta}_i(\tilde{x}) \approx \tilde{0} \text{ for } i = 1,2,\cdots,m \\ &\tilde{x} \succeq \tilde{0} \end{aligned} \tag{2}$$

The fuzzy Lagrange function corresponding to Eq. (2) is given by

$$\tilde{\mathscr{L}}(\tilde{x},\lambda) = \tilde{\delta}(\tilde{x}) + \sum_{i}^{m} \lambda_i \tilde{\theta}_i(\tilde{x})$$

where $\lambda_i, i = 1,2,\cdots,m$, are the fuzzy Lagrange multipliers. The equality constraints Eq.(2) are among the conditions required for a stationary point of $\tilde{\mathscr{L}}(\tilde{x},\lambda)$. The augmented fuzzy Lagrange function, also known as $\tilde{\mathscr{L}}_A(\tilde{x},\lambda,\mu_\rho)$, is defined using the exterior penalty function approach as

$$\tilde{\mathscr{L}}_A(\tilde{x},\lambda,\mu_\rho) = \tilde{\delta}(X) + \sum_{i}^{m} \lambda_i \tilde{\theta}_i(X) + \frac{\mu}{2} \sum_{i}^{m} [\tilde{\theta}_i(X)]^2 \tag{3}$$

where μ_ρ is the penalty parameter.
The augmented fuzzy Lagrange multiplier method with equality constraints effectively transforms a constrained optimization problem into an unconstrained problem by incorporating both fuzzy Lagrange multipliers and penalty terms, facilitating the use of standard optimization techniques. Update the augmented fuzzy Lagrange multipliers using:

$$\lambda_i^{(\rho+1)} = \lambda_i^{(\rho)} + \mu^{(\rho)} \tilde{\theta}_j^{(\rho)}(\tilde{x})$$

Increase the penalty parameter to emphasize constraint satisfaction: $\mu^{(\rho+1)} = \gamma\mu^{(\rho)}$, where $\gamma > 1$.

4.2. Nonlinear Optimization Problem with inequality constraints

Consider a general FNLPP with inequality constraint

$$\begin{aligned} &\min \tilde{\delta}(\tilde{x}) \\ \text{sub. to } &\tilde{\tau}_i(\tilde{x}) \preceq \tilde{0} \text{ for } i = 1,2,\cdots,m \\ &\tilde{x} \succeq \tilde{0} \end{aligned} \tag{4}$$

The Augmented Fuzzy Lagrange Function(AFLF) is

$$\tilde{\mathscr{L}}_A(\tilde{x},\lambda,\mu_\rho) = \tilde{\delta}(\tilde{x}) + \sum_{i=1}^{m} \lambda_i \max(0,\tilde{\tau}_i(\tilde{x})) + \frac{\mu}{2} \sum_{i=1}^{m} [\max(0,\tilde{\tau}_i(X))]^2$$

where λ_i are the fuzzy Lagrange multipliers and μ_ρ is a penalty parameter.
We aim to show how this formulation works and leads to convergence.
The inequality constraints $\tilde{\tau}_i(\tilde{x}) \preceq \tilde{0}$ are convert to equality constraints by introducing slack variables $\tilde{\mathscr{W}}_i \succeq \tilde{0}$.

$\tilde{\tau}_i(\tilde{x}) + \tilde{w}_i \approx \tilde{0}$ for $i = 1, 2, \cdots, m$,

The augmented fuzzy Lagrange function $\tilde{\mathcal{L}}_A(\tilde{x}, \lambda, \tilde{w}, \mu_\rho)$ is constructed as:

$$\tilde{\mathcal{L}}_A(\tilde{x}, \lambda, \tilde{w}, \mu_\rho) = \tilde{\delta}(\tilde{x}) + \sum_{i=1}^{m} \lambda_i[\tilde{\tau}_i(\tilde{x}) + \tilde{w}_i^2] + \sum_{i=1}^{m} \mu_\rho[\tilde{\tau}_i(\tilde{x}) + \tilde{w}_i^2]^2$$

Rewriting the fuzzy Lagrange function:
By minimizing the above expression with respect to $\tilde{w}_i$, we can write

$$(\tilde{\tau}_i(\tilde{x}) + \tilde{w}_i^2) = \frac{1}{\sigma}\max(0, \sigma\tilde{\tau}_i(\tilde{x}) - \mu_i)$$

Simplifying the expression,

$$\begin{aligned}\tilde{\mathcal{L}}_A(\tilde{x}, \mu, \sigma) = \tilde{\delta}(\tilde{x}) &+ \sum_{i=1}^{m} \mu_i[\frac{1}{\sigma}\max(0, \sigma\tilde{\tau}_i(\tilde{x}) - \mu_i)] \\ &+ \frac{\sigma}{2}\sum_{i=1}^{m}[\frac{1}{\sigma}\max(0, \sigma\tilde{\tau}_i(\tilde{x}) - \mu_i)]^2\end{aligned}$$

$$\text{Also} \begin{cases} \max(0, \tilde{\tau}_i(\tilde{x}) - \frac{\mu_i}{\sigma}) = \tilde{\tau}_i(\tilde{x}), & \text{if } \tilde{\tau}_i(\tilde{x}) > \frac{\mu_i}{\sigma} \\ 0, & \text{elsewhere} \end{cases}$$

Let$\lambda = \frac{\mu_i}{\sigma}, \mu_i = \sigma\lambda_i$. The σ is absorbed into the redefinition of λ_i when you set $\lambda = \frac{\mu_i}{\sigma}$, it effectively scales down the contribution of the penalty term, making the transformation.

$$\sigma\lambda_i[\max(0, \tilde{\tau}_i(\tilde{x}) - \lambda_i)] \to \lambda_i[\max(0, \tilde{\tau}_i(\tilde{x})]$$

This step doesn't involve σ directly in the find form because λ_i itself is scaled according to σ. Thus the penalty term is correctly represented in the simplified augmented fuzzy Lagrange form without an explicit σ.

$$\tilde{\mathcal{L}}_A(\tilde{x}, \lambda, \mu) = \tilde{\delta}(\tilde{x}) + \sum_{i=1}^{m} \lambda_i[\max(0, \tilde{\tau}_i(\tilde{x}))] + \frac{\mu_i}{2\lambda_i}[\max(0, \tilde{\tau}_i(\tilde{x}))]^2$$

We have the term $\frac{\mu_i}{2\lambda_i}[\max(0, \tilde{\tau}_i(\tilde{x}))]^2$ Since $\lambda_i = \frac{\mu_i}{\sigma}$, we ultimately end up simplifying λ out of the equation due to our context and given relationships. When σ is expressed in terms of μ_i and λ_i, it effectively cancels out λ_i from the term when considering the simplified augmented fuzzy Lagrange.
The gives us the simplified form: $\frac{\mu_i}{2}(\max(0, \tilde{\tau}_i(\tilde{x})))^2$. $\because \tilde{\alpha} = \max(0, \tilde{\tau}_i(\tilde{x}))$.
The final form of the augmented fuzzy Lagrange function with inequality constraints is:

$$\tilde{\mathcal{L}}_A(\tilde{x}, \lambda, \mu) = \tilde{\delta}(\tilde{x}) + \sum_{i=1}^{m}(\lambda\max(0, \tilde{\tau}_i(\tilde{x})) + \frac{\mu_i}{2}(\max(0, \tilde{\tau}_i(\tilde{x})))^2) \tag{5}$$

$$\Rightarrow \tilde{\mathcal{L}}_A(\tilde{x}, \lambda, \mu) = \tilde{\delta}(\tilde{x}) + \sum_{i=1}^{m}(\lambda\tilde{\alpha}(\tilde{x}) + \frac{\mu_i}{2}(\tilde{\alpha}(\tilde{x}))^2) \tag{6}$$

To systematically handle inequality constraints within the AFLM framework, slack variables are introduced, converting the constraints into an equivalent equality form. The inclusion of slack variables ensures that the inequality constraints are effectively incorporated into the augmented Lagrange function while preserving the fuzzy nature of the problem. By introducing non-negative slack variables, the optimization problem avoids unnecessary crisp conversions, thereby maintaining the problem's inherent fuzziness.This transformation facilitates a smooth reformulation of the fuzzy nonlinear programming problem, allowing for efficient application of the augmented Lagrange method. Through this approach, the AFLM method directly integrates inequality constraints without requiring penalty-based approximations, leading to a more accurate fuzzy optimization framework.

Lemma 1. *Let $\tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu)$ be the augmented fuzzy Lagrange function for the NLPP with objective function $\tilde{\delta}(\tilde{x})$ and constraints $\tilde{\tau}_i(\tilde{x}) \preceq \tilde{0}$, for $i = 1,2,\cdots,m$. Let $\tilde{x}^*$ be the optimal solution and λ^* the optimal fuzzy Lagrange multipliers. Assume $\tilde{\delta}$ and $\tilde{\tau}_i$ are continuously differentiable and $\mu_i(\tilde{\alpha}_i(\tilde{x}))$ are the fuzzy membership functions. Define the sequence $\tilde{\mathcal{L}}_A(\tilde{x}^\rho,\lambda_\rho,\mu_\rho)$ such that $\mu_{\rho+1} \preceq \mu_\rho$ and $\tilde{\mathcal{L}}_A(\tilde{x}^\rho,\lambda_\rho,\mu_\rho) \to 0$ as $\rho \to \infty$.*

(i). $\tilde{\mathcal{L}}_A(\tilde{x}^\rho,\lambda_\rho,\mu_\rho) \preceq \tilde{\mathcal{L}}_A(\tilde{x}^{\rho+1},\lambda_{\rho+1},\mu_{\rho+1})$

(ii). $(\tilde{\alpha}(\tilde{x}^\rho))^2 \succeq (\tilde{\alpha}(\tilde{x}^{\rho+1}))^2$

(iii). $\tilde{\delta}(\tilde{x}^\rho) \preceq \tilde{\delta}(\tilde{x}^{\rho+1})$

(iv). $\tilde{\mathcal{L}}_A(\tilde{x}^*,\lambda_*,\mu_*) \preceq \tilde{\delta}(\tilde{x}^\rho) \preceq \tilde{\mathcal{L}}_A(\tilde{x}^\rho,\lambda_\rho,\mu_\rho)$

Proof. (i). Consider

$$\begin{aligned}
\tilde{\mathcal{L}}_A(\tilde{x}^\rho,\lambda_\rho,\mu_\rho) &= \tilde{\delta}(\tilde{x}^\rho) + \lambda_\rho \alpha(\tilde{x}^\rho) + \frac{\mu_\rho}{2}(\alpha(\tilde{x}^\rho))^2 \\
&\preceq \tilde{\delta}(\tilde{x}^{\rho+1}) + \lambda_\rho \alpha(\tilde{x}^{\rho+1}) + \frac{\mu_\rho}{2}(\alpha(\tilde{x}^{\rho+1}))^2 \\
&\preceq \tilde{\delta}(\tilde{x}^{\rho+1}) + \lambda_{\rho+1} \alpha(\tilde{x}^{\rho+1}) + \frac{\mu_{\rho+1}}{2}(\alpha(\tilde{x}^{\rho+1}))^2 \\
&\approx \tilde{\mathcal{L}}_A(\tilde{x}^{\rho+1},\lambda_{\rho+1},\mu_{\rho+1})
\end{aligned} \tag{7}$$

$$\Rightarrow \tilde{\mathcal{L}}_A(\tilde{x}^\rho,\lambda_\rho,\mu_\rho) \preceq \tilde{\mathcal{L}}_A(\tilde{x}^{\rho+1},\lambda_{\rho+1},\mu_{\rho+1})) \tag{8}$$

(ii). Consider

$$\tilde{\delta}(\tilde{x}^{\rho+1}) + \lambda_{\rho+1}\alpha(\tilde{x}^{\rho+1}) + \frac{\mu_{\rho+1}}{2}(\alpha(\tilde{x}^{\rho+1}))^2 \preceq \tilde{\delta}(\tilde{x}^\rho) + \lambda_{\rho+1}\alpha(\tilde{x}^\rho) + \frac{\mu_{\rho+1}}{2}(\alpha(\tilde{x}^\rho))^2 \tag{9}$$

$$\tilde{\delta}(\tilde{x}^\rho) + \lambda_\rho\alpha(\tilde{x}^\rho) + \frac{\mu_\rho}{2}(\alpha(\tilde{x}^\rho))^2 \preceq \tilde{\delta}(\tilde{x}^{\rho+1}) + \lambda_\rho\alpha(\tilde{x}^{\rho+1}) + \frac{\mu_\rho}{2}(\alpha(\tilde{x}^{\rho+1}))^2 \tag{10}$$

Adding Eq.(9) and Eq.(10) and simplifying, we get

$$[\lambda_{\rho+1}-\lambda_\rho][\tilde{\alpha}(\tilde{x}^{\rho+1})-\tilde{\alpha}(\tilde{x}^{\rho})]+[\frac{\mu_{\rho+1}}{2}-\frac{\mu_\rho}{2}][(\tilde{\alpha}(\tilde{x}^{\rho+1}))^2-(\tilde{\alpha}(\tilde{x}^{\rho}))^2]\succeq\tilde{0}.$$

Since $\lambda_{\rho+1}\succ\lambda_\rho, \frac{\mu_{\rho+1}}{2}\succ\frac{\mu_\rho}{2}$, we have $(\tilde{\alpha}(\tilde{x}^{\rho}))^2-(\tilde{\alpha}(\tilde{x}^{\rho+1}))^2\succeq\tilde{0}\Rightarrow(\tilde{\alpha}(\tilde{x}^{\rho}))^2\succeq(\tilde{\alpha}(\tilde{x}^{\rho+1}))^2$.
(iii). From inequality Eq.(7) we get

$$\tilde{\delta}(\tilde{x}^{\rho})+\lambda_\rho\alpha(\tilde{x}^{\rho})+\frac{\mu_\rho}{2}(\alpha(\tilde{x}^{\rho}))^2\preceq\tilde{\delta}(\tilde{x}^{\rho+1})+\lambda_\rho\alpha(\tilde{x}^{\rho+1})+\frac{\mu_\rho}{2}(\alpha(\tilde{x}^{\rho+1}))^2.$$

$$\Rightarrow\tilde{\delta}(\tilde{x}^{\rho})-\tilde{\delta}(\tilde{x}^{\rho+1})\preceq\lambda_\rho[\alpha(\tilde{x}^{\rho})-\alpha(\tilde{x}^{\rho+1})]+\frac{\mu_\rho}{2}[(\alpha(\tilde{x}^{\rho}))^2-(\alpha(\tilde{x}^{\rho+1}))^2]. \quad (11)$$

When $[\alpha(\tilde{x}^{\rho})-\alpha(\tilde{x}^{\rho+1})]\preceq\tilde{0}$ and $\lambda_\rho,\mu_\rho\succ\tilde{0}$, we have $\tilde{\delta}(\tilde{x}^{\rho})-\tilde{\delta}(\tilde{x}^{\rho+1})\preceq\tilde{0}\Rightarrow\tilde{\delta}(\tilde{x}^{\rho})\preceq\tilde{\delta}(\tilde{x}^{\rho+1})$.

(iv). Suppose that $\tilde{x}^*$ be an optimum solution, then we have $\tilde{\mathcal{L}}_A(\tilde{x}^*,\lambda^*,\mu^*)\preceq\tilde{\delta}(\tilde{x}^{\rho})$ for any feasible solution $\tilde{x}^*$

$$\begin{aligned}\tilde{\mathcal{L}}_A(\tilde{x}^*,\lambda^*,\mu^*)&\preceq\tilde{\delta}\\&\preceq\tilde{\delta}(\tilde{x}^{\rho})+\lambda_\rho\tilde{\alpha}(\tilde{x}^{\rho})+\frac{\mu_\rho}{2}(\tilde{\alpha}(\tilde{x}^{\rho}))^2\\&\approx\tilde{\mathcal{L}}_A(\tilde{x}^{\rho},\lambda_\rho,\mu_\rho)\\&\quad\Rightarrow\tilde{\mathcal{L}}_A(\tilde{x}^*,\lambda^*,\mu^*)\preceq\tilde{\mathcal{L}}_A(\tilde{x}^{\rho},\lambda_\rho,\mu_\rho).\end{aligned}$$

□

Theorem 2. *Let $\tilde{\delta}(\tilde{x}),\tilde{\tau}(\tilde{x}),\tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu)$ be continuous fuzzy valued functions and $\tilde{x}^*$ be the optimal solution. Define the sequence of solution $\tilde{x}^{\rho}$ of $\tilde{\mathcal{L}}_\rho(\tilde{x})$. If there exists an optimal solution $\tilde{x}^*$ and $\tilde{\mathcal{L}}_A(\tilde{x}^*)$, then any limit $\bar{x}$ of any convergent subsequence of $\tilde{x}^{\rho}$ is an optimal solution to original problem, when $\lambda\tilde{\alpha}_i(\tilde{x}^{\rho})\to 0, \frac{\mu}{2}\tilde{\alpha}_i(\tilde{x}^{\rho})\to 0$ as $\rho\to 0$.*

Proof. Let $\tilde{x}^{\rho}$ be a sequence of optimal solution of $\tilde{\mathcal{L}}_A(\tilde{x}$ and suppose that $\bar{x}$ is any limit point of this sequence. Then thee exists a subsequence (denoted again by $\tilde{x}^{\rho}$) such that $\tilde{x}^{\rho}\to\bar{x}$ as $\rho\to\infty$.

Since $\tilde{\delta}(\tilde{x})$ and $\tilde{\tau}(\tilde{x})$ are continuous fuzzy valued functions, taking limits gives

$$\lim_{\rho\to\infty}\tilde{\delta}(\tilde{x}^*,\lambda^*,\mu^*)=\tilde{\delta}(\bar{x}),\qquad\lim_{\rho\to\infty}\tilde{\tau}(\tilde{x}^*)\preceq\tilde{0}.$$

This shows $\bar{x}$ is feasible point for the problem Eq.(4). Then we have

$$\tilde{\mathcal{L}}_A^*=\lim_{\rho\to\infty}\tilde{\mathcal{L}}_A(\tilde{x}^{\rho},\tilde{\lambda}_\rho,\mu\rho)\succeq\tilde{\mathcal{L}}_A(\tilde{x}^*,\lambda^*,\mu^*) \quad (12)$$

and

$$\tilde{\delta}(\bar{x})=\lim_{\rho\to\infty}\tilde{\delta}(\tilde{x}^{\rho})\succeq\tilde{\mathcal{L}}_A(\tilde{x}^*,\lambda^*,\mu^*) \quad (13)$$

From Eq.(12) and Eq.(13)

$$\lim_{\rho\to\infty}[\tilde{\mathcal{L}}_A(\tilde{x}^\rho,\tilde{\lambda}_\rho,\mu\rho)-\tilde{\delta}(\tilde{x}^\rho)]=\tilde{\mathcal{L}}_A^*-\tilde{\delta}(\bar{x})$$

$$\Rightarrow \lim_{\rho\to\infty}[\frac{\mu_\rho}{2}\tilde{\alpha}(\tilde{x}^\rho)+\lambda_\rho\tilde{\alpha}(\tilde{x}^\rho)]=\tilde{\mathcal{L}}_A^*-\tilde{\delta}(\bar{x}) \tag{14}$$

(by continuity of $\tilde{\alpha}$), $\Leftrightarrow$

$$\tilde{\alpha}(\bar{x})=\lim_{\rho\to\infty}[\frac{2}{\mu_\rho}+\frac{1}{\lambda_\rho}][\tilde{\mathcal{L}}_A^*-\tilde{\delta}(\bar{x})]\approx\tilde{0}$$

Since $\tilde{\mathcal{L}}_A^*-\tilde{\delta}(\bar{x})$ is constant, $\frac{2}{\mu_\rho}\to 0, \frac{1}{\lambda}\to 0$ as $k\to\infty$. Hence $\bar{x}$ is a feasible solution to the original problem.

$$\tilde{\mathcal{L}}_A^*\preceq\tilde{\delta}(\bar{x}) \tag{15}$$

From Eq.(12) and Eq.(15)

$$\tilde{\mathcal{L}}_A^*=\tilde{\delta}(\bar{x})$$

Hence the sequence $(\tilde{x}^\rho,\lambda_\rho,\mu_\rho)$ approaches to an optimal solution for the the original constrained problem. From Eq.(12), the following is obtained,

$$\lim_{\rho\to\infty}[\frac{\mu_\rho}{2}\tilde{\alpha}(\tilde{x}^\rho)+\lambda_\rho\tilde{\alpha}(\tilde{x}^\rho)]=\tilde{\mathcal{L}}_A^*-\tilde{\delta}(\bar{x})\approx\tilde{0}$$

Thus, $\frac{\mu_\rho}{2}\tilde{\alpha}(\tilde{x}^\rho)\to 0,\lambda_\rho\tilde{\alpha}(\tilde{x}^\rho)\to 0$ as $\rho\to\infty$. □

Remark 3. *The convergence proof of the proposed method has been expanded to provide a more detailed mathematical justification in the fuzzy setting. Specifically, the handling of fuzziness in constraint transformations has been clarified by explicitly incorporating fuzzy-valued functions and their limit behavior. The revised proof demonstrates that the sequence $\tilde{x}^\rho$ converges to an optimal solution by ensuring the feasibility of the constraints and the satisfaction of the fuzzy KKT conditions. Additionally, the convergence criteria $\lambda_\rho\tilde{\alpha}(\tilde{x}^\rho)\to 0$ and $\frac{\mu_\rho}{2}\tilde{\alpha}(\tilde{x}^\rho\to 0$ have been further analyzed to show their role in the asymptotic stability of the augmented fuzzy Lagrangian function. These refinements provide a rigorous foundation for the effectiveness of the proposed method.*

5. ALGORITHM

This section provides the step-by-step procedure for the Augmented Fuzzy Lagrange Multiplier (AFLM) Method. This algorithm is developed to iteratively improve the solution by considering uncertainties and inconsistencies during the process. They represent the principles of Fuzzy and provide valuable insights into managing uncertainty during optimization.

Algorithm 1: AFLM Method for FNLPP with Equality Constraints

Input: FNLPP with equality constraints

Output: Solution

1 Formulate the FNLPP with equality constraints. Convert the triangular fuzzy number coefficient into an arithmetic number and convert to parametric form.

2 Construct the augmented fuzzy Lagrange function:

$$\tilde{\mathscr{L}}_A(\tilde{x},\lambda^{(\rho)},\mu^{(\rho)}) = \tilde{\delta}(\tilde{x}) + \sum_{i=1}^{m} \lambda_i^{(\rho)}\tilde{\theta}_i(\tilde{x}) + \frac{\mu^{(\rho)}}{2}\sum_{i=1}^{m}\tilde{\theta}_i(\tilde{x})^2$$

Compute the first and second derivatives of the fuzzy Lagrange function.

3 Initialize the fuzzy Lagrange multipliers (FLM) $\lambda_i^{(\rho)}$ and choose an initial solution $\tilde{x}^{(\rho)}$. Set the penalty parameter $\mu^{(\rho)}$ and iteration counter $k = 0$. Initialize $\lambda_i^{(0)} = 0, \mu^{(0)} = 1$.

4 Update the FLM:

$$\lambda_i^{(\rho+1)} = \lambda_i^{(\rho)} + \mu^{(\rho)}\tilde{\theta}_i^{(\rho)}(\tilde{x}^{(\rho)})$$

Check the convergence criteria: **if** $\|\tilde{\theta}(\tilde{x}^{(\rho+1)})\| \preceq \varepsilon_1$ **then**

5 Stop the algorithm.

6 **else**

7 **if** $\|\tilde{x}^{(\rho+1)} - \tilde{x}^{(\rho)}\| \preceq \varepsilon_2$ **then**

8 Stop the algorithm.

9 If convergence criteria are met, stop. Otherwise, go to Step 3.

The accuracy of the obtained results is ensured by employing convergence criteria based on predefined tolerance parameters ε_1 and ε_2. The algorithm terminates only when the constraint violations and the difference between successive iterations fall below these tolerance levels. In addition, the obtained solutions are verified to satisfy the optimality conditions and the problem constraints.

Algorithm 2: AFLM Method for FNLPP with Inequality Constraints

Input: FNLPP with inequality constraints
Output: Solution

1 Formulate the FNLPP with inequality constraints. Convert the triangular fuzzy number coefficient into an arithmetic number and convert to parametric form.

2 Construct the augmented fuzzy Lagrange function:

$$\tilde{\mathcal{L}}_A(\tilde{x},\lambda^{(\rho)},\mu^{(\rho)}) = \tilde{\delta}(\tilde{x}) + \sum_{i=1}^{m} \lambda_i^{(\rho)} \tilde{\alpha}_i(\tilde{x}) + \frac{\mu^{(\rho)}}{2} \sum_{i=1}^{m} (\tilde{\alpha}_i(\tilde{x}))^2$$

Then compute first and second derivatives of the given function.

3 Initialize the fuzzy Lagrange multipliers $\lambda_i^{(\rho)}$ for the inequality constraints $\tilde{\tau}_i(\tilde{x})$ and choose an initial solution $\tilde{x}^{(\rho)}$. Set the penalty parameter $\mu^{(\rho)}$ and iteration counter $k=0$. Initialize $\lambda_i^{(0)} = 0$, $\mu^{(0)} = 1$. Update the penalty parameter if necessary: $\mu^{(\rho+1)} = \gamma\mu^{(\rho)}$ where $\gamma > 1$ is a constant.

4 Update the fuzzy Lagrange multipliers (FLM):

$$\lambda_i^{(\rho+1)} = \max(0, \lambda_i^{(\rho)} + \mu^{(\rho)} \tilde{\tau}_i^{(\rho)}(\tilde{x}^{(\rho)}))$$

Check the convergence criteria: **if** $\|\tilde{\tau}_i(\tilde{x}^{(\rho+1)})\| \preceq \varepsilon_1$ **then**

5 Stop the algorithm.

6 **else**

7 **if** $\|\tilde{x}^{(\rho+1)} - \tilde{x}^{(\rho)}\| \preceq \varepsilon_2$ **then**

8 Stop the algorithm.

9 If convergence criteria are met, stop. Otherwise, go to Step 3.

Remark 4. *Here ε_1 and ε_2 are small positive tolerance parameters used to determine the convergence of the algorithm. The parameter ε_1 controls the feasibility tolerance of the constraint violation, while ε_2 measures the tolerance for the difference between successive iterates. In numerical implementations, these values are typically chosen in the range 10^{-4} to 10^{-6} depending on the required accuracy.*

6. NUMERICAL EXAMPLE

This section the constrained Fuzzy Nonlinear Programming Problem (FNLPP) is solved using the augmented fuzzy Lagrange function method. This approach reformulates the FNLPP by integrating fuzzy logic into the augmented Lagrange multiplier framework, effectively managing uncertainties. The numerical example demonstrates the method's application and robustness, with the discussion integrated for clarity and conciseness.

Example 6.1 Consider a NLPP discussed by Gupta and Hira [7]

$$\begin{aligned} \min \delta(x) =& 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 \\ \text{sub. to } \theta(x) =& x_1 + x_2 + x_3 = 20 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{16}$$

In real life situations the coefficients in the objective function as well as in the constraints may not be precise due to many factors such as error in measurements etc. Hence let us assume that the coefficients are imperious in nature and are assumed to be TFN.

Solution: The given FNLPP becomes,

$$\begin{aligned} \min \tilde{\delta}(\tilde{x}) =& \tilde{2}\tilde{x}_1^2 + \tilde{1}\tilde{x}_2^2 + \tilde{3}\tilde{x}_3^2 + \tilde{10}\tilde{x}_1 + \tilde{8}\tilde{x}_2 + \tilde{6}\tilde{x}_3 - \tilde{100} \\ \text{sub. to } \tilde{\theta}(\tilde{x}) =& \tilde{1}\tilde{x}_1 + \tilde{1}\tilde{x}_2 + \tilde{1}\tilde{x}_3 \approx \tilde{20} \\ & \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \succeq \tilde{0}. \end{aligned} \tag{17}$$

Also, $\tilde{2} = (1,2,3), \tilde{1} = (0,1,2), \tilde{3} = (2,3,4), \tilde{10} = (8,9,10), \tilde{8} = (7,8,9), \tilde{6} = (5,6,7), \tilde{100} = (99,100,101)$ and $\tilde{20} = (19,20,21)$. Now the problem Eq.(17) becomes,

$$\begin{aligned} \min \tilde{\delta}(\tilde{x}) =& (1,2,3)\tilde{x}_1^2 + (0,1,2)\tilde{x}_2^2 + (2,3,4)\tilde{x}_3^2 + (9,10,11)\tilde{x}_1 \\ & + (7,8,9)\tilde{x}_2 + (5,6,7)\tilde{x}_3 - (99,100,101) \\ \text{sub.t to } & (0,1,2)\tilde{x}_1 + (0,1,2)\tilde{x}_2 + (0,1,2)\tilde{x}_3 \approx (19,20,21) \\ & \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \succeq \tilde{0}. \end{aligned} \tag{18}$$

The parametric representation of Eq.(18) is expressed as follows:

$$\begin{aligned} \min \tilde{\delta}(\tilde{x}) =& (2,1-\vartheta,1-\vartheta)\tilde{x}_1^2 + (1,1-\vartheta,1-\vartheta)\tilde{x}_2^2 + (3,1-\vartheta,1-\vartheta)\tilde{x}_3^2 \\ & + (10,1-\vartheta,1-\vartheta)\tilde{x}_1 + (8,1-\vartheta,1-\vartheta)\tilde{x}_2 \\ & + (6,1-\vartheta,1-\vartheta)\tilde{x}_3 - (100,1-\vartheta,1-\vartheta) \\ \text{sub. to } & (1,1-\vartheta,1-\vartheta)\tilde{x}_1 + (1,1-\vartheta,1-\vartheta)\tilde{x}_2 + (1,1-\vartheta,1-\vartheta)\tilde{x}_3 \\ & \approx (20,1-\vartheta,1-\vartheta) \\ & \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \succeq \tilde{0}. \ \ \vartheta \in [0,1]. \end{aligned}$$

The augmented fuzzy Lagrange function is expressed as follows,

$$\tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu) = \tilde{\delta}(\tilde{x}) + \sum_{i=1}^{m} \lambda_i \tilde{\theta}_i(\tilde{x}) + \frac{\mu}{2}\sum_{i=1}^{m}(\tilde{\theta}_i(\tilde{x}))^2$$

$$\begin{aligned} \Rightarrow \tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu) =& (2,1-\vartheta,1-\vartheta)\tilde{x}_1^2 + (1,1-\vartheta,1-\vartheta)\tilde{x}_2^2 + (3,1-\vartheta,1-\vartheta)\tilde{x}_3^2 \\ & + (10,1-\vartheta,1-\vartheta)\tilde{x}_1 + (8,1-\vartheta,1-\vartheta)\tilde{x}_2 + (6,1-\vartheta,1-\vartheta)\tilde{x}_3 \\ & - (100,1-\vartheta,1-\vartheta) + \lambda((1,1-\vartheta,1-\vartheta)\tilde{x}_1 + (1,1-\vartheta,1-\vartheta)\tilde{x}_2 \\ & + (1,1-\vartheta,1-\vartheta)\tilde{x}_3 - (20,1-\vartheta,1-\vartheta)) + \frac{\mu}{2}((1,1-\vartheta,1-\vartheta)\tilde{x}_1 \\ & + (1,1-\vartheta,1-\vartheta)\tilde{x}_2 + (1,1-\vartheta,1-\vartheta)\tilde{x}_3 - (20,1-\vartheta,1-\vartheta))^2 \end{aligned} \tag{19}$$

The necessary condition for $\tilde{\mathbf{x}}$ to be an optimal solution for Eq.(19) is that $\triangledown \tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu) \approx \tilde{0}$. Hence we have

$$\begin{aligned} \frac{\partial \tilde{\mathcal{L}}_A}{\partial \tilde{x}_1} =& (4,1-\vartheta,1-\vartheta)\tilde{x}_1 + (10,1-\vartheta,1-\vartheta) + \lambda(1,1-\vartheta,1-\vartheta) \\ & \mu_\rho((1,1-\vartheta,1-\vartheta)\tilde{x}_1 + (1,1-\vartheta,1-\vartheta)\tilde{x}_2 + (1,1-\vartheta,1-\vartheta)\tilde{x}_3) \approx \tilde{0}. \end{aligned}$$

$$\frac{\partial \tilde{\mathcal{L}}_A}{\partial \tilde{x}_2} =(2,1-\vartheta,1-\vartheta)\tilde{x}_2+(8,1-\vartheta,1-\vartheta)+\lambda(1,1-\vartheta,1-\vartheta)$$
$$\mu_\rho((1,1-\vartheta,1-\vartheta)\tilde{x}_1+(1,1-\vartheta,1-\vartheta)\tilde{x}_2+(1,1-\vartheta,1-\vartheta)\tilde{x}_3) \approx \tilde{0}.$$
$$\frac{\partial \tilde{\mathcal{L}}_A}{\partial \tilde{x}_3} =(6,1-\vartheta,1-\vartheta)\tilde{x}_3+(6,1-\vartheta,1-\vartheta)+\lambda(1,1-\vartheta,1-\vartheta)$$
$$\mu_\rho((1,1-\vartheta,1-\vartheta)\tilde{x}_1+(1,1-\vartheta,1-\vartheta)\tilde{x}_2+(1,1-\vartheta,1-\vartheta)\tilde{x}_3) \approx \tilde{0}.$$

$$\begin{aligned}
\tilde{x}_1 =&\frac{-(1,1-\vartheta,1-\vartheta)\tilde{x}_2\mu_\rho-(1,1-\vartheta,1-\vartheta)\tilde{x}_3\mu_\rho+(20,1-\vartheta,1-\vartheta)\mu_\rho}{(1,1-\vartheta,1-\vartheta)\mu_\rho+(4,1-\vartheta,1-\vartheta)}\\
&+\frac{-\lambda(1,1-\vartheta,1-\vartheta)-(10,1-\vartheta,1-\vartheta)}{(1,1-\vartheta,1-\vartheta)\mu_\rho+(4,1-\vartheta,1-\vartheta)}\\
\tilde{x}_2 =&\frac{-(1,1-\vartheta,1-\vartheta)\tilde{x}_1\mu_\rho-(1,1-\vartheta,1-\vartheta)\tilde{x}_3\mu_\rho+(20,1-\vartheta,1-\vartheta)\mu_\rho}{(1,1-\vartheta,1-\vartheta)\mu_\rho+(2,1-\vartheta,1-\vartheta)}\\
&+\frac{-\lambda(1,1-\vartheta,1-\vartheta)-(8,1-\vartheta,1-\vartheta)}{(1,1-\vartheta,1-\vartheta)\mu_\rho+(2,1-\vartheta,1-\vartheta)}\\
\tilde{x}_3 =&\frac{-(1,1-\vartheta,1-\vartheta)\tilde{x}_1\mu_\rho-(1,1-\vartheta,1-\vartheta)\tilde{x}_2\mu_\rho+(20,1-\vartheta,1-\vartheta)\mu_\rho}{(1,1-\vartheta,1-\vartheta)\mu_\rho+(6,1-\vartheta,1-\vartheta)}\\
&+\frac{-\lambda(1,1-\vartheta,1-\vartheta)-(6,1-\vartheta,1-\vartheta)}{(1,1-\vartheta,1-\vartheta)\mu_\rho+(6,1-\vartheta,1-\vartheta)}
\end{aligned} \tag{20}$$

Solving these equations we get, $\tilde{\mathbf{x}}^* \approx \{\tilde{x}_1,\tilde{x}_2,\tilde{x}_3\}$, where

$$\begin{aligned}
\tilde{x}_1 \approx&\frac{(165,1-\vartheta,1-\vartheta)\mu_\rho^2+(130,1-\vartheta,1-\vartheta)\mu_\rho-(9,1-\vartheta,1-\vartheta)\mu_\rho\lambda}{(33,1-\vartheta,1-\vartheta)\mu_\rho^2+(80,1-\vartheta,1-\vartheta)\mu_\rho+(48,1-\vartheta,1-\vartheta)}+\\
&\frac{-(12,1-\vartheta,1-\vartheta)\lambda-(120,1-\vartheta,1-\vartheta)}{(33,1-\vartheta,1-\vartheta)\mu_\rho^2+(80,1-\vartheta,1-\vartheta)\mu_\rho+(48,1-\vartheta,1-\vartheta)},\\
\tilde{x}_2 \approx&\frac{(363,1-\vartheta,1-\vartheta)\mu_\rho^2+(340,1-\vartheta,1-\vartheta)\mu_\rho-(18,1-\vartheta,1-\vartheta)\mu_\rho\lambda}{(33,1-\vartheta,1-\vartheta)\mu_\rho^2+(80,1-\vartheta,1-\vartheta)\mu_\rho+(48,1-\vartheta,1-\vartheta)}+\\
&\frac{(24,1-\vartheta,1-\vartheta)\lambda-(192,1-\vartheta,1-\vartheta)}{(33,1-\vartheta,1-\vartheta)\mu_\rho^2+(80,1-\vartheta,1-\vartheta)\mu_\rho+(48,1-\vartheta,1-\vartheta)},\\
\tilde{x}_3 \approx&\frac{(44,1-\vartheta,1-\vartheta)\mu_\rho-(2,1-\vartheta,1-\vartheta)\lambda-(12,1-\vartheta,1-\vartheta)}{(11,1-\vartheta,1-\vartheta)\mu_\rho+(12,1-\vartheta,1-\vartheta)}
\end{aligned}$$

Let $\mu_{\rho+1} = \gamma\mu_\rho$. Since $\gamma > 1$, starting with $\gamma = 10, \mu_0 = 1$, we present the following Tables (1),(2).

Note:1-ϑ denotes the parameter of the Triangular Fuzzy Number (TFN), determined using the location index and the left and right fuzziness index functions, as defined in Defn (4).

From the tables above, it is evident that the augmented fuzzy Lagrange multiplier method converges at the 5^th iteration. Therefore, the optimal solution to the given FNLPP

Table 1: FALM Iteration Table with Equality Constraints

ρ	μ_ρ	$\tilde{x}_1^\rho$	$\tilde{x}_2^\rho$	$\tilde{x}_3$
0	1	(1.087,1-ϑ,1-ϑ)	(3.174,1-ϑ,1-ϑ)	(1.391,1-ϑ,1-ϑ)
1	10	(4.615,1-ϑ,1-ϑ)	(10.230,1-ϑ,1-ϑ)	(3.743,1-ϑ,1-ϑ)
2	100	(4.996,1-ϑ,1-ϑ)	(10.992,1-ϑ,1-ϑ)	(3.997,1-ϑ,1-ϑ)
3	1000	(4.999,1-ϑ,1-ϑ)	(10.999,1-ϑ,1-ϑ)	(3.999,1-ϑ,1-ϑ)
4	10000	(5.000,1-ϑ,1-ϑ)	(11.000,1-ϑ,1-ϑ)	(4.000,1-ϑ,1-ϑ)
5	100000	(5.000,1-ϑ,1-ϑ)	(11.000,1-ϑ,1-ϑ)	(4.000,1-ϑ,1-ϑ)

Table 2: Extended Results and Analysis Following Table (1)

ρ	λ^ρ	$\tilde{\theta}(x)$	$\tilde{\delta}(x)^\rho$	$\tilde{\mathcal{L}_A}(x^\rho,\lambda^\rho,\mu^\rho)$
0	(0,1-ϑ,1-ϑ)	(-14.348,1-ϑ,1-ϑ)	(-37.149,1-ϑ,1-ϑ)	(-140.082,1-ϑ,1-ϑ)
1	(-14.348,1-ϑ,1-ϑ)	(-1.412,1-ϑ,1-ϑ)	(239.727,1-ϑ,1-ϑ)	(250.018,1-ϑ,1-ϑ)
2	(-28.468,1-ϑ,1-ϑ)	(-0.015,1-ϑ,1-ϑ)	(280.550,1-ϑ,1-ϑ)	(280.966,1-ϑ,1-ϑ)
3	(-29.968,1-ϑ,1-ϑ)	(-0.003,1-ϑ,1-ϑ)	(-251.132,1-ϑ,1-ϑ)	(-251.047,1-ϑ,1-ϑ)
4	(-32.968,1-ϑ,1-ϑ)	(0,1-ϑ,1-ϑ)	(281,1-ϑ,1-ϑ)	(281,1-ϑ,1-ϑ)
5	(-32.968,1-ϑ,1-ϑ)	(0,1-ϑ,1-ϑ)	(281,1-ϑ,$1-\vartheta$)	(281,1-ϑ,1-ϑ)

in Eq.(18) is achieved is $\tilde{x}_1 = (5.000, 1-\vartheta, 1-\vartheta)$, $\tilde{x}_2 = (11.000, 1-\vartheta, 1-\vartheta)$ and $\tilde{x}_3 =$ 4.000 with $\tilde{\mathcal{L}_A}(\tilde{x},\lambda,\mu) = (281, 1-\vartheta, 1-\vartheta)$. Thus, the optimal solution to the fuzzy nonlinear programming problem is obtained. Eq.(18) is $\tilde{x}_1 = (4.000+\vartheta, 5.000, 6.000-\vartheta)$, $\tilde{x}_2 = (10.000+\vartheta, 11.000, 12.000-\vartheta)$ and $\tilde{x}_3 = (3.000+\vartheta, 4.000, 5.000-\vartheta)$ with $\tilde{\mathcal{L}_A}(\tilde{x},\lambda,\mu) = (280.000+\vartheta, 281.000, 282.000-\vartheta)$.

Example 6.2 Consider a FNLPP discussed by Purnima Raj [21]

$$\begin{aligned} \max \tilde{\delta}(\tilde{x}) =& \tilde{x}_1^2 + \tilde{x}_2^2 \\ \text{sub.t to } \tilde{\tau}_1(\tilde{x}) =& (0,1,2)\tilde{x}_1 + (1,2,3)\tilde{x}_2 \preceq (1,10,27) \\ \tilde{\tau}_2(\tilde{x}) =& (1,2,3)\tilde{x}_1 + (0,1,2)\tilde{x}_2 \preceq (10,11,12) \\ & \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{21}$$

Solution: Let us assume that all the decision parameters and coefficients are represented as Triangular Fuzzy Numbers (TFNs). Then the Eq.(21) becomes

$$\begin{aligned} \max \tilde{\delta}(\tilde{x}) =& (0,1,2)\tilde{x}_1^2 + (0,1,2)\tilde{x}_2^2 \\ \text{sub. to } \tilde{\tau}_1(\tilde{x}) =& (0,1,2)\tilde{x}_1 + (1,2,3)\tilde{x}_2 \preceq (1,10,27) \\ \tilde{\tau}_2(\tilde{x}) =& (1,2,3)\tilde{x}_1 + (0,1,2)\tilde{x}_2 \preceq (10,11,12) \\ & \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{22}$$

The maximization problem in Eq.(22) is transformed into an equivalent minimization FNLPP as follows:

$$\begin{aligned} \min \tilde{\delta}(\tilde{x}) =& -(0,1,2)\tilde{x}_1^2 - (0,1,2)\tilde{x}_2^2 \\ \text{sub. to } \tilde{\tau}_1(\tilde{x}) =& (0,1,2)\tilde{x}_1 + (1,2,3)\tilde{x}_2 \preceq (1,10,27) \\ \tilde{\tau}_2(\tilde{x}) =& (1,2,3)\tilde{x}_1 + (0,1,2)\tilde{x}_2 \preceq (10,11,12) \\ & \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{23}$$

The parametric representation of Eq.(23) is expressed as follows:

$$\begin{aligned} \min \tilde{\delta}(\tilde{x}) =& -(1,1-\vartheta,1-\vartheta)\tilde{x}_1^2-(1,1-\vartheta,1-\vartheta)\tilde{x}_2^2 \\ \text{sub. to } \tilde{\tau}_1(\tilde{x}) =&(1,1-\vartheta,1-\vartheta)\tilde{x}_1+(2,1-\vartheta,1-\vartheta)\tilde{x}_2 \preceq (10,1-\vartheta,1-\vartheta) \\ \tilde{\tau}_2(\tilde{x}) =&(2,1-\vartheta,1-\vartheta)\tilde{x}_1+(1,1-\vartheta,1-\vartheta)\tilde{x}_2 \preceq (11,1-\vartheta,1-\vartheta) \\ & \tilde{x}_1,\tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{24}$$

augmented fuzzy Lagrange multiplier method,

$$\tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu)=\tilde{\delta}(\tilde{x})+\sum_{i=1}^{m}\lambda_i\tilde{\alpha}(\tilde{x})+\frac{\mu}{2}\sum_{i=1}^{m}(\tilde{\alpha}(\tilde{x}))^2$$

where, $\tilde{\alpha}(\tilde{x})=\max(0,\tilde{\tau}_i(\tilde{x}))$. That is

$$\begin{aligned} \min \tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu) =& -(1,1-\vartheta,1-\vartheta)\tilde{x}_1^2-(1,1-\vartheta,1-\vartheta)\tilde{x}_2^2 \\ &+\lambda_1[(1,1-\vartheta,1-\vartheta)\tilde{x}_1+(2,1-\vartheta,1-\vartheta)\tilde{x}_2-(10,1-\vartheta,1-\vartheta)] \\ &+\lambda_2[(2,1-\vartheta,1-\vartheta)\tilde{x}_1+(1,1-\vartheta,1-\vartheta)\tilde{x}_2-(11,1-\vartheta,1-\vartheta)] \\ &+\frac{\mu}{2}[((1,1-\vartheta,1-\vartheta)\tilde{x}_1+(2,1-\vartheta,1-\vartheta)\tilde{x}_2-(10,1-\vartheta,1-\vartheta))^2 \\ &+((2,1-\vartheta,1-\vartheta)\tilde{x}_1+(1,1-\vartheta,1-\vartheta)\tilde{x}_2-(11,1-\vartheta,1-\vartheta))^2] \end{aligned} \tag{25}$$

The necessary condition for $\tilde{\mathbf{x}}$ to be an optimal solution for Eq.(25) is that $\triangledown\tilde{\mathcal{L}}_A(\tilde{x}.\lambda,\mu)\approx\tilde{0}$. Hence we have

$$\begin{aligned} \frac{\partial\tilde{\mathcal{L}}_A}{\partial\tilde{x}_1} =& -(1,1-\vartheta,1-\vartheta)\tilde{x}_1+\lambda_1(1,1-\vartheta,1-\vartheta)+\lambda_2(2,1-\vartheta,1-\vartheta) \\ &+\mu[((1,1-\vartheta,1-\vartheta)\tilde{x}_1+(2,1-\vartheta,1-\vartheta)\tilde{x}_2-(10,1-\vartheta,1-\vartheta)) \\ &+(2,1-\vartheta,1-\vartheta)((2,1-\vartheta,1-\vartheta)\tilde{x}_1+(1,1-\vartheta,1-\vartheta)\tilde{x}_2-(11,1-\vartheta,1-\vartheta))]\approx\tilde{0} \end{aligned}$$

$$\begin{aligned} \frac{\partial\tilde{\mathcal{L}}_A}{\partial\tilde{x}_2} =& -(1,1-\vartheta,1-\vartheta)\tilde{x}_2+\lambda_1(2,1-\vartheta,1-\vartheta)+\lambda_2(1,1-\vartheta,1-\vartheta) \\ &+\mu[(2,1-\vartheta,1-\vartheta)((1,1-\vartheta,1-\vartheta)\tilde{x}_1+(2,1-\vartheta,1-\vartheta)\tilde{x}_2-(10,1-\vartheta,1-\vartheta)) \\ &+((2,1-\vartheta,1-\vartheta)\tilde{x}_1+(1,1-\vartheta,1-\vartheta)\tilde{x}_2-(11,1-\vartheta,1-\vartheta))]\approx\tilde{0}. \end{aligned}$$

This implies that,

$$\begin{aligned} &-(1,1-\vartheta,1-\vartheta)\tilde{x}_1+\lambda_1(1,1-\vartheta,1-\vartheta)+\lambda_2(2,1-\vartheta,1-\vartheta)+\mu_\rho[(5,1-\vartheta,1-\vartheta)\tilde{x}_1 \\ &\qquad +(4,1-\vartheta,1-\vartheta)\tilde{x}_2-(32,1-\vartheta,1-\vartheta)]\approx\tilde{0} \\ &-(1,1-\vartheta,1-\vartheta)\tilde{x}_2+\lambda_1(2,1-\vartheta,1-\vartheta)+\lambda_2(1,1-\vartheta,1-\vartheta)+\mu_\rho[(4,1-\vartheta,1-\vartheta)\tilde{x}_1 \\ &\qquad +(5,1-\vartheta,1-\vartheta)\tilde{x}_2-(31,1-\vartheta,1-\vartheta)]\approx\tilde{0}. \end{aligned}$$

$$\tilde{x}_1 = \frac{\lambda_1(1,1-\vartheta,1-\vartheta)+\lambda_2(2,1-\vartheta,1-\vartheta)+(4,1-\vartheta,1-\vartheta)\tilde{x}_2\mu_\rho-(32,1-\vartheta,1-\vartheta)\mu_\rho}{(2,1-\vartheta,1-\vartheta)-(5,1-\vartheta,1-\vartheta)\mu_\rho}$$

$$\tilde{x}_2 = \frac{\lambda_1(2,1-\vartheta,1-\vartheta)+\lambda_2(1,1-\vartheta,1-\vartheta)+(4,1-\vartheta,1-\vartheta)\tilde{x}_1\mu_\rho-(31,1-\vartheta,1-\vartheta)\mu_\rho}{(2,1-\vartheta,1-\vartheta)-(5,1-\vartheta,1-\vartheta)\mu_\rho} \tag{26}$$

Solving these equations, we get $\tilde{\mathbf{x}}^\rho \approx \{\tilde{x}_1, \tilde{x}_2\}$, where

$$\begin{aligned}
\tilde{x}_1 \approx & \frac{(36,1-\vartheta,1-\vartheta)\mu_\rho^2-(64,1-\vartheta,1-\vartheta)\mu_\rho-(3,1-\vartheta,1-\vartheta)\mu_\rho\lambda_1-(6,1-\vartheta,1-\vartheta)\mu_\rho\lambda_2}{(9,1-\vartheta,1-\vartheta)\mu_\rho^2-(20,1-\vartheta,1-\vartheta)\mu_\rho+(4,1-\vartheta,1-\vartheta)}. \\
& + \frac{(2,1-\vartheta,1-\vartheta)\lambda_1+(4,1-\vartheta,1-\vartheta)\lambda_2}{(9,1-\vartheta,1-\vartheta)\mu_\rho^2-(20,1-\vartheta,1-\vartheta)\mu_\rho+(4,1-\vartheta,1-\vartheta)}, \\
\tilde{x}_2 \approx & \frac{(27,1-\vartheta,1-\vartheta)\mu_\rho^2-(62,1-\vartheta,1-\vartheta)\mu_\rho-(6,1-\vartheta,1-\vartheta)\mu_\rho\lambda_1+(3,1-\vartheta,1-\vartheta)\mu_\rho\lambda_2}{(9,1-\vartheta,1-\vartheta)\mu_\rho^2-(20,1-\vartheta,1-\vartheta)\mu_\rho+(4,1-\vartheta,1-\vartheta)} \\
& + \frac{(4,1-\vartheta,1-\vartheta)\lambda_1+(2,1-\vartheta,1-\vartheta)\lambda_2}{(9,1-\vartheta,1-\vartheta)\mu_\rho^2-(20,1-\vartheta,1-\vartheta)\mu_\rho+(4,1-\vartheta,1-\vartheta)}.
\end{aligned}$$

Let $\mu_{\rho+1} = \gamma\mu_\rho$. Since $\gamma > 1$, starting with $\gamma = 10, \mu_0 = 1$, we present the following Tables (3),(4).

Table 3: FALM Iteration Table with Inequality Constraints

ρ	μ_ρ	$\lambda_1{}^\rho$	$\lambda_2{}^\rho$	$\tilde{x}_1^\rho$	$\tilde{x}_2^\rho$
0	1	(0,1-ϑ,1-ϑ)	(0,1-ϑ,1-ϑ)	(4,1-ϑ,1-ϑ)	(5,1-ϑ,1-ϑ)
1	10	(4,1-ϑ,1-ϑ)	$(2,1-\vartheta,1-\vartheta)$	$(3.886,1-\vartheta,1-\vartheta)$	(2.727,1-ϑ,1-ϑ)
2	100	(0,1-ϑ,1-ϑ)	(0,1-ϑ,1-ϑ)	(4.018,1-ϑ,1-ϑ)	(2.998,1-ϑ,1-ϑ)
3	1000	(1.4,1-ϑ,1-ϑ)	(3.4,1-ϑ,1-ϑ)	(3.999,1-ϑ,1-ϑ)	(2.999,1-ϑ,1-ϑ)
4	10000	(0,1-ϑ,1-ϑ)	(0.4,1-ϑ,1-ϑ)	(4.000,1-ϑ,1-ϑ)	(2.999,1-ϑ,1-ϑ)
5	100000	(0,1-ϑ,1-ϑ)	(0,1-ϑ,1-ϑ)	(4.000,1-ϑ,1-ϑ)	(2.999,1-ϑ,1-ϑ)

Table 4: Extended Results and Analysis Following Table (3)

ρ	$\tilde{\tau}_1^\rho(\tilde{x})$	$\tilde{\tau}_2(\tilde{x})$	$\tilde{\delta}(x)^\rho$	$\tilde{\mathcal{L}}_A(x^\rho,\lambda^\rho,\mu^\rho)$
0	(4.000,1-ϑ,1-ϑ)	(2.000,1-ϑ,1-ϑ)	(-4.000,1-ϑ,1-ϑ)	(-31.000,1-ϑ,1-ϑ)
1	(-0.660,1-ϑ,1-ϑ)	(-0.501,1-ϑ,1-ϑ)	(-22.538,1-ϑ,1-ϑ)	(-22.747,1-ϑ,1-ϑ)
2	(0.014,1-ϑ,1-ϑ)	(0.034,1-ϑ,1-ϑ)	(-25.132,1-ϑ,1-ϑ)	(-25.064,1-ϑ,1-ϑ)
3	(-0.003,1-ϑ,1-ϑ)	(-0.003,1-ϑ,1-ϑ)	(-24.986,1-ϑ,1-ϑ)	(-24.991,1-ϑ,1-ϑ)
4	(-0.002,1-ϑ,1-ϑ)	(-0.001,1-ϑ,1-ϑ)	(-24.994,1-ϑ,1-ϑ)	(-24.969,1-ϑ,1-ϑ)
5	(-0.002,1-ϑ,1-ϑ)	(-0.001,1-ϑ,1-ϑ)	(-24.994,1-ϑ,1-ϑ)	(-24.969,1-ϑ,1-ϑ)

From the tables above, it is evident that the augmented fuzzy Lagrange multiplier method converges at the $5^t h$ iteration. Therefore, the optimal solution to the given FNLPP in Eq.(21) is achieved. is $\tilde{x}_1 = (4.000, 1-\vartheta, 1-\vartheta)$, $\tilde{x}_2 = (2.999, 1-\vartheta, 1-\vartheta)$ with $\max \tilde{\mathcal{L}_A}(\tilde{x}, \lambda, \mu) = (24.969, 1-\vartheta, 1-\vartheta)$. That is the optimal solution of the FNLPP Eq.(21) is $\tilde{x}_1 = (3.000+\vartheta, 4.000, 5.000-\vartheta)$, $\tilde{x}_2 = (1.999+\vartheta, 2.999, 3.999-\vartheta)$ with $\max \tilde{\mathcal{L}_A}(\tilde{x}, \lambda, \mu) = (23.969+\vartheta, 24.969, 25.969-\vartheta)$.

7. Result and Discussion

Tables (5) and (6) present the fuzzy optimal solutions for the FNLPP given by Eq.(18), corresponding to various values of ϑ.

Table 5: FALM for various values of $\vartheta \in [0,1]$

ϑ	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$
0	(4.000,5.000,6.000)	(10.000,11.000,12.000)	(3.000,4.000,5.000)
0.25	(4.250,5.000,5.750)	(10.250,11.000,11.750)	(3.250,4.000,5.250)
0.5	(4.500,5.000,5.500)	(10.500,11.000,11.500)	(3.500,4.000,4.500)
0.75	(4.750,5.000,5.250)	(10.750,11.000,11.250)	(3.750,4.000,4.250)
1	(5.000,5.000,5.000)	(11.000,11.000,11.000)	(4.000,4.000,4.000))
	=5.000	=11.000	=4.000

Table 6: Continuation to Table (5)

ϑ	$\tilde{\mathcal{L}_A}(\tilde{x}, \lambda, \mu)$
0	(280.000,281.000,282.000)
0.25	(280.250,281.000,281.75)
0.5	(280.500,281.000,285.250)
0.75	(280.750,281.000,281.250)
1	(281.000,281.000,281.000)
	=281.000

When $\vartheta = 1$, we see that $\tilde{x}_1 = 5.000, \tilde{x}_2 = 11.000, \tilde{x}_3 = 4.000$ and $\tilde{\mathcal{L}_A}(\tilde{x}, \lambda, \mu) = 281.000$ which is the optimal solution of crisp NLPP Eq.(16). This solution is same as the crisp optimal solution $X^* = (5, 11, 4)$with $\min \delta(x) = 281$ obtained by Gupta and Hira [7].

Also Table (7) present the fuzzy optimal solutions for the FNLPP given by Eq.(21), corresponding to various values of ϑ.

Table 7: FALM for various values of $\vartheta \in [0,1]$

ϑ	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{\mathcal{L}_A}(\tilde{x}, \lambda, \mu)$
0	(3.000,4.000,5.000)	(1.999,2.999,3.999)	(23.969,24.969,25.969)
0.25	(3.250,4.000,4.750)	(2.249,2.999,3.749)	(24.219,24.969,25.719)
0.5	(3.500,4.000,4.500)	(2.499,2.999,3.499)	(24.469,24.969,25.469)
0.75	(3.75,4.000,4.250)	(2.749,2.999,3.249)	(25.719,24.969,25.219)
1	(4.000,4.000,4.000)	(2.999,2.999,2.999)	(24.969,24.969,24.969)
	=4.000	=2.999	=24.969

The fuzzy optimal solution obtained by proposed method is vagueness less comparing with fuzzy optimal solution $x_1 = (4/3, 4, 6), x_2 = (1, 4, 17/3)$with $\delta = (25/9, 32, 613/9)$ obtained by Purnima Raj [21]. By applying proposed ranking method the given FNLPP can be reduce to crisp NLPP and by applying any existing method, the optimal solution obtained will be $x_1 = 4, x_2 = 3$ with $\delta = 25$. It is to be noted that the optimal solution at $\vartheta = 1$ is also same by the proposed method as given in **Table** (7). Additionally, from **Table** (7), it can be observed that the proposed method provides the decision maker with the flexibility to select an appropriate ϑ based on the situation and preferences. Extending the proposed method to multi-objective settings could be explored as future work, incorporating techniques like fuzzy Pareto front construction.

8. CONCLUSION

This work presents a new approach to the solution of FNLPP that utilizes the augmented fuzzy Lagrange multiplier technique and incorporates triangular fuzzy numbers. The location index, left fuzziness index, and right fuzziness index are the defining characteristics of these numbers. By applying the augmented Lagrangian multiplier method, the fuzzy optimal solution is efficiently achieved while addressing both equality and inequality constraints without converting the problem into its crisp form. An illustration is provided to show that the suggested approach yields solutions with less ambiguity compared to previous approaches. Based on the context and personal choices, individuals can select a suitable outcome by choosing an appropriate parameter of $\vartheta \in [0, 1]$. Fuzzy nonlinear programming models are widely used in engineering design problems such as optimal structural design, energy management systems, transportation planning, and manufacturing process optimization where system parameters are uncertain and can be modeled using fuzzy numbers.

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